ALMOST DISJOINTNESS AND ITS GENERALIZATIONS

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SUMMARY

Recently, some modifications of the almost disjointness number $\mathfrak{a}$ with respect to a given ideal $\mathcal{I}$ were introduced and a new cardinal characteristic was defined — $\mathfrak{a}(\mathcal{I})$, the minimal cardinality of a maximal $\mathcal{I}$-almost disjoint family.

We investigate $\mathfrak{a}(\mathcal{I})$ in the case $\mathcal{I} = \mathcal{W}$, where $\mathcal{W}$ is the van der Waerden ideal which consists of the sets which do not contain arbitrarily long arithmetic progressions. We prove, in particular, that $\mathfrak{a}(\mathcal{W})$ is less than or equal to $\mathfrak{a}$ and greater than or equal to the pseudointersection number $\mathfrak{p}$.

CARDINAL CHARACTERISTICS

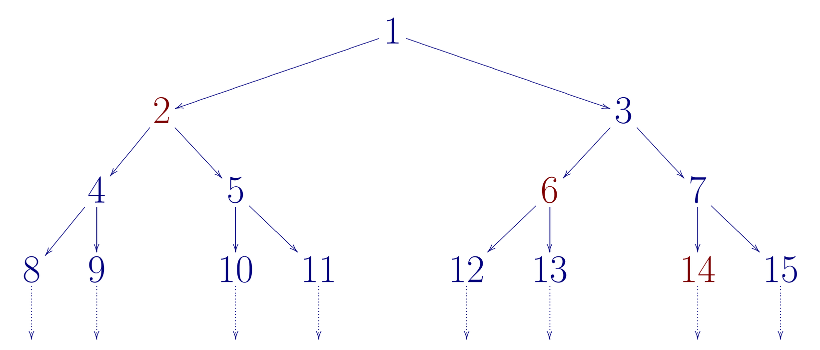
There are several cardinal characteristics which reflect in different ways the combinatorics of the infinite subsets of the natural numbers $\mathbb{N}$. An excellent survey on cardinal characteristics was written by Blass [1].

Almost disjointness number $\mathfrak{a}$

Two subsets of $\mathbb{N}$ are almost disjoint if their intersection is a finite set. A family $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$ is called an **almost disjoint family (= AD family)** if any two sets in $\mathcal{A}$ are almost disjoint.

Whereas any family consisting of disjoint subsets of $\mathbb{N}$ must be countable because every set has a unique minimal element and $\mathbb{N}$ has only countably many elements, the cardinality of AD families can be much greater.

Example 1. Enumerate the nodes of a complete binary tree as the picture below suggests. The numbers along each branch of the tree represent an infinite subset of $\mathbb{N}$ and these sets are almost disjoint for any two distinct branches. This implies that the cardinality of an almost disjoint family can be $\mathfrak{c}$, the cardinality of all branches in the tree and also the cardinality of $\mathcal{P}(\mathbb{N})$.



Although the resulting AD family has the maximal possible cardinality, there are many infinite sets which are almost disjoint with all its members. The red numbers in the pictures indicate the first three elements of one of those sets.

An AD family $\mathcal{A}$ is **maximal (= MAD family)** if for every infinite set $F \subseteq \mathbb{N}$ there exists a set $A \in \mathcal{A}$ such that the intersection $A \cap F$ is infinite.

$\mathfrak{a}$ = minimal cardinality of infinite MAD family

The almost disjointness number $\mathfrak{a}$ cannot be greater than $\mathfrak{c}$ and it has to be uncountable because for any countable collection $\{A_k : k \in \mathbb{N}\}$ of almost disjoint sets one can easily construct an infinite set which is almost disjoint with all its members by induction.

Pseudointersection number $\mathfrak{p}$

We say that $A \subseteq \mathbb{N}$ is almost contained in $B \subseteq \mathbb{N}$ if all but finitely many elements of A belong to B. A set $A \subseteq \mathbb{N}$ is a **pseudointersection** of a family $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$ if A is almost contained in every $F \in \mathcal{F}$.

A finite family $\mathcal{F}$ has an infinite pseudointersection if and only if it has an infinite intersection. However, the situation is different for infinite families $\mathcal{F}$.

Example 2. For $k \in \mathbb{N}$ put

$F_k = \{kn : n \in \mathbb{N}\}$ ... the set of all multiples of k.

It is obvious that any finite subfamily of $\mathcal{F}$ has an infinite intersection because there are infinitely many numbers divisible by finitely many given numbers, but the intersection of the family $\mathcal{F} = \{F_k : k \in \mathbb{N}\}$ is an empty set because no number is divisible by all natural numbers.

However, $\mathcal{F}$ has an infinite pseudointersection:

$F = \{k! : k \in \mathbb{N}\}$ is almost contained in every $F_k \in \mathcal{F}$

A family $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$ is called **centered** if every finite subfamily of $\mathcal{F}$ has an infinite intersection.

$\mathfrak{p}$ = minimal cardinality of centered family with no infinite pseudointersection

Complements of sets of a MAD family form a centered family which does not have an infinite pseudointersection. Therefore the pseudointersection number $\mathfrak{p}$ cannot be greater than $\mathfrak{a}$. It must be also uncountable since for any countable centered family $\{F_k : k \in \mathbb{N}\}$ one can easily construct an infinite pseudointersection by induction.

IDEALIZED CHARACTERISTICS

AD families and pseudointersections share the idea that the finite sets are in certain sense "negligible". This idea generalizes in the concept of an ideal which represent a larger collection of "negligible" sets.

A family $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ is an **ideal** on $\mathbb{N}$ if

- $\emptyset \in \mathcal{I}$ and $\mathcal{I} \neq \mathcal{P}(\mathbb{N})$
- if $A_1, A_2 \dots A_k \in \mathcal{I}$ then $A_1 \cup A_2 \cup \dots \cup A_k \in \mathcal{I}$
- if $A \in \mathcal{I}$ and $B \subseteq A$ then $B \in \mathcal{I}$

Example 3. For every function $g : \mathbb{N} \to (0, +\infty)$ such that $\sum_{n \in \mathbb{N}} g(n) = +\infty$ the family

$$\mathcal{I}_g = \{A \subseteq \mathbb{N} : \sum_{a \in A} g(a) < +\infty\}$$

is an ideal on $\mathbb{N}$, which is called the summable ideal determined by function g.

For any ideal $\mathcal{I}$ on $\mathbb{N}$ the collection of $\mathcal{I}$-positive sets $\mathcal{P}(\mathbb{N}) \setminus \mathcal{I}$ (=sets which are not "negligible" with respect to $\mathcal{I}$) is denoted by $\mathcal{I}^+$.

A family $\mathcal{A} \subseteq \mathcal{I}^+$ is **$\mathcal{I}$-AD family** if $A \cap B \in \mathcal{I}$ for every $A \neq B$, A, $B \in \mathcal{A}$. The family $\mathcal{A} \subseteq \mathcal{I}^+$ is an **$\mathcal{I}$-MAD family** if for every $F \in \mathcal{I}^+$ there exists $A \in \mathcal{A}$ such that $F \cap A \in \mathcal{I}^+$.

$\mathfrak{a}(\mathcal{I})$ = minimal cardinality of infinite $\mathcal{I}$-MAD family

A family $\mathcal{F} \subseteq \mathcal{I}^+$ is **$\mathcal{I}^+$-centered** if the intersection of any finite subfamily of $\mathcal{F}$ is an $\mathcal{I}$-positive set. A set $A \in \mathcal{I}^+$ is a **$\mathcal{I}^+$-pseudointersection** of a family $\mathcal{F} \subseteq \mathcal{I}^+$ if $A \setminus F \in \mathcal{I}$ for every $F \in \mathcal{F}$.

$\mathfrak{p}(\mathcal{I})$ = minimal cardinality of $\mathcal{I}^+$-centered family with no $\mathcal{I}^+$-pseudointersection

By a similar argument as for the standard characteristics, it is possible to show to that the modified cardinal characteristics satisfy the same inequality:

$$\mathfrak{p}(\mathcal{I}) \leq \mathfrak{a}(\mathcal{I}) \leq \mathfrak{c}$$

In general, $\mathfrak{p}(\mathcal{I})$ and $\mathfrak{a}(\mathcal{I})$ need not be uncountable.

Farkas and Soukup studied in [2] the cardinal characteristics $\mathfrak{a}(\mathcal{I})$ for certain classes of ideals on $\mathbb{N}$ and they showed among others that $\mathfrak{a}(\mathcal{I}_g)$ is uncountable for each tall summable ideal $\mathcal{I}_g$.

CHARACTERISTICS OF $\mathcal{P}(\mathbb{N})/\mathcal{W}$

If a set $A \subseteq \mathbb{N}$ contains arbitrarily long arithmetic progressions, then A is called an **AP-set**. The famous van der Waerden theorem states that sets which are not AP-sets form an ideal on $\mathbb{N}$. We call this ideal **van der Waerden ideal** and denote it as $\mathcal{W}$.

What can be proved about $\mathfrak{a}(\mathcal{W})$ and $\mathfrak{p}(\mathcal{W})$ apart from the obviously true inequality $\mathfrak{p}(\mathcal{W}) \leq \mathfrak{a}(\mathcal{W})$?

Proposition 1. $\mathfrak{p}(\mathcal{W}) \geq \mathfrak{p}$

The idea of the proof. The proposition follows from the fact that van der Waerden ideal is a tall F_σ ideal and $\mathfrak{p}(\mathcal{I}) \geq \mathfrak{p}$ for any tall F_σ ideal.

Proposition 2. $\mathfrak{p}(\mathcal{W}) \leq \mathfrak{p}$

Proposition 3. $\mathfrak{a}(\mathcal{W}) \leq \mathfrak{a}$

The proofs of both propositions 2 and 3 are based on the fact that a centered family with no infinite pseudointersection and a MAD-family can be transformed into a $\mathcal{W}^+$-centered family with no $\mathcal{W}^+$-pseudointersection and a $\mathcal{W}$-MAD family respectively.

Lemma 4. *For every AP-set A there is an increasing sequence $\langle n_k \rangle_{k=1}^{\infty}$ such that $A \cap [2^{n_k}, 2^{n_k+1})$ contains an arithmetic progression of length at least k.*

So one can assign to every AP-set A an infinite set $W(A) = \{n_k : k \in \mathbb{N}\}$ which "witnesses" the fact that $A \in \mathcal{W}^+$ in a strong way: Whenever $M \subseteq W(A)$ is infinite then $\bigcup_{m \in M} A \cap [2^m, 2^{m+1})$ is an AP-set.

Proof of Proposition 2. Assume $\mathcal{F}$ is a centered system of cardinality $\mathfrak{p}$ with no infinite pseudointersection. For every $F \in \mathcal{F}$ put

$$A_F = \bigcup_{n \in F} [2^n, 2^{n+1})$$

It is easy to see that the family $\mathcal{A} = \{A_F : F \in \mathcal{F}\}$ is $\mathcal{W}^+$-centered.

If $M \subseteq \mathbb{N}$ is an AP-set then $W(M) = \{n_k : k \in \mathbb{N}\}$ is infinite. Since $\mathcal{F}$ has no infinite pseudointersection, there exists $F \in \mathcal{F}$ such that $W(M) \setminus F$ is infinite. However, this means that $M \setminus A_F$ is an AP-set, i.e. M is not a $\mathcal{W}^+$-pseudointersection of $\mathcal{A}$. □

Proof of Proposition 3. Assume $\mathcal{A}$ is a MAD family of cardinality $\mathfrak{a}$ and put

$$B_A = \bigcup_{n \in A} [2^n, 2^{n+1}).$$

Then $\mathcal{B} = \{B_A : A \in \mathcal{A}\}$ is an AD family consisting of AP-sets.

For every AP-set M there exists $A \in \mathcal{A}$ such that $W(M) \cap A$ is infinite because $\mathcal{A}$ was a MAD family. Therefore $M \cap B_A$ is an AP-set which proves that $\mathcal{B}$ is $\mathcal{W}$-MAD family. □

References.

[1] A. Blass: Combinatorial cardinal characteristics of the continuum, in: *Handbook of set theory*, Vols. 1, 2, 3, 395—489, Springer, Dordrecht, 2010.

[2] B. Farkas, L. Soukup: More on cardinal invariants of analytic P-ideals, *Comment. Math. Univ. Carolin.*, **50**, no. 2, 281—295, 2009.