$\mathcal{I}$-ultrafilters and summable ideals

Jana Flašková

flaskova@kma.zcu.cz

University of West Bohemia, Pilsen
Czech Republic

$\mathcal{I}$-ultrafilters

Definition A. (Baumgartner)

Let $\mathcal{I}$ be a family of subsets of a set X such that $\mathcal{I}$ contains all singletons and is closed under subsets. An ultrafilter $\mathcal{U}$ on ω is called an $\mathcal{I}$-ultrafilter if for every $F : \omega \to X$ there exists $A \in \mathcal{U}$ such that $F[A] \in \mathcal{I}$.

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- if $\mathcal{I} \subseteq \mathcal{J}$ then every $\mathcal{I}$-ultrafilter is a $\mathcal{J}$-ultrafilter

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- if $\mathcal{I} \subseteq \mathcal{J}$ then every $\mathcal{I}$-ultrafilter is a $\mathcal{J}$-ultrafilter
- if $\mathcal{U} \leq_{RK} \mathcal{V}$ and $\mathcal{V}$ is an $\mathcal{I}$-ultrafilter then $\mathcal{U}$ is also an $\mathcal{I}$-ultrafilter

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- if $\mathcal{I} \subseteq \mathcal{J}$ then every $\mathcal{I}$-ultrafilter is a $\mathcal{J}$-ultrafilter
- if $\mathcal{U} \leq_{RK} \mathcal{V}$ and $\mathcal{V}$ is an $\mathcal{I}$-ultrafilter then $\mathcal{U}$ is also an $\mathcal{I}$-ultrafilter
- $\mathcal{I}$-ultrafilters and $\langle\mathcal{I}\rangle$-ultrafilters coincide

where $\langle\mathcal{I}\rangle$ is the ideal generated by $\mathcal{I}$

Weak $\mathcal{I}$-ultrafilters

Definition B.

Let $\mathcal{I}$ be a family of subsets of a set X such that $\mathcal{I}$ contains all singletons and is closed under subsets. An ultrafilter $\mathcal{U}$ on ω is called

weak $\mathcal{I}$-ultrafilter if for every *finite-to-one* function $F : \omega \longrightarrow X$ there exists $A \in \mathcal{U}$ such that $F[A] \in \mathcal{I}$.

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Definition B.

Let $\mathcal{I}$ be a family of subsets of a set X such that $\mathcal{I}$ contains all singletons and is closed under subsets. An ultrafilter $\mathcal{U}$ on ω is called

- weak $\mathcal{I}$-ultrafilter if for every *finite-to-one* function $F : \omega \longrightarrow X$ there exists $A \in \mathcal{U}$ such that $F[A] \in \mathcal{I}$.
- $\mathcal{I}$-friendly ultrafilter if for every *one-to-one* function $F : \omega \longrightarrow X$ there exists $A \in \mathcal{U}$ such that $F[A] \in \mathcal{I}$.

Ideals on ω

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Lemma 1.

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Proposition 2.

($\mathfrak{p} = \mathfrak{c}$) If $\mathcal{I}$ is a tall ideal then $\mathcal{I}$-ultrafilters exist.

Summable ideals

Definition.

Given a function $g : \omega \to [0, \infty)$ such that $\sum_{n \in \omega} g(n) = \infty$ then the family

$$\mathcal{I}_g = \{A \subseteq \omega : \sum_{a \in A} g(a) < +\infty\}$$

is a proper ideal which we call summable ideal determined by function g.

Summable ideals

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Observation.

A summable ideal is tall if and only if $\lim_{n\to\infty} g(n) = 0$.

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We consider only tall summable ideals.

Rapid ultrafilters

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An ultrafilter $\mathcal{U} \in \omega^*$ is rapid if and only if $\mathcal{I}_g^* \not\leq_{KB} \mathcal{U}$ for every summable ideal $\mathcal{I}_g$.

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Theorem (Vojtáš)

An ultrafilter $\mathcal{U} \in \omega^*$ is rapid if and only if $\mathcal{U} \cap \mathcal{I}_g \neq \emptyset$ for every summable ideal $\mathcal{I}_g$.

Rapid ultrafilters

Theorem 3.

For an ultrafilter $\mathcal{U} \in \omega^*$ the following are equivalent:

- $\mathcal{U}$ is rapid
- $\mathcal{U}$ is a weak $\mathcal{I}_g$-ultrafilter for every summable ideal $\mathcal{I}_g$
- $\mathcal{U}$ is an $\mathcal{I}_g$-friendly ultrafilter for every summable ideal $\mathcal{I}_g$
- $\mathcal{U} \cap \mathcal{I}_g \neq \emptyset$ for every summable ideal $\mathcal{I}_g$

Rapid ultrafilters

Theorem 4.

($\mathrm{MA}_{\mathbf{ctble}}$) There is an $\mathcal{I}_g$-ultrafilter which is not a rapid ultrafilter.

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Theorem 5.

($MA_{\mathbf{ctble}}$) There is a rapid ultrafilter which is not an $\mathcal{I}_g$-ultrafilter.

Theorem 5*.

($MA_{\mathbf{ctble}}$) For every tall ideal $\mathcal{I}$ there is a Q-point which is not an $\mathcal{I}$-ultrafilter.

$\mathcal{I}_g$-ultrafilters

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ZFC $\Downarrow \not\Uparrow$???

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Theorem 6.

($\mathrm{MA}_{\mathbf{ctble}}$) For every summable ideal $\mathcal{I}_g$ there is an $\mathcal{I}_g$-ultrafilter.

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Observation

For every summable ideal $\mathcal{I}_g$ there is an ultrafilter $\mathcal{U} \in \omega^*$ such that $\mathcal{U} \cap \mathcal{I}_g \neq \emptyset$.

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At least for some summable ideals $\mathcal{I}_g$-friendly ultrafilters exist in ZFC.

Some results

Theorem 7.

$\mathcal{I}_{1/n}$-friendly ultrafilters exist in ZFC.

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Corollary 8.

If $\mathcal{I}_{1/n} \subseteq \mathcal{I}_g$ then $\mathcal{I}_g$-friendly ultrafilters exist in ZFC.

Examples: $g(n) = \frac{1}{n \ln n}$

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If $\mathcal{I}_{1/n} \subseteq \mathcal{I}_g$ then $\mathcal{I}_g$-friendly ultrafilters exist in ZFC.

Examples: $g(n) = \frac{1}{n \ln n}$

Theorem 9.

If $g(n) = \frac{\ln^p n}{n}$, $p \in \omega$, then $\mathcal{I}_g$-friendly ultrafilters exist in ZFC.

Construction

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Definition.

A family $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is called

- a k-linked family if $F_1 \cap \ldots \cap F_k$ is infinite whenever $F_i \in \mathcal{F}$, $i \leq k$.
- a centered system if $\mathcal{F}$ is k-linked for every k i.e., if any finite subfamily of $\mathcal{F}$ has an infinite intersection.

Construction

We say that $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is a summable family if for every one-to-one function $f : \omega \longrightarrow \mathbb{N}$ there is $A \in \mathcal{F}$ such that $f[A] \in \mathcal{I}_{1/n}$.

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Proposition 10.

For every $k \in \mathbb{N}$ there exists a summable k-linked family $\mathcal{F}_k \subseteq \mathcal{P}(\omega)$.

Construction

Lemma 11.

If $\mathcal{F}_k \subseteq \mathcal{P}(\omega)$ is a k-linked family then

$\mathcal{F} = \{F \subseteq \omega : (\forall k)(\exists U^k \in \mathcal{F}_k)\, U^k \subseteq^* F\}$

is a centered system.

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If every $\mathcal{F}_k$ is summable then $\mathcal{F}$ is summable.

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is a centered system.

If every $\mathcal{F}_k$ is summable then $\mathcal{F}$ is summable.

More generally, if $\mathcal{I}$ is a P-ideal and for every one-to-one function $f \in {}^\omega\mathbb{N}$ and for every $k \in \mathbb{N}$ there exists $U^k \in \mathcal{F}_k$ such that $f[U^k] \in \mathcal{I}$ then there exists $U \in \mathcal{F}$ such that $f[U] \in \mathcal{I}$.

Some questions

Question.

Do $\mathcal{I}_g$-friendly ultrafilters exist in ZFC for every summable ideal $\mathcal{I}_g$? What about $g(n) = \frac{1}{\sqrt{n}}$ or $\frac{1}{\ln n}$?

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Question.
Do $\mathcal{I}_g$-friendly ultrafilters exist in ZFC for every summable ideal $\mathcal{I}_g$? What about $g(n) = \frac{1}{\sqrt{n}}$ or $\frac{1}{\ln n}$?

Question.
Is there an $\mathcal{I}_g$-friendly ultrafilter which is not an $\mathcal{I}_h$-friendly ultrafilter whenever $\mathcal{I}_g \not\subseteq \mathcal{I}_h$?

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