$\mathcal{I}$-ULTRAFILTERS AND SUMMABLE IDEALS

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ABSTRACT. We study $\mathcal{I}$-ultrafilters and two weaker notions for the setting where $\mathcal{I}$ is a tall summable ideal $\mathcal{I}_g$ on ω. Such ultrafilters are closely related to rapid ultrafilters. We consider this relation from various aspects and show some similarities and differences between the two classes of ultrafilters. Under Martin's axiom for countable posets we construct a rapid ultrafilter which is not an $\mathcal{I}_g$-ultrafilter for arbitrary summable ideal $\mathcal{I}_g$ or an ultrafilter which is $\mathcal{I}_g$-ultrafilter for every tall summable ideal $\mathcal{I}_g$, but is not a Q-point.

1. INTRODUCTION

In this paper we study $\mathcal{I}$-ultrafilters and their two modifications – weak $\mathcal{I}$-ultrafilters and $\mathcal{I}$-friendly ultrafilters – for the case where $\mathcal{I}$ is a tall summable ideal on ω. In the first section we provide necessary definitions and present some basic facts about $\mathcal{I}$-ultrafilters and the other notions. We recall in the first section also the definition of summable ideals and mention some known results connecting summable ideals and rapid ultrafilters.

The definition of $\mathcal{I}$-ultrafilter was given by Baumgartner in [1]: Let $\mathcal{I}$ be a family of subsets of a set X such that $\mathcal{I}$ contains all singletons and is closed under subsets. Given an ultrafilter $\mathcal{U}$ on ω, we say that $\mathcal{U}$ is an *$\mathcal{I}$-ultrafilter* if for every function $F : \omega \to X$ there exists $A \in \mathcal{U}$ such that $F[A] \in \mathcal{I}$.

We modify Baumgartner's definition so that we restrict the class of considered functions to finite-to-one or one-to-one functions: Let $\mathcal{I}$ be a family of subsets of a set X such that $\mathcal{I}$ contains all singletons and is closed under subsets. We say that an ultrafilter $\mathcal{U}$ on ω is

– *weak $\mathcal{I}$-ultrafilter* if for every finite-to-one function $F : \omega \to X$ there exists $A \in \mathcal{U}$ such that $F[A] \in \mathcal{I}$.

– *$\mathcal{I}$-friendly ultrafilter* if for every one-to-one function $F : \omega \to X$ there exists $A \in \mathcal{U}$ such that $F[A] \in \mathcal{I}$.

It follows immediately from the definition that whenever $\mathcal{I} \subseteq \mathcal{J}$ then every $\mathcal{I}$-ultrafilter is a $\mathcal{J}$-ultrafilter and the same holds for weak $\mathcal{I}$-ultrafilters and $\mathcal{I}$-friendly ultrafilters.

From now on we only consider $X = \omega$ and $\mathcal{I}$ is an ideal on $\mathbb{N}$ although some of the propositions in the first section are true in other settings as well.

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Recall the definition of Rudin-Keisler order $\leq_{RK}$ on ω^*: For $\mathcal{U}, \mathcal{V} \in \beta\omega$ we write $\mathcal{U} \leq_{RK} \mathcal{V}$ iff there is $f \in {}^\omega\omega$ such that $\beta f(\mathcal{V}) = \mathcal{U}$, which is equivalent to $(\forall U \in \mathcal{U})\ f^{-1}[U] \in \mathcal{V}$ or $(\forall V \in \mathcal{V})\ f[V] \in \mathcal{U}$.

The family of all $\mathcal{I}$-ultrafilters is downwards closed in Rudin-Keisler order, i.e. whenever $\mathcal{U}$ is an $\mathcal{I}$-ultrafilter and $\mathcal{V} \leq_{RK} \mathcal{U}$ then $\mathcal{V}$ is also an $\mathcal{I}$-ultrafilter. In fact, one can state this observation as equivalence:

Lemma 1.1. *$\mathcal{U} \in \omega^*$ is an $\mathcal{I}$-ultrafilter if and only if every $\mathcal{V} \in \omega^*$ such that $\mathcal{V} \leq_{RK} \mathcal{U}$ is an $\mathcal{I}$-ultrafilter too.*

It is important to mention that the existence of $\mathcal{I}$-ultrafilters has not been established in ZFC, but it is consistent with ZFC for a large class of ideals on ω. Assuming Martin's axiom for σ-centered posets we proved in [4] that $\mathcal{I}$-ultrafilters exist for every tall ideal $\mathcal{I}$ on ω. Remember that an ideal on ω is called *tall* (or *dense*) if for every infinite set $A \subseteq \omega$ there exists infinite $B \subseteq A$ such that $B \in \mathcal{I}$. It is not difficult to check that for ideals which are not tall the corresponding $\mathcal{I}$-ultrafilters cannot exist.

For a function $g : \omega \to (0, \infty)$ such that $\sum_{n\in\omega} g(n) = +\infty$ the family

$$\mathcal{I}_g = \{A \subseteq \omega : \sum_{a\in A} g(a) < +\infty\}$$

is an ideal on ω which we call summable ideal determined by function g. A summable ideal is tall if and only if $\lim_{n\to\infty} g(n) = 0$.

It is important for our purposes that the summable ideals are P-ideals, i.e. if $A_n, n \in \omega$, belong to $\mathcal{I}_g$ then there exists $A \in \mathcal{I}_g$ such that $A_n \subseteq^* A$ for every $n \in \omega$ where $A_n \subseteq^* A$ means A contains all but finitely many elements of A_n.

From now on we will consider only (weak) $\mathcal{I}_g$-ultrafilters and $\mathcal{I}_g$-friendly ultrafilters for a tall summable ideal $\mathcal{I}_g$ if not explicitly stated otherwise. Our aim is to compare such ultrafilters to rapid ultrafilters and Q-points, so let us recall the definitions:

A free ultrafilter $\mathcal{U}$ is called a *rapid ultrafilter* if the enumeration functions of its sets form a dominating family in $({}^\omega\omega, \leq^*)$ where enumeration function of a set A is the unique strictly increasing function e_A from ω onto A.

A free ultrafilter $\mathcal{U}$ is called a *Q-point* if for every partition $\{Q_n : n \in \omega\}$ of ω into finite sets there is $U \in \mathcal{U}$ such that $|U \cap Q_n| \leq 1$ for every $n \in \omega$.

It is known that every Q-point is a rapid ultrafilter, but not vice versa.

There exists a certain connection of weak $\mathcal{I}_g$-ultrafilters to rapid ultrafilters. Some results published in [7] and other presented in [6] concerning rapid ultrafilters may be translated in terms of $\mathcal{I}$-ultrafilters as follows

Theorem 1.2. *For an ultrafilter $\mathcal{U} \in \omega^*$ the following are equivalent:*

1. *$\mathcal{U}$ is rapid*
2. *$\mathcal{U}$ is a weak $\mathcal{I}$-ultrafilter for every tall summable ideal $\mathcal{I}$*
3. *$\mathcal{U}$ is an $\mathcal{I}$-friendly ultrafilter for every tall summable ideal $\mathcal{I}$*

(4) *$\mathcal{U} \cap \mathcal{I} \neq \emptyset$ for every tall summable ideal $\mathcal{I}$*

In section 3. we will show that rapid ultrafilters need not be $\mathcal{I}_g$-ultrafilters. So it is consistent that for every summable ideal $\mathcal{I}_g$ one cannot reverse the obvious implication that every $\mathcal{I}$-ultrafilter is a weak $\mathcal{I}$-ultrafilter.

On the other hand, one would like to construct an $\mathcal{I}_g$-ultrafilter which is not rapid. This has been proved for $\mathcal{I}_{\frac{1}{n}}$-ultrafilters in [4], but not yet for an arbitrary summable ideal $\mathcal{I}_g$. Leaving this problem open we present in section 4. a weaker result that it is consistent that there exists an ultrafilter which is an $\mathcal{I}_g$-ultrafilter for all summable ideals and not a Q-point.

All the results in this paper are consistency results. It is not known whether (weak) $\mathcal{I}_g$-ultrafilters exist in ZFC. For $\mathcal{I}_g$-friendly ultrafilters the situation is a bit different. It has been proved in [4] that $\mathcal{I}_g$-friendly ultrafilters exist in ZFC for $g = \frac{1}{n}$ and the construction may be easily modified for $g = \frac{\ln^p n}{n}$, where $p \geq 0$.

2. The existence of $\mathcal{I}_g$-ultrafilters

Assuming Martin's axiom for σ-centered posets one can construct $\mathcal{I}$-ultrafilters for arbitrary tall ideal $\mathcal{I}$. Thus for every tall summable ideal $\mathcal{I}_g$ it is consistent that $\mathcal{I}_g$-ultrafilters (and weak $\mathcal{I}_g$-ultrafilters and $\mathcal{I}_g$-friendly ultrafilters) exist. However, it is sufficient to assume Martin's axiom for countable posets when we consider summable ideals and one can even require more — we will construct an ultrafilter $\mathcal{U} \in \omega^*$ which is an $\mathcal{I}_g$-ultrafilter for all summable ideals $\mathcal{I}_g$.

Theorem 2.1. (MA_{ctble}) *There exists $\mathcal{U} \in \omega^*$ such that $\mathcal{U}$ is an $\mathcal{I}_g$-ultrafilter for every tall summable ideal $\mathcal{I}_g$.*

Proof. Enumerate all pairs $\{\langle f_\alpha, \mathcal{I}_{g_\alpha}\rangle : \alpha < \mathfrak{c}\}$ where $f_\alpha \in {}^\omega\omega$ and $\mathcal{I}_{g_\alpha}$ is a tall summable ideal. By transfinite induction on $\alpha < \mathfrak{c}$ we will construct filter bases $\mathcal{F}_\alpha$ so that the following conditions are satisfied:

(i) $\mathcal{F}_0$ is the Fréchet filter
(ii) $\mathcal{F}_\alpha \subseteq \mathcal{F}_\beta$ whenever $\alpha \leq \beta$
(iii) $\mathcal{F}_\gamma = \bigcup_{\alpha<\gamma} \mathcal{F}_\alpha$ for γ limit
(iv) $(\forall\alpha)$ $|\mathcal{F}_\alpha| \leq |\alpha| \cdot \omega$
(v) $(\forall\alpha)$ $(\exists U \in \mathcal{F}_{\alpha+1})$ $f_\alpha[U] \in \mathcal{I}_{g_\alpha}$

Conditions (i)–(iii) take care of the beginning and continuation of the induction. Because of (iii) condition (iv) is satisfied at limit stages of the construction and one only has to check that conditions (iv) and (v) hold at non-limit induction steps.

Induction step: Suppose we know already $\mathcal{F}_\alpha$. If there is $F \in \mathcal{F}_\alpha$ such that $f_\alpha[F] \in \mathcal{I}_{g_\alpha}$ then simply put $\mathcal{F}_{\alpha+1} = \mathcal{F}_\alpha$. If $f_\alpha[F] \notin \mathcal{I}_{g_\alpha}$ (in particular, $f_\alpha[F]$ is infinite) for every $F \in \mathcal{F}_\alpha$ we will construct a suitable set eventually making use of Martin's Axiom.

If there exists $K \in [\omega]^{<\omega}$ such that $F \cap f_\alpha^{-1}[K]$ is infinite for every $F \in \mathcal{F}_\alpha$ then we let $\mathcal{F}_{\alpha+1}$ be the filter base generated by $\mathcal{F}_\alpha$ and $U = f_\alpha^{-1}[K]$. In the following we will assume that no such set exists, i.e. (♣) for every $K \in [\omega]^{<\omega}$ there is $F_K \in \mathcal{F}_\alpha$ such that $F_K \cap f_\alpha^{-1}[K]$ is finite. This means also that $F \setminus f_\alpha^{-1}[K]$ is infinite for every $F \in \mathcal{F}_\alpha$.

Consider $P = \{K \in [\omega]^{<\omega} : (\forall u, v \in f_\alpha[K]) \text{ if } u < v \text{ then } g_\alpha(v) < \frac{1}{2} g_\alpha(u)\}$ and define a partial order $\leq_P$ on P in the following way: $K \leq_P L$ if and only if $K = L$ or $K \supset L$ and $\min K \setminus L > \max L$. For every $F \in \mathcal{F}_\alpha$ and $k \in \omega$ define $D_{F,k} = \{K \in P : |F \cap K| \geq k\}$.

Claim. $D_{F,k} = \{K \in P : |F \cap K| \geq k\}$ *is dense in* P *for every* $F \in \mathcal{F}_\alpha$ *and* $k \in \omega$.
Take arbitrary $L \in P$. Since $\lim_{n\to\infty} g_\alpha(n) = 0$ there exists $n_L > \max f_\alpha[L]$ such that for every $n \geq n_L$ we have $g_\alpha(n) \leq \frac{1}{2}\, g_\alpha(\max f_\alpha[L])$. According to the assumption (♣) the set $F \setminus f_\alpha^{-1}[0, n_L)$ is infinite and we can inductively pick $m_i \in F \setminus f_\alpha^{-1}[0, n_L)$ for $i = 1, \ldots, k$ so that $m_1 > \max L$, $m_{i+1} > m_i$, $f_\alpha(m_{i+1}) > f_\alpha(m_i)$ and $g_\alpha(f_\alpha(m_{i+1})) < \frac{1}{2}\, g_\alpha(f_\alpha(m_i))$. Finally, let $K = L \cup \{m_i : i = 1, \ldots, k\}$. It is obvious that $K \leq_P L$ and $K \in D_{F,k}$.

The family $\mathscr{D} = \{D_{F,k} : F \in \mathcal{F}_\alpha, k \in \omega\}$ consists of dense subsets of P and has cardinality less than $\mathfrak{c}$. So there exists a $\mathscr{D}$-generic filter $\mathcal{G}$ on P. Let $U = \bigcup\{K : K \in \mathcal{G}\}$. It remains to check that U is the set to add to $\mathcal{F}_\alpha$.

(1) $(\forall F \in \mathcal{F}_\alpha)\ U \cap F$ is infinite
For every $k \in \omega$ and every $K \in \mathcal{G} \cap D_{F,k}$ we have $U \supset K$ and $|K \cap F| \geq k$. Thus $U \cap F$ is infinite.

(2) $f_\alpha[U] \in \mathcal{I}_{g_\alpha}$
Enumerate $f_\alpha[U] = \{u_n : n \in \omega\}$. For every $n \in \omega$ there exists $K_n \in \mathcal{G}$ such that $u_n, u_{n+1} \in f_\alpha[K_n]$. Definition of P implies $g_\alpha(u_{n+1}) < \frac{1}{2}\, g_\alpha(u_n)$. Therefore

$$\sum_{n\in\omega} g_\alpha(u_n) \leq \sum_{n\in\omega} \frac{1}{2^n}\, g_\alpha(u_0) = 2\, g_\alpha(u_0)$$

and $f_\alpha[U]$ belongs to the ideal $\mathcal{I}_{g_\alpha}$.

Due to (1) filter base generated by $\mathcal{F}_\alpha$ and U is a uniform filter base, which will be taken as $\mathcal{F}_{\alpha+1}$ to complete the induction step.

Finally, let $\mathcal{F} = \bigcup_{\alpha<\mathfrak{c}} \mathcal{F}_\alpha$ and consider an ultrafilter $\mathcal{U} \supseteq \mathcal{F}$. Because of condition (v) in the construction, $\mathcal{U}$ is an $\mathcal{I}_g$-ultrafilter for every tall summable ideal $\mathcal{I}_g$. □

We say that a free ultrafilter $\mathcal{U}$ is a *hereditarily rapid ultrafilter* if it is a rapid ultrafilter such that for every $\mathcal{V} \leq_{RK} \mathcal{U}$ the ultrafilter $\mathcal{V}$ is again a rapid ultrafilter.

It follows from Theorem 1.2 and Lemma 1.1 that the ultrafilter constructed in Theorem 2.1 is a hereditarily rapid ultrafilter. Although hereditarily rapid ultrafilters have been already constructed in [4], we present here a different construction because the existence of such ultrafilters is an important prerequisite for the Theorem 4.2 in this paper.

3. Rapid ultrafilters need not be $\mathcal{I}_g$-ultrafilters

It has been proved in [4] that assuming Martin's axiom for countable posets for every tall ideal $\mathcal{I}$ there exists a Q-point (and hence of course rapid ultrafilter) which is not an $\mathcal{I}$-ultrafilter. It follows that for every summable ultrafilter $\mathcal{I}_g$ there exists under $\mathrm{MA}_{\mathrm{ctble}}$ a Q-point, which is not an $\mathcal{I}_g$-ultrafilter. In this chapter we strengthen the result and show that assuming Martin's Axiom for countable posets there is a Q-point which is not an $\mathcal{I}_g$-ultrafilter for all summable ideals at once. It is a consequence of the more general Theorem 3.3. Several lemmas precede the main theorem in this section to make its proof more transparent.

Let us recall that a family $\mathcal{A} = \{A_{\alpha,n} : \alpha \in I, n \in \omega\} \subseteq \mathscr{P}(\omega)$ is called *independent with respect to a filter* (or filter base) $\mathcal{F}$ if $\{A_{\alpha,n} : n \in \omega\}$ is a partition of ω into infinite sets for every $\alpha \in I$ and $(\forall F \in \mathcal{F})$ $(\forall M \in [I]^{<\omega})$ $(\forall f : M \to \omega)$ $|F \cap \bigcap_{\beta \in M} A_{\beta,f(\beta)}| = \omega$.

Lemma 3.1. (MA_{ctble}) *Assume $\mathcal{F}$ is filter base on ω with $|\mathcal{F}| < \mathfrak{c}$ and family $\mathcal{A} = \{A_{\beta,n} : \beta \leq \alpha, n \in \omega\}$, $\alpha < \mathfrak{c}$, is independent with respect to $\mathcal{F}$. Then there exists a partition of ω into infinite sets $\{A_{\alpha+1,n} : n \in \omega\}$ such that $\mathcal{A}' = \mathcal{A} \cup \{A_{\alpha+1,n} : n \in \omega\}$ is independent with respect to $\mathcal{F}$.*

Proof. Consider the set of all finite functions $P = ({}^{<\omega}\omega, \leq_P)$ with partial order defined by $\sigma_2 \leq_P \sigma_1$ if σ_2 is an end extension of σ_1. For every $F \in \mathcal{F}$, $M \in [\alpha]^{<\omega}$, $f : M \to \omega$ and $m, n \in \omega$ let $D_{F,M,f,m,n} = \{\sigma \in P : (\exists k \geq m)$ $k \in F \cap \bigcap_{\beta \in M} A_{\beta,f(\beta)}$, $k \in \operatorname{dom} \sigma$ and $\sigma(k) = n\}$.

Claim. *$D_{F,M,f,m,n}$ is dense in $(P, \leq_P)$ for every $F \in \mathcal{F}$, $M \in [\alpha]^{<\omega}$, $f : M \to \omega$ and $m, n \in \omega$.*
Take arbitrary $\rho \in P$. Since $F \cap \bigcap_{\beta \in M} A_{\beta,f(\beta)}$ is infinite there exists $k \in F \cap \bigcap_{\beta \in M} A_{\beta,f(\beta)}$ such that $k > \max \operatorname{dom} \rho$ and $k \geq m$. Put $\sigma = \rho^\frown\langle k, n\rangle$. Obviously, $\sigma \leq_P \rho$ and $\sigma \in D_{F,M,f,m,n}$.

The family $\mathscr{D} = \{D_{F,M,f,m,n} : F \in \mathcal{F}, M \in [\alpha]^{<\omega}, f : M \to \omega, m, n \in \omega\}$ consists of dense subsets of P and has cardinality less than $\mathfrak{c}$. So there exists a $\mathscr{D}$-generic filter $\mathcal{G}$ on P.

Let $g = \bigcup\{\sigma : \sigma \in \mathcal{G}\}$. Observe that there are infinitely many $k \in F \cap \bigcap_{\beta \in M} A_{\beta,f(\beta)}$ such that $g(k) = n$. Thus $\{g^{-1}(n) : n \in \omega\}$ is a partition of ω into infinite sets. Let $A_{\alpha+1,n} = g^{-1}(n)$ for every $n \in \omega$. It follows from the definition of $\{A_{\alpha+1,n} : n \in \omega\}$ that $\mathcal{A}' = \mathcal{A} \cup \{A_{\alpha+1,n} : n \in \omega\}$ is independent with respect to $\mathcal{F}$. □

Lemma 3.2. (MA_{ctble}) *Assume $\mathcal{F}$ is filter base on ω with $|\mathcal{F}| < \mathfrak{c}$, family $\mathcal{A} = \{A_{\beta,n} : \beta \leq \alpha, n \in \omega\}$, $\alpha < \mathfrak{c}$, is independent with respect to $\mathcal{F}$ and $\mathcal{Q} = \{Q_i : i \in \omega\}$ is a partition of ω into finite sets.*

Then there exists $H \subseteq \omega$ such that $|H \cap Q| \leq 1$ for every $Q \in \mathcal{Q}$ and $\mathcal{A}$ is independent with respect to the filter base $\mathcal{F}'$ generated by $\mathcal{F}$ and H.

Proof. Consider the set $P = \{K \in [\omega]^{<\omega} : (\forall Q \in \mathcal{Q})\ |K \cap Q| \leq 1\}$ with partial order defined by $K \leq_P L$ if $K = L$ or $K \supset L$ and $\min(K \setminus L) > \max L$. For every $F \in \mathcal{F}$, $M \in [\alpha]^{<\omega}$, $f : M \to \omega$ and $k \in \omega$ let $D_{F,M,f,k} = \{K \in P : |K \cap F \cap \bigcap_{\beta \in M} A_{\beta,f(\beta)}| \geq k\}$.

Claim: $D_{F,M,f,k}$ is dense in $(P, \leq_P)$ for every $F \in \mathcal{F}$, $M \in [\alpha]^{<\omega}$, $f : M \to \omega$ and $k \in \omega$.
Whenever we take $L \in P$ there is a finite set $S \subseteq \omega$ such that $L \subseteq \bigcup_{i \in S} Q_i$. Since $F \cap \bigcap_{\beta \in M} A_{\beta,f(\beta)}$ is infinite there exists $L' \subseteq F \cap \bigcap_{\beta \in M} A_{\beta,f(\beta)}$ such that $\min L' > \max \bigcup_{i \in S} Q_i$, $|L'| \geq k$ and $|L' \cap Q| \leq 1$ for every $Q \in \mathcal{Q}$. Put $K = L \cup L'$. It is easy to see that $K \leq_P L$ and $K \in D_{F,M,f,k}$. □

The family $\mathscr{D} = \{D_{F,M,f,k} : F \in \mathcal{F}, M \in [\alpha]^{<\omega}, f : M \to \omega, k \in \omega\}$ consists of dense subsets of P and has cardinality less than $\mathfrak{c}$. So there exists a $\mathscr{D}$-generic filter $\mathcal{G}$ on P. Let $H = \bigcup\{K : K \in \mathcal{G}\}$. It remains to check that H has the required properties:

(1) $(\forall Q \in \mathcal{Q}_\alpha)\ |H \cap Q| \leq 1$
If $u, v \in H$ then there is $K \in \mathcal{G}$ such that $u, v \in K$ and according to the definiton of P elements u, v belong to distinct sets from partition $\mathcal{Q}$.

(2) $(\forall F \in \mathcal{F})\ (\forall M \in [\alpha]^{<\omega})\ (\forall f : M \to \omega)\ |H \cap F \cap \bigcap_{\beta \in M} A_{\beta,f(\beta)}| = \omega$
For every $k \in \omega$ and every $K \in \mathcal{G} \cap D_{F,M,f,k}$ we have $H \supset K$ and $|K \cap F \cap \bigcap_{\beta \in M} A_{\beta,f(\beta)}| \geq k$. Thus $H \cap F \cap \bigcap_{\beta \in M} A_{\beta,f(\beta)}$ is infinite and family $\mathcal{A}$ is independent with respect to the filter base $\mathcal{F}'$ generated by $\mathcal{F}$ and H. □

Theorem 3.3. (MA_{ctble}) *Assume $\{\mathcal{I}_\alpha : \alpha < \mathfrak{c}\}$ is a family of tall ideals. There is a Q-point which is not an $\mathcal{I}_\alpha$-ultrafilter for every $\alpha < \mathfrak{c}$.*

Proof. Enumerate all partitions of ω into finite sets as $\{\mathcal{Q}_\alpha : \alpha < \mathfrak{c}\}$. By transfinite induction on $\alpha < \mathfrak{c}$ we will construct filter bases $\mathcal{F}_\alpha$ and families $\mathcal{A}_\alpha = \{A_{\beta,n} : \beta \leq \alpha, n \in \omega\}$ such that $\{A_{\alpha,n} : n \in \omega\}$ is a partition of ω into infinite sets for every $\alpha < \mathfrak{c}$ and the following conditions are satisfied:

(i) $\mathcal{F}_0$ is the Fréchet filter, $\mathcal{A}_0$ is partition of ω into infinite sets
(ii) $\mathcal{F}_\alpha \subseteq \mathcal{F}_\beta$, $\mathcal{A}_\alpha \subseteq \mathcal{A}_\beta$ whenever $\alpha \leq \beta$
(iii) $\mathcal{F}_\gamma = \bigcup_{\alpha<\gamma} \mathcal{F}_\alpha$, $\mathcal{A}_\gamma = \bigcup_{\alpha<\gamma} \mathcal{A}_\alpha$ for γ limit
(iv) $(\forall \alpha)\ |\mathcal{F}_\alpha| \leq |\alpha| \cdot \omega$ and $|\mathcal{A}_\alpha| \leq |\alpha| \cdot \omega$
(v) $(\forall \alpha)\ \mathcal{A}_\alpha$ is independent with respect to $\mathcal{F}_\alpha$
(vi) $(\forall \alpha)\ (\exists F \in \mathcal{F}_{\alpha+1})\ (\forall Q \in \mathcal{Q}_\alpha)\ |F \cap Q| \leq 1$

Conditions (i)–(iii) allow us to start and continue the induction, moreover (iii) ensures that (iv)–(vi) are satisfied at limit stages of the construction. It is neccesary to check conditions (iv)–(vi) at non-limit induction step only.

Induction step: Suppose we know already $\mathcal{F}_\alpha$ and $\mathcal{A}_\alpha$. Due to (iv) and (v) we may apply Lemma 3.1 to $\mathcal{F}_\alpha$ and $\mathcal{A}_\alpha$ and put $\mathcal{A}_{\alpha+1} = \mathcal{A}'$. The family $\mathcal{A}_{\alpha+1}$ satisfies condition (iv) and is independent with respect to $\mathcal{F}_\alpha$. Now, we may apply Lemma 3.2 to $\mathcal{F}_\alpha$, $\mathcal{A}_{\alpha+1}$ and $\mathcal{Q}_\alpha$ and put $\mathcal{F}_{\alpha+1} = \mathcal{F}'$. The filter base $\mathcal{F}_{\alpha+1}$ satisfies (iv)–(vi).

Finally, let $\mathcal{F} = \bigcup_{\alpha<\mathfrak{c}} \mathcal{F}_\alpha$. Every ultrafilter which extends $\mathcal{F}$ is a Q-point because of condition (vi). Let us show that $\mathcal{F}$ can be extended to an ultrafilter which is not an I_α-ultrafilter for every $\alpha < \mathfrak{c}$. For every $X \in \mathcal{I}_\alpha$ let $G^X_\alpha = \bigcup\{A_{\alpha,n} : n \notin X\}$. Family $\mathcal{G}_\alpha = \{G^X_\alpha : X \in \mathcal{I}_\alpha\}$ is a filter base for every $\alpha < \mathfrak{c}$ because it is closed under finite intersections.

Claim 1. $\mathcal{G} = \mathcal{F} \cup \bigcup\{\mathcal{G}_\alpha : \alpha < \mathfrak{c}\}$ *has finite intersection property.*
We have to check that $G^{X_1}_{\alpha_1} \cap G^{X_2}_{\alpha_2} \cap \cdots \cap G^{X_k}_{\alpha_k} \cap F$ is infinite whenever $F \in \mathcal{F}$, $G^{X_i}_{\alpha_i} \in \mathcal{G}_{\alpha_i}$. We may assume $\alpha_1 < \alpha_2 < \cdots < \alpha_k < \mathfrak{c}$ and $F \in \mathcal{F}_\beta$. Let $\alpha = \max\{\beta, \alpha_i : i = 1, 2, \dots, k\}$ and for every $i = 1, 2, \dots, k$ choose $n_i \notin X_i$. The set $F \cap \bigcap_{i=1}^k A_{\alpha_i,n_i}$ is infinite because family $\mathcal{A}_\alpha$ is independent with respect to $\mathcal{F}_\alpha \supseteq \mathcal{F}_\beta$. It follows from $G^{X_i}_{\alpha_i} \supseteq A_{\alpha_i,n_i}$ that $F \cap \bigcap_{i=1}^k G^{X_i}_{\alpha_i}$ is also infinite.

Let $\mathcal{U}$ be an ultrafilter which extends $\mathcal{G}$. The ultrafilter $\mathcal{U}$ is obviously a Q-point because it extends $\mathcal{F}$. To complete the proof it remains to check that $\mathcal{U}$ is not an $\mathcal{I}_\alpha$-ultrafilter for every $\alpha < \mathfrak{c}$.

Claim 2. $(\forall \alpha < \mathfrak{c})$ $\mathcal{U}$ *is not an* $\mathcal{I}_\alpha$*-ultrafilter.*
Given $\alpha < \mathfrak{c}$ define $f_\alpha : \omega \to \omega$ by $f_\alpha[A_{\alpha,n}] = \{n\}$. Assume for the contrary that there exists $U \in \mathcal{U}$ such that $f_\alpha[U] = X \in \mathcal{I}_\alpha$. Then $G^X_\alpha = f^{-1}_\alpha[\omega \setminus X]$ is disjoint from U and $\mathcal{U}$ is not an ultrafilter — a contradiction. □

Corollary 3.4. (MA_{ctble}) *There is a Q-point which is not an $\mathcal{I}_g$-ultrafilter for every (tall) summable ideal $\mathcal{I}_g$.*

Proof. This is an immediate consequence of Theorem 3.3 because there are only $\mathfrak{c}$-many summable ideals. □

4. $\mathcal{I}_g$-ultrafilters need not be Q-points

Unlike Q-points, $\mathcal{I}_g$-ultrafilters are closed under sums and products. We will use this fact to construct an $\mathcal{I}_g$-ultrafilter which is not a Q-point. So let us first recall the definition of ultrafilter sums and products (for details see [2]).

Assume $\mathcal{U}$ and $\mathcal{V}_n$, $n \in \omega$, are ultrafilters on ω. There is an ultrafilter on $\omega \times \omega$ denoted as $\sum_{\mathcal{U}} \langle \mathcal{V}_n : n \in \omega\rangle$ defined by $M \in \sum_{\mathcal{U}} \langle \mathcal{V}_n : n \in \omega\rangle$ if and only if $\{n : \{m : \langle n, m\rangle \in A\} \in \mathcal{V}_n\} \in \mathcal{U}$. Ultrafilter $\sum_{\mathcal{U}} \langle \mathcal{V}_n : n \in \omega\rangle$ is called the *$\mathcal{U}$-sum of ultrafilters* $\mathcal{V}_n$, $n \in \omega$. If $\mathcal{V}_n = \mathcal{V}$ for every $n \in \omega$ then we write $\sum_{\mathcal{U}} \langle \mathcal{V}_n : n \in \omega\rangle = \mathcal{U} \cdot \mathcal{V}$ and ultrafilter $\mathcal{U} \cdot \mathcal{V}$ is called the *product of ultrafilters $\mathcal{U}$ and $\mathcal{V}$.*

The ultrafilter sum of $\mathcal{I}_g$-ultrafilters is again an $\mathcal{I}_g$-ultrafilter and product of $\mathcal{I}_g$-ultrafilters is again an $\mathcal{I}_g$-ultrafilter for arbitrary summable ideal $\mathcal{I}_g$ because summable ideals are P-ideals and the following proposition holds.

Proposition 4.1. *Let $\mathcal{I}$ be a P-ideal on ω. If $\mathcal{U}$ is an $\mathcal{I}$-ultrafilter and $\{n : \mathcal{V}_n$ is $\mathcal{I}$-ultrafilter$\} \in \mathcal{U}$ then $\sum_{\mathcal{U}} \langle \mathcal{V}_n : n \in \omega\rangle$ is an $\mathcal{I}$-ultrafilter.*

Proof. Suppose $\mathcal{U}$ is an $\mathcal{I}$-ultrafilter and for some $U_0 \in \mathcal{U}$ the ultrafilters $\mathcal{V}_n$, $n \in U_0$, are also $\mathcal{I}$-ultrafilters. Let $f : \omega \times \omega \to \omega$ be an arbitrary function. We want to find $M \in \sum_{\mathcal{U}} \langle \mathcal{V}_n : n \in \omega \rangle$ such that $f[M] \in \mathcal{I}$.

For every $n \in U_0$ define $f_n : \omega \to \omega$ by $f_n(m) = f(\langle n, m \rangle)$ for every $m \in \omega$. According to the assumption for every $n \in U_0$ there exists $V_n \in \mathcal{V}_n$ such that $f_n[V_n] \in \mathcal{I}$. Since $\mathcal{I}$ is a P-ideal there is a set $A \in \mathcal{I}$ such that $f_n[V_n] \subseteq^* A$ for every $n \in U_0$.

The set $f_n^{-1}[f_n[V_n]]$ belongs obviously to $\mathcal{V}_n$. Therefore either $f_n^{-1}[f_n[V_n] \cap A]$ or $f_n^{-1}[f_n[V_n] \setminus A]$ belongs to $\mathcal{V}_n$ for every $n \in U_0$. Define $I_0 = \{n \in U_0 : f_n^{-1}[f_n[V_n] \cap A] \in \mathcal{V}_n\}$ and $I_1 = \{n \in U_0 : f_n^{-1}[f_n[V_n] \setminus A] \in \mathcal{V}_n\}$. Since $\mathcal{U}$ is an ultrafilter one of the sets I_0, I_1 belongs to the ultrafilter $\mathcal{U}$.

Case A. $I_0 \in \mathcal{U}$

Put $M = \{\{n\} \times f_n^{-1}[f_n[V_n] \cap A] : n \in I_0\}$. It is easy to see that $M \in \sum_{\mathcal{U}} \langle \mathcal{V}_n : n \in \omega \rangle$ and $f[M] = \bigcup_{n \in I_0} f_n[V_n] \cap A \subseteq A \in \mathcal{I}$.

Case B. $I_1 \in \mathcal{U}$

Since $f_n[U_n] \setminus A$ is finite and $\mathcal{V}_n$ is an ultrafilter, there exists $k_n \in f_n[U_n] \setminus A$ such that $f_n^{-1}\{k_n\} \in \mathcal{V}_n$ for every $n \in I_1$. Define $g : \omega \to \omega$ by $g(n) = k_n$ if $n \in I_1$ and arbitrarily otherwise. Since $\mathcal{U}$ is an $\mathcal{I}$-ultrafilter there exists $U \in \mathcal{U}$ such that $g[U] \in \mathcal{I}$. It remains to put $M = \{\{n\} \times f_n^{-1}\{k_n\} : n \in I_1 \cap U\}$. It is easy to check that $M \in \sum_{\mathcal{U}} \langle \mathcal{V}_n : n \in \omega \rangle$ and $f[M] \subseteq g[U] \in \mathcal{I}$. □

Theorem 4.2. (MA_{ctble}) *There exists $\mathcal{U} \in \omega^*$ such that $\mathcal{U}$ is an $\mathcal{I}_g$-ultrafilter for every tall summable ideal $\mathcal{I}_g$ and $\mathcal{U}$ is not a Q-point.*

Proof. According to Theorem 2.1 there exists an ultrafilter $\mathcal{U} \in \omega^*$ such that $\mathcal{U}$ is an $\mathcal{I}_g$-ultrafilter for every tall summable ideal $\mathcal{I}_g$. By Proposition 4.1 the ultrafilter $\mathcal{U} \times \mathcal{U}$ is again an $\mathcal{I}_g$-ultrafilter for every tall summable ideal $\mathcal{I}_g$, but it is not a Q-point because no product of two ultrafilters is a Q-point. □

Unfortunately, one cannot use ultrafilter sums to construct $\mathcal{I}_g$-ultrafilters which are not rapid because rapid ultrafilters are closed under sums over rapid ultrafilters. Thus it remains an open problem whether for every tall summable ideal $\mathcal{I}_g$ there exists an $\mathcal{I}_g$-ultrafilter which is not rapid.

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References

[1] Baumgartner, J., Ultrafilters on ω, *J. Symbolic Logic* **60**, no. 2, 624–639, 1995.

[2] W. W. Comfort, S. Negrepontis, *The Theory of Ultrafilters*, Springer, Berlin, New York, 1974

[3] Farah, I., Analytic Quotients: Theory of Liftings for Quotients over Analytic Ideals on the Integers, *Mem. Amer. Math. Soc.* **148**, no. 702, 2000.

[4] Flašková, J., Ultrafilters and small sets, Ph.D. Thesis, Charles University, Prague, 2006.

[5] Flašková, J., More than a 0-point, *Comment. Math. Univ. Carolinae* **47**, no. 4, 617–621, 2006.
[6] Hrušák, M., talk in Prague Set theory seminar, August 2007.
[7] Vojtáš, P., On ω^* and absolutely divergent series, *Topology Proceedings* **19**, 335 – 348, 1994.

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