THIN ULTRAFILTERS

Jana Flašková

Abstract. Two ideals on ω are presented which determine the same class of $\mathscr{I}$-ultrafilters. It is proved that the existence of these ultrafilters is independent of ZFC and the relation between this class and other well-known classes of ultrafilters is shown.

Let $\mathscr{I}$ be a family of subsets of a set X such that $\mathscr{I}$ contains all singletons and is closed under subsets. Given an ultrafilter $\mathscr{U}$ on ω, we say that $\mathscr{U}$ is an *$\mathscr{I}$-ultrafilter* if for any mapping $F : \omega \to X$ there is $A \in \mathscr{U}$ such that $F(A) \in \mathscr{I}$.

Concrete examples of $\mathscr{I}$-ultrafilters are nowhere dense ultrafilters, measure zero ultrafilters or countably closed ultrafilters defined by taking $X = 2^\omega$ and $\mathscr{I}$ to contain all the nowhere dense sets, the sets with closure of measure zero, or the sets with countable closure, respectively. The class of α-ultrafilters was defined for an indecomposable countable ordinal α by taking $X = \omega_1$ and $\mathscr{I}$ to consist of the subsets of ω_1 with order type less than α.

Consistency results about existence of these ultrafilters and some inclusions among the appropriate classes of ultrafilters were obtained by Baumgartner [1], Brendle [4], Barney [2]. It was proved by Shelah [8] that consistently there are no nowhere dense ultrafilters, consequently all mentioned ultrafilters (except the α-ultrafilters for which the question is still open) may not exist.

In this paper, we focus on free ultrafilters on ω defined by taking $X = \omega$ and $\mathscr{I}$ to be two different collections of subsets of natural numbers.

I. Basic facts and definitions.

It was noticed in [1] that for a given family $\mathscr{I}$ the $\mathscr{I}$-ultrafilters are closed downward under the Rudin-Keisler ordering $\leq_{RK}$ (recall that $\mathscr{U} \leq_{RK} \mathscr{V}$ if there is a function $f : \omega \to \omega$ whose Stone extension $\beta f : \beta\omega \to \beta\omega$ maps $\mathscr{V}$ on $\mathscr{U}$, see [5]).

Replacing $f[U] \in \mathscr{I}$ by $f[U] \in \langle \mathscr{I} \rangle$ in the definition of $\mathscr{I}$-ultrafilter, where $\langle \mathscr{I} \rangle$ is the ideal generated by $\mathscr{I}$, we get the same concept (see [2]), i.e. $\mathscr{I}$-ultrafilters and $\langle \mathscr{I} \rangle$-ultrafilters coincide. Obviously, if $\mathscr{U}$ is an $\mathscr{I}$-ultrafilter then $\mathscr{U} \cap \mathscr{I} \neq \emptyset$ (the converse is not true) and if $\mathscr{I} \subseteq \mathscr{J}$ then every $\mathscr{I}$-ultrafilter is a $\mathscr{J}$-ultrafilter.

Lemma. *If $\mathcal{C}$ is a class of ultrafilters closed downward under $\leq_{RK}$ and $\mathscr{I}$ an ideal on ω then the following are equivalent:*

(i) There exists $\mathscr{U} \in \mathcal{C}$ which is not an $\mathscr{I}$-ultrafilter

(ii) There exists $\mathscr{V} \in \mathcal{C}$ which extends $\mathscr{I}^*$, the dual filter of $\mathscr{I}$

Proof. No ultrafilter extending $\mathscr{I}^*$ is an $\mathscr{I}$-ultrafilter, so (ii) implies (i) trivially. To prove (i) implies (ii) assume that $\mathscr{U} \in \mathcal{C}$ is not an $\mathscr{I}$-ultrafilter. Hence there is a function $f \in {}^\omega\omega$ such that $(\forall A \in \mathscr{I})$ $f^{-1}[A] \notin \mathscr{U}$. Let $\mathscr{V} = \{V \subseteq \omega : f^{-1}[V] \in \mathscr{U}\}$. Obviously $\mathscr{V}$ extends $\mathscr{I}^*$ and $\mathscr{V} \leq_{RK} \mathscr{U}$. Since $\mathcal{C}$ is closed downward under $\leq_{RK}$ and $\mathscr{U} \in \mathcal{C}$ we get $\mathscr{V} \in \mathcal{C}$. $\square$

An infinite set $A \subseteq \omega$ with enumeration $A = \{a_n : n \in \omega\}$ is called *almost thin* if $\limsup_n \frac{a_n}{a_{n+1}} < 1$ and *thin* (see [3]) if $\lim_n \frac{a_n}{a_{n+1}} = 0$. (Notice that by enumeration of a set of natural numbers we always mean an order preserving enumeration.)

We will denote the ideal generated by finite and thin sets by $\mathscr{T}$ and the ideal generated by finite and almost thin sets by $\mathscr{A}$. The corresponding $\mathscr{I}$-ultrafilters will be called *thin ultrafilters* and *almost thin ultrafilters* respectively. We prove in Section I. that in fact these two classes of ultrafilters coincide, although the ideals $\mathscr{T}$ and $\mathscr{A}$ differ as the set $\{2^n : n \in \omega\} \in \mathscr{A} \setminus \mathscr{T}$.

We show in Section II. the relation between thin ultrafilters and selective ultrafilters and the relation between thin ultrafilters and Q-points in Section III. Let us recall the definitions:

A free ultrafilter $\mathscr{U}$ is called a *selective ultrafilter* if for all partitions of ω, $\{R_i : i \in \omega\}$, either for some i, $R_i \in \mathscr{U}$, or $\exists U \in \mathscr{U}$ such that $(\forall i \in \omega)$ $|U \cap R_i| \leq 1$.

A free ultrafilter $\mathscr{U}$ is called a *Q-point* if for all partitions of ω consisting of finite sets, $\{Q_i : i \in \omega\}$, $\exists U \in \mathscr{U}$ such that $(\forall i \in \omega)$ $|U \cap Q_i| \leq 1$.

II. Thin ultrafilters and almost thin ultrafilters.

The existence of thin ultrafilters is independent of ZFC. It is easy to construct a thin ultrafilter if we assume the Continuum Hypothesis (in Section II. we prove that the strictly weaker assumption (see [6]) that selective ultrafilters exist is sufficient) and we prove in Section III. that every thin ultrafilter is a Q-point whose existence is not provable in ZFC (see [7]).

Proposition 1. *(CH) There is a thin ultrafilter.*

Proof. Enumerate ${}^{\omega}\omega = \{f_\alpha : \alpha < \omega_1\}$. By transfinite induction on $\alpha < \omega_1$ we construct countable filter bases $\mathscr{F}_\alpha$ satisfying

(i) $\mathscr{F}_0$ is the Fréchet filter

(ii) $\mathscr{F}_\alpha \subseteq \mathscr{F}_\beta$ whenever $\alpha \leq \beta$

(iii) $\mathscr{F}_\gamma = \bigcup_{\alpha<\gamma} \mathscr{F}_\alpha$ for γ limit

(iv) $(\forall \alpha)$ $(\exists F \in \mathscr{F}_{\alpha+1})$ $f_\alpha[F] \in \mathscr{T}$

Suppose we have constructed $\mathscr{F}_\alpha$. If there exists a set $F \in \mathscr{F}_\alpha$ such that $f_\alpha[F] \in \mathscr{T}$ then put $\mathscr{F}_{\alpha+1} = \mathscr{F}_\alpha$. If $f_\alpha[F] \notin \mathscr{T}$ (in particular, $f_\alpha[F]$ is infinite) for every $F \in \mathscr{F}_\alpha$ then enumerate $\mathscr{F}_\alpha = \{F_n : n \in \omega\}$ and construct by induction a set $U = \{u_n : n \in \omega\}$ which we extend the filter base by:

Choose arbitrary $u_0 \in F_0$ such that $f_\alpha(u_0) > 0$ (such an element exists since $f_\alpha[F_0]$ is infinite). If $u_0, u_1, \ldots, u_{k-1}$ are already known we can choose $u_k \in \bigcap_{i \leq k} F_i$ so that $f_\alpha(u_k) > k \cdot f_\alpha(u_{k-1})$.

It is obvious that $U \subseteq^* F_n$, i.e. all but finitely many elements of U are contained in F_n, for all $n \in \omega$. We can check immediately that

$f_\alpha[U]$ is thin:

$$\lim_{n\to\infty} \frac{f_\alpha(u_n)}{f_\alpha(u_{n+1})} \leq \lim_{n\to\infty} \frac{f_\alpha(u_n)}{(n+1)\cdot f_\alpha(u_n)} = \lim_{n\to\infty} \frac{1}{(n+1)} = 0$$

To complete the induction step let $\mathscr{F}_{\alpha+1}$ be the countable filter base generated by $\mathscr{F}_\alpha$ and the set U.

It is clear that any ultrafilter extending $\bigcup_{\alpha<\omega_1} \mathscr{F}_\alpha$ is thin. $\square$

Since $\mathscr{T} \subset \mathscr{A}$ every thin ultrafilter has to be almost thin. The following proposition states that the converse also holds true and the classes of almost thin and thin ultrafilters coincide.

Proposition 2. *Every almost thin ultrafilter is a thin ultrafilter.*

Proof. Because of the Lemma it suffices to prove that every almost thin ultrafilter contains a thin set. To that end assume that $\mathscr{U}$ is an almost thin ultrafilter and $U_0 \in \mathscr{U}$ is an almost thin set which is not thin with enumeration $U_0 = \{u_n : n \in \omega\}$. Denote $\limsup_n \frac{u_n}{u_{n+1}} = q_0 < 1$. We may assume that the set of even numbers belongs to $\mathscr{U}$ (otherwise the role of even and odd numbers interchange).

Define $g : \omega \to \omega$ so that $g(u_n) = 2n$, $g[\omega \setminus U_0] = \{2n+1 : n \in \omega\}$.

Since $\mathscr{U}$ is an almost thin ultrafilter there exists $U_1 \in \mathscr{U}$ such that $g[U_1]$ is almost thin. Let $U = U_0 \cap U_1 = \{u_{n_k} : k \in \omega\}$. Almost thin sets are closed under subsets, therefore $g[U] = \{g(u_{n_k}) : k \in \omega\} \subseteq g[U_1]$ is almost thin and $1 > \limsup_k \frac{g(u_{n_k})}{g(u_{n_{k+1}})} = \limsup_k \frac{2n_k}{2n_{k+1}}$.

We know that there is n_0 such that $(\forall n \geq n_0)\ \frac{u_n}{u_{n+1}} \leq \frac{q_0+1}{2}$ and that there is k_0 such that $(\forall k \geq k_0)\ n_k \geq n_0$. Hence for $k \geq k_0$ we have

$$\frac{u_{n_k}}{u_{n_{k+1}}} = \frac{u_{n_k}}{u_{n_k+1}} \cdot \cdots \cdot \frac{u_{n_{k+1}-1}}{u_{n_{k+1}}} \leq \left(\frac{q_0+1}{2}\right)^{n_{k+1}-n_k}$$

It follows from $\limsup_k \frac{n_k}{n_{k+1}} < 1$ that $\lim_k (n_{k+1} - n_k) = +\infty$. Hence

$$\lim_{k\to\infty} \frac{u_{n_k}}{u_{n_{k+1}}} \leq \lim_{k\to\infty} \left(\frac{q_0+1}{2}\right)^{n_{k+1}-n_k} = 0$$

and the set $U \in \mathscr{U}$ is thin. $\square$

III. Thin ultrafilters and selective ultrafilters.

Proposition 3. *Every selective ultrafilter is thin.*

Proof. Selective ultrafilters are minimal points in Rudin-Keisler ordering (see [5]), hence the class is downward closed under $\leq_{RK}$ and we may apply the Lemma. It suffices to prove that there is no selective ultrafilter extending the dual filter of $\mathscr{T}$.

Claim: Every selective ultrafilter contains a thin set.
Assume $\mathscr{U}$ is a selective ultrafilter and consider the partition of ω, $\{R_n : n \in \omega\}$, where $R_0 = \{0\}$ and $R_n = [n!, (n+1)!)$ for $n > 0$. Since $\mathscr{U}$ is selective there exists $U_0 \in \mathscr{U}$ such that $|U_0 \cap R_n| \leq 1$ for every $n \in \omega$. Since $\mathscr{U}$ is an ultrafilter either $A_0 = \bigcup\{R_n : n \text{ is even}\}$ or $A_1 = \bigcup\{R_n : n \text{ is odd}\}$ belongs to $\mathscr{U}$. Without loss of generality, assume $A_0 \in \mathscr{U}$. Enumerate $U = U_0 \cap A_0 \in \mathscr{U}$ as $\{u_k : k \in \omega\}$. If $u_k \in [(2m_k)!, (2m_k+1)!)$ then $u_{k+1} \geq (2m_k+2)!$ and we have $\frac{u_k}{u_{k+1}} \leq \frac{(2m_k+1)!}{(2m_k+2)!} = \frac{1}{2m_k+2} \leq \frac{1}{2k+2}$. Hence U is thin. $\square$

Proposition 4. *(CH) Not every thin ultrafilter is selective.*

Proof. Fix a partition $\{R_n : n \in \omega\}$ of ω into infinite sets and enumerate ${}^{\omega}\omega = \{f_\alpha : \alpha < \omega_1\}$. By transfinite induction on $\alpha < \omega_1$ we will construct countable filter bases $\mathscr{F}_\alpha$ satisfying

(i) $\mathscr{F}_0$ is the countable filter base generated by Fréchet filter and $\{\omega \setminus R_n : n \in \omega\}$

(ii) $\mathscr{F}_\alpha \subseteq \mathscr{F}_\beta$ whenever $\alpha \leq \beta$

(iii) $\mathscr{F}_\gamma = \bigcup_{\alpha<\gamma} \mathscr{F}_\alpha$ for γ limit

(iv) $(\forall\alpha)$ $(\forall F \in \mathscr{F}_\alpha)$ $\{n : |F \cap R_n| = \omega\}$ is infinite

(v) $(\forall\alpha)$ $(\exists F \in \mathscr{F}_{\alpha+1})$ $f_\alpha[F] \in \mathscr{T}$

Suppose we know already $\mathscr{F}_\alpha$. If there is a set $F \in \mathscr{F}_\alpha$ such that $f_\alpha[F] \in \mathscr{T}$ then put $\mathscr{F}_{\alpha+1} = \mathscr{F}_\alpha$. If $(\forall F \in \mathscr{F}_\alpha)$ $f_\alpha[F] \notin \mathscr{T}$ then one of the following cases occurs.

Case A. $(\forall F \in \mathscr{F}_\alpha)$ $\{n : |f_\alpha[F \cap R_n]| = \omega\}$ is infinite

Fix an enumeration $\{F_m : m \in \omega\}$ of $\mathscr{F}_\alpha$. According to the assumption the set $I_m = \{n : |f_\alpha[\bigcap_{i \leq m} F_i \cap R_n]| = \omega\}$ is infinite for all $m \in \omega$. Let us enumerate the set $\{\langle m, n\rangle : n \in I_m, m \in \omega\}$ as $\{p_k : k \in \omega\}$ so that each ordered pair $\langle m, n\rangle$ occurs infinitely many times. By induction we will construct a set $U = \{u_k : k \in \omega\}$ which we may add to $\mathscr{F}_\alpha$.

Consider $p_0 = \langle m, n\rangle$. Since $|f_\alpha[\bigcap_{i \leq m} F_i \cap R_n]| = \omega$ we can choose $u_0 \in \bigcap_{i \leq m} F_i \cap R_n$ such that $f_\alpha(u_0) > 0$. Suppose we know $u_0, \dots, u_{k-1}$ such that $u_{i+1} > u_i$, $u_i \in \bigcap_{j \leq m} F_j \cap R_n$ where $p_i = \langle m, n\rangle$ and $f_\alpha(u_{i+1}) > (i+1) \cdot f_\alpha(u_i)$ for $i < k-1$. Consider $p_k = \langle m, n\rangle$. Since $\bigcap_{i \leq m} F_i \cap R_n$ and its image under f_α is infinite we may choose $u_k \in \bigcap_{i \leq m} F_i \cap R_n$ so that $u_k > u_{k-1}$ and $f_\alpha(u_k) > k \cdot f_\alpha(u_{k-1})$.

It remains to verify that $f_\alpha[U]$ is thin and $\mathscr{F}_\alpha \cup \{U\}$ generates a filter base satisfying (iv).

• $f_\alpha[U]$ is thin:

$$\lim_{k\to\infty} \frac{f_\alpha(u_k)}{f_\alpha(u_{k+1})} \leq \lim_{k\to\infty} \frac{f_\alpha(u_k)}{(k+1) \cdot f_\alpha(u_k)} = \lim_{k\to\infty} \frac{1}{(k+1)} = 0$$

• $(\forall F \in \mathscr{F}_\alpha)$ $\{n : |U \cap F \cap R_n| = \omega\}$ is infinite:

For every $F \in \mathscr{F}_\alpha$ there is $m_F \in \omega$ such that $U \cap F \cap R_n \supseteq U \cap \bigcap_{i \leq m_F} F_i \cap R_n$ which is infinite whenever $n \in I_{m_F}$ since $U \cap \bigcap_{i \leq m_F} F_i \cap R_n \supseteq \{u_k : p_k = \langle m_F, n\rangle\}$.

To complete the induction step let $\mathscr{F}_{\alpha+1}$ be the countable filter base generated by $\mathscr{F}_\alpha$ and U.

Case B. $(\exists F_0 \in \mathscr{F}_\alpha)$ $\{n : |f_\alpha[F_0 \cap R_n]| = \omega\}$ is finite

Enumerate $\mathscr{F}_\alpha \setminus \{F_0\} = \{F_m : m > 0\}$. Since F_0 satisfies (iv) there is $n_0 \in \omega$ such that $|f_\alpha[F_0 \cap R_{n_0}]| < \omega$ and $|F_0 \cap R_{n_0}| = \omega$. It follows that there is $z_0 \in \omega$ such that $f_\alpha^{-1}[\{z_0\}] \cap F_0 \cap R_{n_0}$ is infinite. The set $\bigcap_{i \leq m} F_i$ satisfies (iv) for any $m \in \omega$ and for all but finitely many n the set $f_\alpha[\bigcap_{i \leq m} F_i \cap R_n]$ is finite. So we may choose $n_m > n_{m-1}$ such that $f_\alpha[\bigcap_{i \leq m} F_i \cap R_{n_m}]$ is finite and $\bigcap_{i \leq m} F_i \cap R_{n_m}$ infinite. We find z_m such that $f_\alpha^{-1}[\{z_m\}] \cap \bigcap_{i \leq m} F_i \cap R_{n_m}$ is infinite.

Consider the sequence $\langle z_m : m \in \omega \rangle$. We can find a subsequence $\langle z_{m_j} : j \in \omega \rangle$ which is either constant or satisfies $z_{m_{j+1}} > (j+1) \cdot z_{m_j}$ for every $j \in \omega$. Set $U = \bigcup_{j \in \omega} f_\alpha^{-1}[\{z_{m_j}\}]$. It is obvious that $f_\alpha[U] \in \mathscr{T}$ and $(\forall F \in \mathscr{F}_\alpha)\ \{n : |U \cap F \cap R_n| = \omega\}$ is infinite.
To complete the induction step let $\mathscr{F}_{\alpha+1}$ be the countable filter base generated by $\mathscr{F}_\alpha$ and U.

The filter base $\mathscr{F} = \bigcup_{\alpha<\omega_1} \mathscr{F}_\alpha$ satisfies (iv) and every ultrafilter which extends $\mathscr{F}$ is thin because of condition (v).

Claim: Every filter satisfying (iv) may be extended to an ultrafilter satisfying (iv).
Whenever $\mathscr{F}$ is a filter satisfying (iv) and $A \subseteq \omega$ then either for every $F \in \mathscr{F}$ exist infinitely many $n \in \omega$ such that $|A \cap F \cap R_n| = \omega$, so the filter generated by $\mathscr{F}$ and A satisfies (iv) or there is $F_0 \in \mathscr{F}$ such that for all but finitely many $n \in \omega$ we have $|A \cap F_0 \cap R_n| < \omega$. Then since for every $F \in \mathscr{F}$ exist infinitely many $n \in \omega$ for which $|F \cap F_0 \cap R_n| = \omega$ the filter generated by $\mathscr{F}$ and $\omega \setminus A$ satisfies (iv). Hence for every subset of ω we may extend $\mathscr{F}$ either by the set itself or its complement. Consequently, $\mathscr{F}$ may be extended to an ultrafilter satisfying (iv). $\square$

IV. Thin ultrafilters and *Q*-points.

Proposition 5. *Every thin ultrafilter is a Q-point.*

Proof. Let $\mathscr{U}$ be a thin ultrafilter and $\mathscr{Q} = \{Q_n : n \in \omega\}$ a partition of ω into finite sets. Enumerate $Q_n = \{q_i^n : i = 0, \dots, k_n\}$ (where $k_n = |Q_n| - 1$).

Define a one-to-one function $f : \omega \to \omega$ in the following way:
$f(q_0^0) = 0$, $f(q_0^{n+1}) = (n+2) \cdot \max\{f(q_{k_n}^n), k_{n+1}\}$ for $n \in \omega$ and $f(q_i^n) = f(q_0^n) + i$ for $i \le k_n, n \in \omega$

Notice that $f \restriction Q_n$ is strictly increasing for every n.

Since $\mathscr{U}$ is a thin ultrafilter there exists $U_0 \in \mathscr{U}$ such that $f[U_0] = \{v_m : m \in \omega\}$ is a thin set. Hence there is $m_0 \in \omega$ such that $\frac{v_m}{v_{m+1}} < \frac{1}{2}$ for every $m \ge m_0$.

Since f is one-to-one and $\mathcal{Q}$ a partition of ω we find $K \subseteq \omega$ of size at most m_0 such that $\{f^{-1}(v_i) : i < m_0\} \subseteq \bigcup_{n \in K} Q_n$. The latter set is finite. Therefore $U = U_0 \setminus \bigcup_{n \in K} Q_n \in \mathcal{U}$.
Claim: $(\forall n \in \omega)\ |U \cap Q_n| \leq 1$
The intersection is clearly empty for $n \in K$. Assume for the contrary that for some $n \notin K$ there are two distinct elements $u_1, u_2 \in U \cap Q_n$, $u_1 < u_2$. Then $f(u_1) = v_m$ for some $m \geq m_0$ and $f(u_2) = v_n$ for some $n \geq m + 1$. We get $\frac{v_m}{v_{m+1}} \geq \frac{v_m}{v_n} = \frac{f(u_1)}{f(u_2)} \geq \frac{f(q_0^n)}{f(q_0^n)+k_n} \geq \frac{(n+1)\cdot M}{(n+1)\cdot M + M} = \frac{n+1}{n+2}$ where $M = \max\{f(q_{k_{n-1}}^{n-1}), k_n\}$. But $\frac{n+1}{n+2} \geq \frac{1}{2}$, a contradiction. $\square$

Note that while proving that every selective ultrafilter contains a thin set we actually proved that every Q-point contains a thin set as the partition under consideration consists of finite sets. However, Q-points need not be thin ultrafilters.

Proposition 6. *(CH) Not every Q-point is a thin ultrafilter.*

Proof. Fix $\{R_n : n \in \omega\}$ a partition of ω into infinite sets. Let $\{\mathcal{Q}_\alpha : \alpha < \omega_1\}$ be the list of all partitions of ω into finite sets. By transfinite induction on $\alpha < \omega_1$ we will construct countable filter bases $\mathscr{F}_\alpha$ so that the following are satisfied:

(i) $\mathscr{F}_0$ is the Fréchet filter
(ii) $\mathscr{F}_\alpha \subseteq \mathscr{F}_\beta$ whenever $\alpha < \beta$
(iii) $\mathscr{F}_\gamma = \bigcup_{\alpha < \gamma} \mathscr{F}_\alpha$ for γ limit
(iv) $(\forall \alpha)\ (\forall F \in \mathscr{F}_\alpha)\ (\forall n)\ |F \cap R_n| = \omega$
(v) $(\forall \alpha)\ (\exists F \in \mathscr{F}_{\alpha+1})\ (\forall Q \in \mathcal{Q}_\alpha)\ |F \cap Q| \leq 1$

Suppose we already know $\mathscr{F}_\alpha$. If there is a set $F \in \mathscr{F}_\alpha$ such that $|F \cap Q| \leq 1$ for each $Q \in \mathcal{Q}_\alpha$ then let $\mathscr{F}_{\alpha+1} = \mathscr{F}_\alpha$. If $(\forall F \in \mathscr{F}_\alpha)$ $(\exists Q \in \mathcal{Q}_\alpha)\ |F \cap Q| > 1$, we construct by induction a set U compatible with $\mathscr{F}_\alpha$ such that $|U \cap Q| \leq 1$ for each $Q \in \mathcal{Q}_\alpha$.

Enumerate $\mathscr{F}_\alpha = \{F_k : k \in \omega\}$, $\mathcal{Q}_\alpha = \{Q_n : n \in \omega\}$ and list $\{R_n : n \in \omega\} = \{M_k : k \in \omega\}$ so that each R_n is listed infinitely often. To start the construction of U choose $u_0 \in M_0$ arbitrarily. Since $\bigcup \mathcal{Q}_\alpha = \omega$ there exists $n_0 \in \omega$ such that $u_0 \in Q_{n_0}$.

Suppose we already know $u_0, \ldots, u_{k-1}$ and $n_0, \ldots, n_{k-1}$ such that $u_i \in Q_{n_i}$ for $i < k$. Since $\bigcup_{i<k} Q_{n_i}$ is finite and $\bigcap_{i<k} F_i \cap M_k$ is infinite according to (iv) we may choose $u_k \in (\bigcap_{i<k} F_i \cap M_k) \setminus \bigcup_{i<k} Q_{n_i}$. Let Q_{n_k} be the unique element of $\mathscr{Q}_\alpha$ which contains u_k.

Finally, let $U = \{u_k : k \in \omega\}$. For every $n \in \omega$ we have either $U \cap Q_n = \{u_k\}$ (if $n = n_k$) or $U \cap Q_n = \emptyset$. It remains to check that U is compatible with $\mathscr{F}_\alpha$ and satisfies (iv). However, for every $F \in \mathscr{F}_\alpha$ and for every R_n there is $n_F \in \omega$ such that $U \cap F \cap R_n \supseteq U \cap \bigcap_{i<n_F} F_i \cap R_n \supseteq \{u_k : k \geq n_F, M_k = R_n\}$. The latter set is infinite. Hence the countable filter base generated by $\mathscr{F}_\alpha$ and U satisfies (ii), (iv), (v) as required and it may be taken as $\mathscr{F}_{\alpha+1}$.

Because of condition (v) every ultrafilter which extends the filter base $\mathscr{F} = \bigcup_{\alpha<\omega_1} \mathscr{F}_\alpha$ is a Q-point. Let $\mathscr{G} = \{\bigcup_{n\in M} R_n : \omega \setminus M \in \mathscr{T}\}$. Condition (iv) in induction assumption guarantees that $\mathscr{F} \cup \mathscr{G}$ generates a free filter on ω. Every ultrafilter $\mathscr{U}$ which extends $\mathscr{F} \cup \mathscr{G}$ is a Q-point but not a thin ultrafilter because for every $U \in \mathscr{U}$ $f[U] \notin \mathscr{T}$ where $f : \omega \to \omega$ is defined by $f \restriction R_n = n$. $\square$

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