ON CERTAIN $\mathcal{I}$-ULTRAFILTERS AND IP RICH SETS

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SUMMARY

We study special classes of ultrafilters on natural numbers and we focus on $\mathcal{I}$-ultrafilters which were introduced by Baumgartner. We are particularly interested in the situation where $\mathcal{I}$ is an ideal on ω. Summable ideals and the Hindman ideal are considered and the relations between the corresponding classes of ultrafilters are investigated. We show that there is no inclusion between the two classes of ultrafilters and that selective ultrafilters belong to both of them. Further, we consider also the ideal generated by sets which are not ip rich and describe how the class of corresponding $\mathcal{I}$-ultrafilters is related to the previous ones and to P-points. Assuming Martin's axiom for countable posets we can construct several ultrafilters with various combinations of (not) being an $\mathcal{I}$-ultrafilter and/or (not) being a $\mathcal{J}$-ultrafilter where $\mathcal{I}$, $\mathcal{J} \in \{$summable ideal, Hindman ideal, ideal $\mathcal{I}_{ipr}\}$.

INTRODUCTION

The definition of an $\mathcal{I}$-ultrafilter was introduced by Baumgartner in [1] : Let $\mathcal{I}$ be a family of subsets of a set X such that $\mathcal{I}$ contains all singletons and is closed under subsets. Given an ultrafilter $\mathcal{U}$ on ω, we say that $\mathcal{U}$ is an *$\mathcal{I}$-ultrafilter* if for every function $F : \omega \to X$ there exists $A \in \mathcal{U}$ such that $F[A] \in \mathcal{I}$.

For our purposes also a modification of Baumgartner's definition is relevant, in which the class of considered functions is restricted to finite-to-one functions: Let $\mathcal{I}$ be a family of subsets of a set X such that $\mathcal{I}$ contains all singletons and is closed under subsets. We say that an ultrafilter $\mathcal{U}$ on ω is a *weak $\mathcal{I}$-ultrafilter* if for every finite-to-one function $F : \omega \to X$ there exists $A \in \mathcal{U}$ such that $F[A] \in \mathcal{I}$.

We are particularly interested in the situation where $X = \omega$ and $\mathcal{I}$ is an ideal on ω. Since every principal ultrafilter is obviously an $\mathcal{I}$-ultrafilter for arbitrary $\mathcal{I}$, our attention pays to free ultrafilters only. We use the notation ω^* for the set of all free ultrafilters on ω.

Basic facts about $\mathcal{I}$-ultrafilters

It follows from the definition that every $\mathcal{I}$-ultrafilter is a weak $\mathcal{I}$-ultrafilter and every weak $\mathcal{I}$-ultrafilter has nonempty intersection with the ideal $\mathcal{I}$. Hence whenever $\mathcal{U} \in \omega^*$ satisfies $\mathcal{U} \cap \mathcal{I} = \emptyset$, i.e. $\mathcal{I}^* \subseteq \mathcal{U}$ where $\mathcal{I}^*$ is the dual filter to the given ideal $\mathcal{I}$, the ultrafilter $\mathcal{U}$ is not an $\mathcal{I}$-ultrafilter.

In order to generalize this observation recall the definition of Katětov order $\leq_K$ on filters. Given two filters $\mathcal{F}$ and $\mathcal{G}$ on ω we write $\mathcal{F} \leq_K \mathcal{G}$ if there is a function $f : \omega \to \omega$ such that $f^{-1}[F] \in \mathcal{G}$ for every $F \in \mathcal{F}$.

Lemma ([2], Lemma 2.1.6.)**.** *An ultrafilter $\mathcal{U}$ is an $\mathcal{I}$-ultrafilter if and only if $\mathcal{I}^* \not\leq_K \mathcal{U}$.*

$\mathcal{I}$-ultrafilters and $\mathcal{J}$-ultrafilters

Assume $\mathcal{I}$ and $\mathcal{J}$ are two ideals on ω. Obviously, if $\mathcal{I} \subseteq \mathcal{J}$ then every $\mathcal{I}$-ultrafilter is a $\mathcal{J}$-ultrafilter and an analogous result holds for weak $\mathcal{I}$-ultrafilters.

If there is no inclusion between the two ideals $\mathcal{I}$ and $\mathcal{J}$ then in general all possible combinations of being/not being an $\mathcal{I}$-ultrafilter and/or a $\mathcal{J}$-ultrafilter may occur. However, it is possible that the function witnessing the fact that an $\mathcal{I}$-ultrafilter is not a $\mathcal{J}$-ultrafilter is always the identity function.

Lemma ([2], Corollary 2.1.5.)**.** *Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on ω. An $\mathcal{I}$-ultrafilter which is not a $\mathcal{J}$-ultrafilter exists if and only if there exists an $\mathcal{I}$-ultrafilter which extends the dual filter $\mathcal{J}^*$.*

IDEALS ON NATURAL NUMBERS

An ideal on ω is called a *tall ideal* if for every infinite set $A \subseteq \omega$ there exists infinite $B \subseteq A$ such that $B \in \mathcal{I}$. It is not difficult to show that for ideals which are not tall the corresponding $\mathcal{I}$-ultrafilters and weak $\mathcal{I}$-ultrafilters do not exist.

We say that an ideal $\mathcal{I}$ is a *P-ideal* if whenever $A_n \in \mathcal{I}$, $n \in \omega$, then there is $A \in \mathcal{I}$ such that $A_n \subseteq^* A$ for every n where $A_n \subseteq^* A$ means "A contains all but finitely many elements of A_n".

If we identify a subset of ω with its characteristic function we may regard ideals on natural numbers as subsets of the Cantor set 2^ω. Hence the Borel hierarchy on the Cantor set induces Borel hierarchy on ideals and so we may speak about F_σ ideals, Borel ideals etc.

Summable ideals

For an arbitrary function $g : \omega \to (0, \infty)$ such that $\sum_{n\in\omega} g(n) = +\infty$ the family

$$\mathcal{I}_g = \{A \subseteq \omega : \sum_{a\in A} g(a) < +\infty\}$$

is an ideal on ω, which we call the *summable ideal determined by function g*. All summable ideals are P-ideals and F_σ ideals. A summable ideal is tall if and only if $\lim_{n\to\infty} g(n) = 0$.

Ideal $\mathcal{I}_{ipr}$ and Hindman ideal

We say that $A \subseteq \mathbb{N}$ is an *ip-rich set* if for every $k \in \mathbb{N}$ there is $D \subseteq \mathbb{N}$ with $|D| = k$ and $FS(D) \subseteq A$ where $FS(D)$ denotes the set of all finite sums of elements from D. It is known that sets which are not ip-rich form an ideal, which we denote by $\mathcal{I}_{ipr}$.

A set $A \subseteq \mathbb{N}$ is an *IP-set* if there exists an infinite set $D \subseteq \mathbb{N}$ such that $FS(D) \subseteq A$. Hindman Theorem implies that sets which are not IP-sets form an ideal, which we refer to as *Hindman ideal* and denote by $\mathcal{I}_H$.

Both $\mathcal{I}_{ipr}$ and $\mathcal{I}_H$ are tall ideals. Unlike summable ideals, neither $\mathcal{I}_{ipr}$ nor the Hindman ideal are P-ideals. The ideal $\mathcal{I}_{ipr}$ is an F_σ ideal, whereas the Hindman ideal is of higher Borel complexity.

A DIAGRAM FOR ULTRAFILTERS

Since we investigate not only the relations between the classes of $\mathcal{I}_g$-ultrafilters, $\mathcal{I}_{ipr}$-ultrafilters and $\mathcal{I}_H$-ultrafilters alone, but also their relations to the well known classes of selective ultrafilters and P-points let us recall the definition.

A free ultrafilter $\mathcal{U}$ is called a *P-point* if for all partitions of ω, $\{R_i : i \in \omega\}$, either for some i, $R_i \in \mathcal{U}$, or $(\exists U \in \mathcal{U})\ (\forall i \in \omega)\ |U \cap R_i| < \omega$.

A free ultrafilter $\mathcal{U}$ is called a *selective ultrafilter* if for all partitions of ω, $\{R_i : i \in \omega\}$, either for some i, $R_i \in \mathcal{U}$, or $(\exists U \in \mathcal{U})\ (\forall i \in \omega)\ |U \cap R_i| \leq 1$.

Provable inclusions

No inclusion relations between the considered classes of ultrafilters are known apart from the obvious ones: Every $\mathcal{I}_g$-ultrafilter is a weak $\mathcal{I}_g$-ultrafilter and every $\mathcal{I}_{ipr}$-ultrafilter is an $\mathcal{I}_H$-ultrafilter because $\mathcal{I}_{ipr} \subset \mathcal{I}_H$. However, some inclusion relations hold for selective ultrafilters and P-points.

Theorem. *Every selective ultrafilter is an $\mathcal{I}_{ipr}$-ultrafilter and an $\mathcal{I}_g$-ultrafilter for arbitrary summable ideal $\mathcal{I}_g$.*

It was proved in [2] that every P-point contains a set A such that $\lim_{n\to\infty} a_{n+1} - a_n = \infty$ for its increasing enumeration $A = \{a_n : n \in \omega\}$. Such a set is obviously not an IP-set.

Theorem. *Every P-point is an $\mathcal{I}_H$-ultrafilter.*

Consistent counterexamples

Theorem ([3], Theorem 3.1.)**.** (MA_{ctble}) *For arbitrary F_σ ideal $\mathcal{I}$ there exists a P-point which is not an $\mathcal{I}$-ultrafilter.*

Hence, it is consistent that there exists a P-point which is not an $\mathcal{I}_g$-ultrafilter for arbitrary tall summable ideal $\mathcal{I}_g$ and also a P-point which is not an $\mathcal{I}_{ipr}$-ultrafilter. The converse inclusions also consistently fail.

Theorem (follows from [2], Proposition 2.3.4.)**.** (MA_{ctble}) *There exists an $\mathcal{I}_{ipr}$-ultrafilter which is not a P-point.*

Theorem (follows from [3], Theorem 3.2.)**.** (MA_{ctble}) *For arbitrary tall summable ideal $\mathcal{I}_g$ there exists an $\mathcal{I}_g$-ultrafilter which is not a P-point.*

At the moment we cannot construct an $\mathcal{I}_g$-ultrafilter which is not an $\mathcal{I}_H$-ultrafilter, but we have weaker results in this direction.

Theorem. (MA_{ctble}) *For arbitrary tall summable ideal $\mathcal{I}_g$ there exists a weak $\mathcal{I}_g$-ultrafilter $\mathcal{U} \in \omega^*$ with $\mathcal{U} \cap \mathcal{I}_H = \emptyset$.*

Theorem. (MA_{ctble}) *For arbitrary tall summable ideal $\mathcal{I}_g$ there exists an $\mathcal{I}_g$-ultrafilter $\mathcal{U} \in \omega^*$ with $\mathcal{U} \cap \mathcal{I}_{ipr} = \emptyset$.*

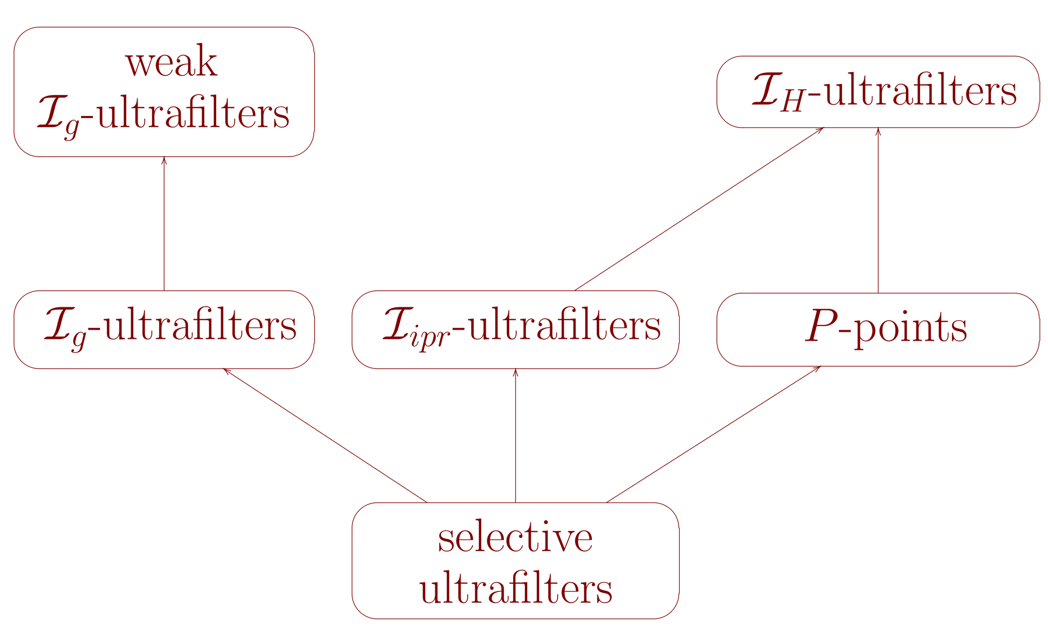
The construction proceeds by transfinite induction on $\alpha < \mathfrak{c}$. After enumerating ${}^\omega\omega = \{f_\alpha : \alpha < \mathfrak{c}\}$, filter bases $\mathcal{F}_\alpha$, $\alpha < \mathfrak{c}$ are constructed so that the following conditions are satisfied:

(i) $\mathcal{F}_0$ is the Fréchet filter
(ii) $\mathcal{F}_\alpha \subseteq \mathcal{F}_\beta$ whenever $\alpha \leq \beta$
(iii) $\mathcal{F}_\gamma = \bigcup_{\alpha<\gamma} \mathcal{F}_\alpha$ for γ limit
(iv) $(\forall\alpha)\ |\mathcal{F}_\alpha| \leq |\alpha| \cdot \omega$
(v) $(\forall\alpha)\ (\forall F \in \mathcal{F}_\alpha)\ F$ is ip-rich
(vi) $(\forall\alpha)\ (\exists F \in \mathcal{F}_{\alpha+1})\ f_\alpha[F] \in \mathcal{I}_g$

Conjecture. (MA_{ctble}) *For arbitrary summable ideal $\mathcal{I}_g$ there is an $\mathcal{I}_{ipr}$-ultrafilter which is not an $\mathcal{I}_g$-ultrafilter.*

Overview

The diagram summarizes known inclusions between $\mathcal{I}_g$-ultrafilters, $\mathcal{I}_{ipr}$-ultrafilters and $\mathcal{I}_H$-ultrafilters.



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[2] Flašková J. (2006): Ultrafilters and small sets, Ph.D. Thesis, Charles University, Prague.
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