On sums and products of certain $\mathcal{I}$-ultrafilters

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$\mathcal{I}$-ultrafilters

Definition A. (Baumgartner)

Let $\mathcal{I}$ be a family of subsets of a set X such that $\mathcal{I}$ contains all singletons and is closed under subsets. An ultrafilter $\mathcal{U}$ on ω is called an $\mathcal{I}$-ultrafilter if for every $F : \omega \longrightarrow X$ there exists $A \in \mathcal{U}$ such that $F[A] \in \mathcal{I}$.

- if $\mathcal{I} \subseteq \mathcal{J}$ then every $\mathcal{I}$-ultrafilter is a $\mathcal{J}$-ultrafilter
- $\mathcal{I}$-ultrafilters and $\langle \mathcal{I} \rangle$-ultrafilters coincide

  where $\langle \mathcal{I} \rangle$ is the ideal generated by $\mathcal{I}$

$\mathcal{I}$-ultrafilters for $X = 2^\omega$

family $\mathcal{I}$	corresponding $\mathcal{I}$-ultrafilters
converging sequences and finite sets	P-points
discrete sets	discrete ultrafilters
scattered sets	scattered ultrafilters
$\{A : \mu(\bar{A}) = 0\}$	measure zero ultrafilters
nowhere dense sets	nowhere dense ultrafilters

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Theorem (Shelah)
There may be no nowhere dense ultrafilters.

$\mathcal{I}$-ultrafilters for $X = \omega$

Theorem 1.

If $\mathcal{I}$ is a maximal ideal and $\chi(\mathcal{I}) = \mathfrak{c}$ then $\mathcal{I}$-ultrafilters exist in ZFC.

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$\mathcal{I} \subseteq \mathcal{P}(\omega)$ is a dense ideal if for every $A \in [\omega]^\omega$ there exists an infinite set $B \subseteq A$ such that $B \in \mathcal{I}$.

Proposition 2.

If $\mathcal{I}$ is not a dense ideal then there are no $\mathcal{I}$-ultrafilters.

$\mathcal{I}$-ultrafilters for $X = \omega$

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If $\mathcal{I}$ is a (dense) F_σ-ideal or analytic P-ideal then every selective ultrafilter is an $\mathcal{I}$-ultrafilter.

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If $\mathcal{I}$ is a (dense) F_σ-ideal or analytic P-ideal then every selective ultrafilter is an $\mathcal{I}$-ultrafilter.

Theorem 5.

Let $\mathcal{I}$ be a P-ideal. If there is an $\mathcal{I}$-ultrafilter then there is an $\mathcal{I}$-ultrafilter that is not a P-point.

Small subsets of ω or $(\mathbb{N})$

Let A be a subset of ω with an increasing enumeration $A = \{a_n : n \in \omega\}$. We say that A is

thin if $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 0$

almost thin if $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} < 1$

(SC)-set if $\lim_{n\to\infty} a_{n+1} - a_n = \infty$

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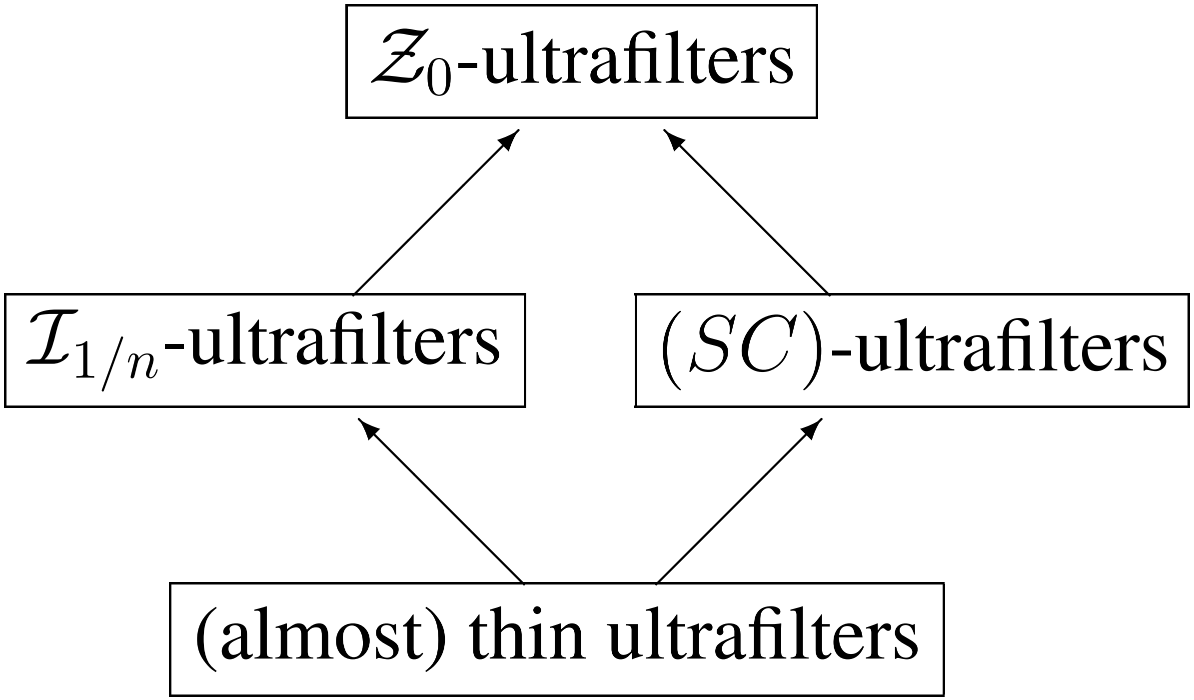
$\mathcal{I}_{1/n} = \{A \subseteq \mathbb{N} : \sum_{a\in A} \frac{1}{a} < \infty\}$

$\mathcal{Z}_0 = \{A \subseteq \mathbb{N} : \limsup_{n\to\infty} \frac{|A\cap n|}{n} = 0\}$

Connections



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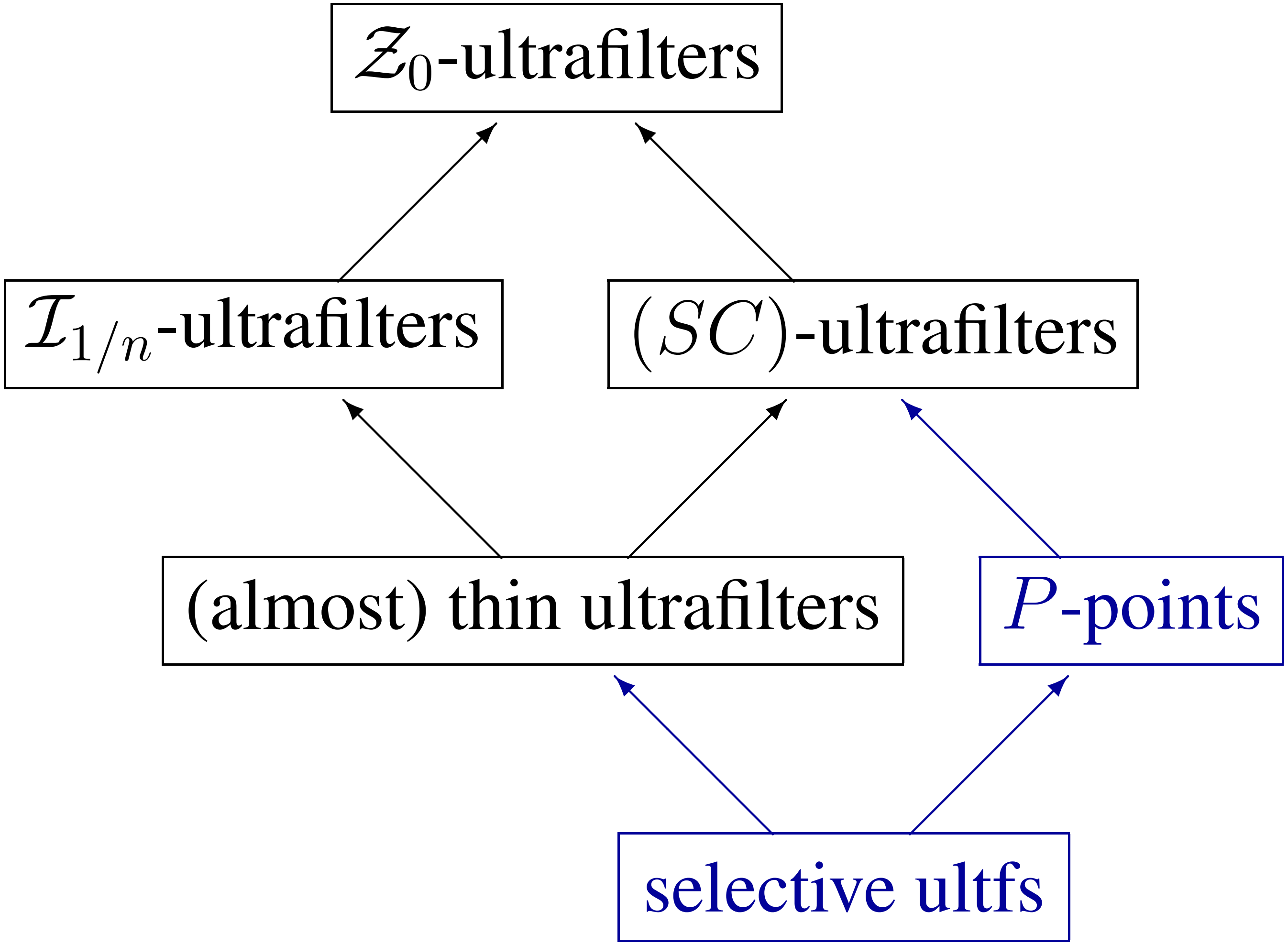

assuming $\mathrm{MA}_{\mathrm{ctble}}$
no arrow can be added

Ultrafilter sums and products

Definition B.

Let $\mathcal{U}$ and $\mathcal{V}_n$, $n \in \omega$, be ultrafilters on ω.
$\mathcal{U}$-sum of ultrafilters $\mathcal{V}_n$, $\sum_{\mathcal{U}}\langle \mathcal{V}_n : n \in \omega\rangle$,
is an ultrafilter on $\omega \times \omega$ defined by
$M \in \sum_{\mathcal{U}}\langle \mathcal{V}_n : n \in \omega\rangle$ if and only if
$\{n : \{m : \langle n, m\rangle \in A\} \in \mathcal{V}_n\} \in \mathcal{U}$.

If $\mathcal{V}_n = \mathcal{V}$ for every $n \in \omega$ then we write
$\sum_{\mathcal{U}}\langle \mathcal{V}_n : n \in \omega\rangle = \mathcal{U} \cdot \mathcal{V}$ and the ultrafilter $\mathcal{U} \cdot \mathcal{V}$
is called the product of ultrafilters $\mathcal{U}$ and $\mathcal{V}$.

Ultrafilter sums and products

Definition C. (Baumgartner)

Let $\mathcal{C}$ and $\mathcal{D}$ be two classes of ultrafilters.
We say that $\mathcal{C}$ is closed under $\mathcal{D}$-sums provided that whenever $\{\mathcal{V}_n : n \in \omega\} \subseteq \mathcal{C}$ and $\mathcal{U} \in \mathcal{D}$ then $\sum_{\mathcal{U}} \langle \mathcal{V}_n : n \in \omega \rangle \in \mathcal{C}$.

- If $\mathcal{D}$ is a class of $\mathcal{I}$-ultrafilters then we say that $\mathcal{C}$ is closed under $\mathcal{I}$-sums.

Some results

Theorem 6.

Let $\mathcal{I}$ be a P-ideal on ω (or $\mathbb{N}$).
If $\mathcal{U}$ is an $\mathcal{I}$-ultrafilter and $\{n : \mathcal{V}_n \text{ is } \mathcal{I}\text{-ultrafilter}\} \in \mathcal{U}$ then $\sum_{\mathcal{U}}\langle \mathcal{V}_n : n \in \omega\rangle$ is an $\mathcal{I}$-ultrafilter.

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Corollary 7.

$\mathcal{I}$-ultrafilters are closed under $\mathcal{I}$-sums if $\mathcal{I}$ is a P-ideal.

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$\mathcal{I}$-ultrafilters are closed under $\mathcal{I}$-sums if $\mathcal{I}$ is a P-ideal.

Proposition 8.

For arbitrary $\mathcal{U} \in \omega^*$ the ultrafilter $\mathcal{U} \cdot \mathcal{U}$ is not an (SC)-ultrafilter.

Weak $\mathcal{I}$-ultrafilters

Definition A. (Baumgartner)

Let $\mathcal{I}$ be a family of subsets of a set X such that $\mathcal{I}$ contains all singletons and is closed under subsets. An ultrafilter $\mathcal{U}$ on ω is called an $\mathcal{I}$-ultrafilter if for every $F : \omega \to X$ there exists $A \in \mathcal{U}$ such that $F[A] \in \mathcal{I}$.

Weak $\mathcal{I}$-ultrafilters

Definition D.

Let $\mathcal{I}$ be a family of subsets of a set X such that $\mathcal{I}$ contains all singletons and is closed under subsets.
An ultrafilter $\mathcal{U}$ on ω is called an weak $\mathcal{I}$-ultrafilter if for every finite-to-one $F : \omega \to X$ there exists $A \in \mathcal{U}$ such that $F[A] \in \mathcal{I}$.

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Let $\mathcal{I}$ be a family of subsets of a set X such that $\mathcal{I}$ contains all singletons and is closed under subsets.
An ultrafilter $\mathcal{U}$ on ω is called an weak $\mathcal{I}$-ultrafilter if for every finite-to-one $F : \omega \to X$ there exists $A \in \mathcal{U}$ such that $F[A] \in \mathcal{I}$.

An ultrafilter $\mathcal{U}$ on ω is called an $\mathcal{I}$-close ultrafilter if for every one-to-one $F : \omega \to X$ there exists $A \in \mathcal{U}$ such that $F[A] \in \mathcal{I}$.

More results

Proposition 8^*.

For arbitrary $\mathcal{U} \in \omega^*$ the ultrafilter $\mathcal{U} \cdot \mathcal{U}$ is not an (SC)-close ultrafilter.

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Theorem 6.

Let $\mathcal{I}$ be a P-ideal on ω (or $\mathbb{N}$).

- If $\{n : \mathcal{V}_n \text{ is } \mathcal{I}\text{-ultrafilter}\} \in \mathcal{U}$ and $\mathcal{U}$ is an $\mathcal{I}$-ultrafilter then $\sum_{\mathcal{U}} \langle \mathcal{V}_n : n \in \omega \rangle$ is an $\mathcal{I}$-ultrafilter.

More results

Proposition 8*.

For arbitrary $\mathcal{U} \in \omega^*$ the ultrafilter $\mathcal{U} \cdot \mathcal{U}$ is not an (SC)-close ultrafilter.

Theorem 9.

Let $\mathcal{I}$ be a P-ideal on ω (or $\mathbb{N}$).

- If $\{n : \mathcal{V}_n$ is weak $\mathcal{I}$-ultrafilter$\} \in \mathcal{U}$ then $\sum_{\mathcal{U}} \langle \mathcal{V}_n : n \in \omega \rangle$ is a weak $\mathcal{I}$-ultrafilter.
- If $\{n : \mathcal{V}_n$ is $\mathcal{I}$-close ultrafilter$\} \in \mathcal{U}$ then $\sum_{\mathcal{U}} \langle \mathcal{V}_n : n \in \omega \rangle$ is an $\mathcal{I}$-close ultrafilter.

More results

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Let $\mathcal{I}$ be a P-ideal on ω (or $\mathbb{N}$).
If $\{n : \mathcal{V}_n \text{ is } \mathcal{I}\text{-ultrafilter}\} \in \mathcal{U}$
and $\mathcal{U}$ is an $\mathcal{I}$-ultrafilter
then $\sum_{\mathcal{U}}\langle \mathcal{V}_n : n \in \omega\rangle$ is an $\mathcal{I}$-ultrafilter.

More results

Theorem 10.

Assume $\mathcal{I}$ is an ideal on $\mathbb{N}$.
If $\sum_{\mathcal{U}}\langle \mathcal{V}_n : n \in \omega\rangle$ is an $\mathcal{I}$-ultrafilter
then $\{n : \mathcal{V}_n$ is an $\mathcal{I}$-ultrafilter$\} \in \mathcal{U}$
and $\mathcal{U}$ is an $\mathcal{I}$-ultrafilter.

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and $\mathcal{U}$ is an $\mathcal{I}$-ultrafilter.

More results

Theorem 11.

Assume $\mathcal{I}$ is an ideal on $\mathbb{N}$ and there exists $g : \mathbb{N} \to \mathbb{N}$ with $g(n) > n$ for every $n \in \mathbb{N}$ and $A \notin \mathcal{I}$ implies $g[A] \notin \mathcal{I}$ for every $A \subseteq \mathbb{N}$.
If $\sum_{\mathcal{U}} \langle \mathcal{V}_n : n \in \omega \rangle$ is a weak $\mathcal{I}$-ultrafilter then $\{n : \mathcal{V}_n$ is a weak $\mathcal{I}$-ultrafilter$\} \in \mathcal{U}$.

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Theorem 12.

Assume $\mathcal{I}$ is an ideal on $\mathbb{N}$ and $A \notin \mathcal{I}$ implies $A + 1 \notin \mathcal{I}$ and $2A \notin \mathcal{I}$ for each $A \subseteq \mathbb{N}$.
If $\sum_{\mathcal{U}} \langle \mathcal{V}_n : n \in \omega \rangle$ is an $\mathcal{I}$-close ultrafilter then $\{n : \mathcal{V}_n$ is an $\mathcal{I}$-close ultrafilter$\} \in \mathcal{U}$.

A "problematic" ideal

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- If $\sum_{\mathcal{U}} \langle \mathcal{V}_n : n \in \omega \rangle$ is an $\mathcal{W}$-close ultrafilter then $\{n : \mathcal{V}_n$ is an $\mathcal{W}$-close ultrafilter$\} \in \mathcal{U}$.

Question:
Are $\mathcal{W}$-ultrafilters closed under $\mathcal{W}$-sums or products?

References

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Flašková, J., Ultrafilters and small sets, Ph.D. Thesis, Charles University, Prague, 2006.