Some cardinal invariants related to analytic quotients

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Fact. $\aleph_1 \leq \mathfrak{p} \leq \mathfrak{a} \leq \mathfrak{c}$

Cardinal invariants of analytic quotients

All ideals are tall, analytic and contain Fin.

Given an ideal $\mathcal{I}$ on ω we write $B \subseteq_{\mathcal{I}} A$ if $B \setminus A \in \mathcal{I}$.
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A family $\mathcal{F} \subseteq \mathcal{I}^+$ is ($\mathcal{I}^+$-)centered if $\bigcap_{i=1}^{k} F_i \in \mathcal{I}^+$ for every $k \in \omega$, $F_i \in \mathcal{F}$, $i = 1, \ldots, k$.

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$\mathfrak{p}(\mathcal{I})$ is the minimal cardinality of an $\mathcal{I}^+$-centered family $\mathcal{F} \subseteq \mathcal{I}^+$ with no pseudointersection in $\mathcal{I}^+$ i.e.
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In general, $\mathfrak{p}(\mathcal{I})$ and $\mathfrak{a}(\mathcal{I})$ need not be uncountable

Some known results

Theorem (Farkas, Soukup).

$\mathfrak{a}(\mathcal{Z}_\mu) = \aleph_0$ for any tall density zero ideal $\mathcal{Z}_\mu$.

$\mathfrak{a}(\mathcal{I}_h) \geq \aleph_1$ for any tall summable ideal $\mathcal{I}_h$.

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Definition (Farkas, Soukup).

Let $\bar{\mathfrak{a}}(\mathcal{I})$ be the cardinality of an uncountable $\mathcal{I}$-MAD family.

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Let $\bar{\mathfrak{a}}(\mathcal{I})$ be the cardinality of an uncountable $\mathcal{I}$-MAD family.

Theorem (Farkas, Soukup).

$\bar{\mathfrak{a}}(\mathcal{I}) \geq \mathfrak{b}$ for any tall analytic P-ideal $\mathcal{I}$.

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Combining results of Farkas, Soukup and Brendle we obtain for F_σ P-ideals:

$$\mathfrak{p} \leq \mathfrak{p}(\mathcal{I}) \leq \mathfrak{b} \leq \mathfrak{a}(\mathcal{I})$$

AP-sets and van der Waerden theorem

Definition.

A set $A \subseteq \mathbb{N}$ is called an AP-set if it contains arbitrary long arithmetic progressions.

Van der Waerden Theorem.

If an AP-set is partitioned into finitely many pieces then at least one of them is again an AP-set.

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Sets which are not AP-sets form a proper ideal on $\mathbb{N}$
— van der Waerden ideal denoted by $\mathcal{W}$

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$$\mathcal{W}_n = \{A \subseteq \mathbb{N} : A \text{ contains no a. p. of length } n\}$$
- not a P-ideal — consider for example the sets
$$A_k = \{2^n + k : n \in \omega\} \text{ for } k \in \omega$$

Van der Waerden ideal $\mathcal{W}$

Szemerédi Theorem.

$$\mathcal{W} \subseteq \mathcal{Z} \quad \text{where} \quad \mathcal{Z} = \{A \subseteq \mathbb{N} : \limsup_{n\to\infty} \frac{|A \cap n|}{n} = 0\}$$

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Erdős Conjecture.

$$\mathcal{W} \subseteq \mathcal{I}_{1/n} \quad \text{where } \mathcal{I}_{1/n} = \{A \subseteq \mathbb{N} : \sum_{a\in A} \frac{1}{a} < \infty\}$$

Pseudointersection number of $\mathcal{P}(\omega)/\mathcal{W}$

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Corollary 2. $\mathfrak{p}(\mathcal{W}) = \mathfrak{p}$

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Define $f : [\omega]^\omega \to \mathcal{W}^+$ by

$$f(A) = \bigcup_{n \in A} [2^n, 2^{n+1})$$

More about the almost disjointness number

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We have seen $\bar{\mathfrak{a}}(\mathcal{Z}) \leq \mathfrak{a}$ and $\mathfrak{a}(\mathcal{W}) \leq \mathfrak{a}$ hold.

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Theorem (Farkas, Soukup).

$\mathfrak{a}(\mathcal{I}) \geq \mathfrak{b}$ for any tall F_σ P-ideal.

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We have seen $\bar{\mathfrak{a}}(\mathcal{Z}) \leq \mathfrak{a}$ and $\mathfrak{a}(\mathcal{W}) \leq \mathfrak{a}$ hold.

Question B. Is $\mathfrak{a}(\mathcal{I}_{1/n}) \leq \mathfrak{a}$?

More about the pseudointersection number

We observed that $\mathfrak{p}(\mathcal{W}) \leq \mathfrak{p}$ and thus $\mathfrak{p}(\mathcal{W}) = \mathfrak{p}$ hold.

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Question C. Is $\mathfrak{p}(\mathcal{I}_{1/n}) = \mathfrak{p}$?

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Question C. Is $\mathfrak{p}(\mathcal{I}_{1/n}) = \mathfrak{p}$?

Question D.

Is there an F_σ ideal $\mathcal{I}$ such that $\mathfrak{p}(\mathcal{I}) > \mathfrak{p}$ is consistent?

References

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