More than a 0-point

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Abstract

We construct in ZFC an ultrafilter $\mathscr{U} \in \mathbb{N}^*$ such that for every one-to-one function $f : \mathbb{N} \to \mathbb{N}$ there exists $U \in \mathscr{U}$ with $f[U]$ in summable ideal, i.e. the sum of reciprocals of its elements converges. This strengthens Gryzlov's result concerning the existence of 0-points.

1 Introduction

In his talk during the 12th Winter School on Abstract Analysis in Srní, A. Gryzlov defined 0-points and he constructed such ultrafilters in ZFC (see [2], [3]). Let us recall that an ultrafilter $\mathscr{U} \in \mathbb{N}^*$ is called a 0-point if for every one-to-one function $f : \mathbb{N} \to \mathbb{N}$ there exists a set $U \in \mathscr{U}$ such that $f[U]$ has asymptotic density zero.

We strengthen Gryzlov's result and construct a summable ultrafilter which we define as an ultrafilter $\mathscr{U} \in \mathbb{N}^*$ such that for every one-to-one function $f : \mathbb{N} \to \mathbb{N}$ there exists $U \in \mathscr{U}$ with $f[U]$ in the summable ideal. Our proof was motivated by Gryzlov's original construction as it was written down by K. P. Hart [4].

The summable ideal is the family $\{A \subseteq \mathbb{N} : \sum_{a \in A} \frac{1}{a} < +\infty\}$. It is not difficult to prove that every set in the summable ideal has asymptotic density zero, but the converse is not true (consider, e.g., the set of all prime numbers). It is also known that the summable ideal is a P-ideal, i.e., whenever A_n, $n \in \mathbb{N}$, are sets from the ideal there exists A in the summable ideal that contains all but finitely many elements of each A_n (we use the notation $A_n \subseteq^* A$ for this).

We call a family $\mathscr{F} \subseteq \mathscr{P}(\mathbb{N})$ summable if for every one-to-one function $f : \mathbb{N} \to \mathbb{N}$ there is $A \in \mathscr{F}$ such that $f[A]$ belongs to the summable ideal.

Let us recall that a family $\mathscr{F} \subseteq \mathscr{P}(\mathbb{N})$ is called k-linked if $F_0 \cap F_1 \cap \cdots \cap F_k$ is infinite whenever $F_i \in \mathscr{F}$, $i \le k$, and it is called centered if any finite subfamily of $\mathscr{F}$ has an infinite intersection, i.e., it is k-linked for every $k \in \mathbb{N}$.

During the construction we make use of the following upper bound for partial sums of the harmonic series:

Fact 1.1. $1 + \frac{1}{2} + \cdots + \frac{1}{N} \le 1 + \ln N \le 1 + \log_2 N$ *for every* $N \in \mathbb{N}$.

2 Construction

Lemma 2.1 is fairly general, but it will enable us to construct a summable centered system by applying Proposition 2.2 to get summable k-linked families for every k. The summable centered system may then be extended to a summable ultrafilter.

Lemma 2.1. *If $\mathscr{F}_k$ is a k-linked family of infinite subsets of $\mathbb{N}$ for every $k \in \mathbb{N}$ then $\mathscr{F} = \{F \subseteq \mathbb{N} : (\forall k)(\exists U_k \in \mathscr{F}_k)\, U_k \subseteq^* F\}$ is a centered system.*

If moreover, $\mathscr{I}$ is a P-ideal, $f \in {}^{\mathbb{N}}\mathbb{N}$ a one-to-one function and for every $k \in \mathbb{N}$ there exists $U_k \in \mathscr{F}_k$ such that $f[U_k] \in \mathscr{I}$ then there exists $U \in \mathscr{F}$ such that $f[U] \in \mathscr{I}$. In particular, if $\mathscr{F}_k$ is summable for every k then $\mathscr{F}$ is summable.

Proof. Take $F^1, F^2, \dots, F^n \in \mathscr{F}$ and for every $j = 1, \dots, n$ choose $U_k^j \in \mathscr{F}_k$ such that $U_k^j \subseteq^* F^j$ for every k. For every $k \geq n$ family $\mathscr{F}_k$ is n-linked, hence $\bigcap_{j=1}^n U_k^j$ is an infinite set. We have

$$\bigcap_{j=1}^{n} U_k^j \subseteq^* \bigcap_{j=1}^{n} F^j$$

for every $k \geq n$ and it follows that family $\mathscr{F}$ is centered.

For the moreover part, consider $A \in \mathscr{I}$ such that $f[U_k] \subseteq^* A$ for every $k \in \mathbb{N}$. We get $U_k \subseteq^* f^{-1}[A]$ for every $k \in \mathbb{N}$. According to the definition set $U = f^{-1}[A]$ belongs to $\mathscr{F}$ and $f[U] = A \in \mathscr{I}$. $\square$

In the proof of the next proposition we treat the natural numbers as both numbers and sets. In order to help the reader we use $\prod$ to denote a product of sets and $\bigodot$ to denote a product of numbers.

Proposition 2.2. *Let A be an infinite subset of $\mathbb{N}$. For every $k \in \mathbb{N}$ there exists a summable k-linked family $\mathscr{F}_k \subseteq \mathscr{P}(A)$.*

Proof. Fix $k \in \mathbb{N}$. We divide A into disjoint finite blocks, $A = \bigcup_{n \in \mathbb{N}} B_n$, and for every n enumerate B_n, faithfully, as $\{b(\varphi) : \varphi \in \prod_{j=0}^k Q(j,n)\}$ where $Q(j,n)$ is defined by $Q(j,n) = 2^{n \cdot 2^j}$. Notice that for every $i \leq k$ we have $Q(i,n) = 2^n \cdot \bigodot_{j=0}^{i-1} Q(j,n)$.

For every $i \leq k$, $x \in Q(i,n)$ and $s \in \prod_{j=i+1}^k Q(j,n)$ define $B_n(i,x,s) = \{b(\varphi^\frown\langle x\rangle^\frown s) : \varphi \in \prod_{j=0}^{i-1} Q(j,n)\}$. For every one-to-one function $f : \mathbb{N} \to \mathbb{N}$ let $m_x^f = \min f[B_n(i,x,s)]$. Finally, let $x(f,s) \in Q(i,n)$ be that x for which m_x^f is maximal, i.e., $m_{x(f,s)}^f = \max\{m_x^f : x \in Q(i,n)\}$. Now, we may define $A^f \subseteq A$ block by block as the union $A^f = \bigcup_{n\in\mathbb{N}} B^f_n$, where $B^f_n \subseteq B_n$ is defined in two stages: first $B^f_n = \bigcup_{i=0}^k B^f_n(i)$ and second $B^f_n(i) = \bigcup\{B^f_n(i,s) : s \in \prod_{j=i+1}^k Q(j,n)\}$, where $B^f_n(i,s) = B_n(i, x(f,s), s)$.

Claim 1. The family $\mathscr{F}_k = \{A^f : f \in {}^{\mathbb{N}}\mathbb{N} \text{ one-to-one}\}$ is k-linked.

Consider $f_0, f_1, \dots, f_k$ distinct one-to-one functions from $\mathbb{N}$ to $\mathbb{N}$. Since

$$\bigcap_{j=0}^k A^{f_j} \supseteq \bigcup_{n=1}^\infty \bigcap_{j=0}^k B^{f_j}_n$$

it suffices to show that $\bigcap_{j=0}^k B^{f_j}_n \neq \emptyset$ for every $n \in \mathbb{N}$. To see this fix n and define $\varphi \in \prod_{j=0}^k Q(j,n)$ recursively: put $s_0 = \emptyset$ and set $\varphi(k) = x(f_0, s_0)$, next $s_1 = \langle \varphi(k) \rangle$ and $\varphi(k-1) = x(f_1, s_1)$, and so on. It follows that $b(\varphi) \in \bigcap_{j=0}^k B^{f_j}_n(k-j, s_j) \subseteq \bigcap_{j=0}^k B^{f_j}_n(k-j) \subseteq \bigcap_{j=0}^k B^{f_j}_n$.

Claim 2. The set $f[A^f]$ belongs to the summable ideal for every one-to-one function f.

Our aim is to bound the sum $\sum_{a\in B^f_n} \frac{1}{f(a)}$ from above by elements of a convergent series because $f[A^f] = \bigcup_{n\in\mathbb{N}} f[B^f_n]$. At first, we estimate the sum of the reciprocals of elements in $f[B^f_n(i,s)]$ for every $i \le k$ and $s \in \prod_{j=i+1}^k Q(j,n)$.

Since $|f[B^f_n(i,s)]| = \bigodot_{j=0}^{i-1} Q(j,n)$ we have

$$\sum_{a\in B^f_n(i,s)} \frac{1}{f(a)} \le \bigodot_{j=0}^{i-1} Q(j,n) \cdot \frac{1}{\min f[B^f_n(i,s)]} = \frac{2^{n\cdot(2^i-1)}}{m^f_{x(f,s)}} \tag{1}$$

Put $q_{i,n} = \bigodot_{j=i+1}^k Q(j,n)$ and enumerate $\{m^f_{x(f,s)} : s \in \prod_{j=i+1}^k Q(j,n)\}$ increasingly as $\{m_l : l = 1, \dots, q_{i,n}\}$. It is easy to see that $m_l \ge l \cdot Q(i,n)$ for every l and it follows that

$$\sum_{l=1}^{q_{i,n}} \frac{1}{m_l} \le \frac{1}{Q(i,n)} \cdot \sum_{l=1}^{q_{i,n}} \frac{1}{l} \le \frac{1 + \log_2 q_{i,n}}{Q(i,n)} = \frac{1 + \sum_{j=i+1}^k \log_2 Q(j,n)}{Q(i,n)} \tag{2}$$

where we used Fact 1.1.

Now, observe that

$$1 + \sum_{j=i+1}^k \log_2 Q(j,n) \le 1 + n\sum_{j=0}^k 2^j = 1 + n(2^{k+1}-1) \le n2^{k+1} \tag{3}$$

and putting together (1), (2) and (3) we obtain

$$\sum_{a\in B_n^f(i)} \frac{1}{f(a)} \leq \bigodot_{j=0}^{i-1} Q(j,n) \cdot \frac{1+\sum_{j=i+1}^{k} \log_2 Q(j,n)}{Q(i,n)} = \frac{n2^{k+1}}{2^n}. \tag{4}$$

Thus we get for every n

$$\sum_{a\in B_n^f} \frac{1}{f(a)} \leq \sum_{i=0}^{k} \frac{n2^{k+1}}{2^n} = \frac{n(k+1)2^{k+1}}{2^n} \tag{5}$$

and finally

$$\sum_{a\in A^f} \frac{1}{f(a)} \leq \sum_{n=1}^{\infty} \frac{n(k+1)2^{k+1}}{2^n} \leq 2(k+1)2^{k+1}, \tag{6}$$

i.e., the set $f[A^f]$ belongs to the summable ideal. □

While constructing a 0-point Gryzlov made use of function $Q(j,n) = n^{2^j}$. We cannot use this function for our purpose because it “grows too slowly”. Its polynomial growth with respect to n provides in formula (4) (or (5)) a divergent series as an upper bound for $\sum_{a\in B_n^f} \frac{1}{f(a)}$. So it seems to be necessary that $Q(j,n)$ depends exponentially on n. In formula (4) occurs $\bigodot_{j=0}^{i-1} Q(j,n) \cdot Q(i,n)^{-1}$, which excludes functions of type $2^n \cdot p(j)$ or $2^{n\cdot p(j)}$ where $p(j)$ is a polynomial in j. Hence our definition $Q(j,n) = 2^{n\cdot 2^j}$ seems to be the best possible to use while constructing a summable ultrafilter.

Theorem 2.3. *There is a summable ultrafilter on* $\mathbb{N}$.

Proof. Consider an arbitrary countable family $\{A_k : k \in \mathbb{N}\}$ of infinite subsets of natural numbers and apply Proposition 2.2 to obtain a summable k-linked family $\mathscr{F}_k$ on A_k for every k. From Lemma 2.1 we obtain a summable centered system $\mathscr{F}$ on $\mathbb{N}$. It is obvious that any ultrafilter that extends $\mathscr{F}$ is summable. □

Corollary 2.4. *There are* 2^{2^ω} *distinct summable ultrafilters on* $\mathbb{N}$.

Proof. Assume $\{A_k : k \in \mathbb{N}\}$ is a countable family of disjoint infinite subsets of $\mathbb{N}$ and $\mathscr{F}_k$ is a summable k-linked family on A_k for every k. For every free ultrafilter $\mathscr{U}$ on $\mathbb{N}$ let $\mathscr{F}_\mathscr{U} \subseteq \mathscr{P}(\mathbb{N})$ consist of sets F such that $\{k : F \cap A_k \in \mathscr{F}_k\} \in \mathscr{U}$. It is easy to see that $\mathscr{F}_\mathscr{U}$ is a summable filter and $\mathscr{F}_\mathscr{U} \neq \mathscr{F}_\mathscr{V}$ whenever $\mathscr{U} \neq \mathscr{V}$. It follows that there are 2^{2^ω} distinct summable ultrafilters. □

3 Open questions

The construction relies strongly on the fact that functions in question are one-to-one. It is a limiting assumption, but it is not known at the moment whether it is possible to construct in ZFC a summable ultrafilter if we enlarge the family of functions considered in the definition of a summable ultrafilter to all finite-to-one functions, or even more, to all functions from $\mathbb{N}$ to $\mathbb{N}$ (examples constructed under Martin's Axiom for countable posets can be found in [1]).

Another interesting question arises if we replace the summable ideal in the definition of a summable ultrafilter by a generalized summable ideal that is defined for any (decreasing) function $g : \mathbb{N} \to [0, \infty)$ with $\lim_{n\to\infty} g(n) = 0$ by $\mathscr{I}_g = \{A \subseteq \mathbb{N} : \sum_{a\in A} g(a) < \infty\}$ where we assume $\sum_{n\in\mathbb{N}} g(n) = \infty$ to obtain a proper ideal. It is easy to see that the ideal $\mathscr{I}_g$ is a P-ideal that extends the ideal of all finite sets. Again, a straightforward modification of construction in [1] provides examples of such ultrafilters under Martin's Axiom for countable posets, but there are no examples in ZFC at the moment.

Acknowledgements

The author wishes to thank K. P. Hart for sharing his notes [4] and the referee for providing a better argument in the proof of Corollary 2.4 and for many helpful suggestions about how to make the article more readable.

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