Hindman spaces and summable ultrafilters

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Van der Waerden spaces

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Definition A. (Kojman)

A topological space X is called van der Waerden if for every sequence $\langle x_n \rangle_{n\in\omega}$ in X there exists a converging subsequence $\langle x_{n_k} \rangle_{k\in\omega}$ so that $\{n_k : k \in \omega\}$ is an AP-set.

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!!! only finite T_2 spaces fullfill the condition!!!

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Definition B. (Kojman)

A topological space X is called Hindman if for every sequence $\langle x_n \rangle_{n \in \omega}$ in X there exists an infinite set $D \subseteq \mathbb{N}$ such that $\langle x_n \rangle_{n \in FS(D)}$ IP-converges to some $x \in X$.

Known facts

Theorem (Kojman)

- There exists a sequentially compact space which is not van der Waerden.

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Proof. Consider the one-point compactification of $\Psi(\mathcal{A})$ for a suitable MAD family $\mathcal{A}$.

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Note: $\Psi(\mathcal{A})$ is regular, first countable and separable.

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For example, compact metric spaces or every succesor ordinal with the order topology satisfy ($*$).

Known facts + questions

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Theorem (Jones)
($\mathrm{MA}_{\sigma-\mathrm{cent.}}$) There exists a van der Waerden space which is not Hindman.

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Is it possible to strentghen the result of Jones?

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Definition C.

A topological space X is called $\mathcal{I}_{1/n}$-space if for every sequence $\langle x_n \rangle_{n\in\omega}$ in X there exists a converging subsequence $\langle x_{n_k} \rangle_{k\in\omega}$ so that $\{n_k : k \in \omega\}$ does not belong to $\mathcal{I}_{1/n}$.

$\mathcal{I}_{1/n}$-spaces

Theorem 1.

If a Hausdorff space X satisfies the following condition

(∗) The closure of every countable set in X is compact and first-countable.

Then X is an $\mathcal{I}_{1/n}$-space.

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Theorem 1.

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Theorem 2.

There exists a sequentially compact space which is not an $\mathcal{I}_{1/n}$-space.

$\mathcal{I}_{1/n}$ & van der Waerden spaces

Erdős-Turán Conjecture.

Every set $A \notin \mathcal{I}_{1/n}$ is an AP-set.

If Erdős-Turán Conjecture is true then every $\mathcal{I}_{1/n}$-space is van der Waerden.

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If Erdős-Turán Conjecture is true then every $\mathcal{I}_{1/n}$-space is van der Waerden.

Theorem 3.

$(\mathrm{MA}_{\sigma-\mathrm{cent.}})$ There exists a van der Waerden space which is not an $\mathcal{I}_{1/n}$-space.

$\mathcal{I}_{1/n}$ & Hindman spaces

There is no inclusion between $\mathcal{I}_{1/n}$ and Hindman ideal.

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Theorem 4.

($MA_{\sigma-\mathrm{cent.}}$) There exists an $\mathcal{I}_{1/n}$-space which is not Hindman.

$\mathcal{I}_{1/n}$ & Hindman spaces

There is no inclusion between $\mathcal{I}_{1/n}$ and Hindman ideal.

Theorem 4.
($\mathrm{MA}_{\sigma-\mathrm{cent.}}$) There exists an $\mathcal{I}_{1/n}$-space which is not Hindman.

Proposition
($\mathrm{MA}_{\sigma-\mathrm{cent.}}$) There exists a MAD family $\mathcal{A}$ consisting of non-IP-sets so that for every $B \subseteq \mathbb{N}$, $B \notin \mathcal{I}_{1/n}$ and every finite-to-one function $f : B \to \mathbb{N}$ there exists $C \subseteq B$, $C \notin \mathcal{I}_{1/n}$ and $A \in \mathcal{A}$ so that $f[C] \subseteq A$.

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$A \subseteq \mathbb{N}$ is an ip-rich set if A contains all finite sums of elements of arbitrarily large finite sets.

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- (Folkman-Rado-Sanders)
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- (Folkman-Rado-Sanders)
  Sets that are not ip-rich form an ideal
- Ideal $\mathcal{I}_{ipr}$ is an F_σ-ideal

$\mathcal{I}_{\mathbf{1/n}}$ & $\mathcal{I}_{ipr}$-spaces

Definition D.

A topological space X is called $\mathcal{I}_{ipr}$-space if for every sequence $\langle x_n \rangle_{n\in\omega}$ in X there exists a converging subsequence $\langle x_{n_k} \rangle_{k\in\omega}$ so that $\{n_k : k \in \omega\}$ is an ip-rich set.

$\mathcal{I}_{\mathbf{1/n}}$ & $\mathcal{I}_{ipr}$-spaces

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A topological space X is called $\mathcal{I}_{ipr}$-space if for every sequence $\langle x_n \rangle_{n\in\omega}$ in X there exists a converging subsequence $\langle x_{n_k} \rangle_{k\in\omega}$ so that $\{n_k : k \in \omega\}$ is an ip-rich set.

Theorem 7.

($\mathrm{MA}_{\sigma-\mathrm{cent.}}$) There exists an $\mathcal{I}_{ipr}$-space which is not an $\mathcal{I}_{1/n}$-space.

Weak $\mathcal{I}$-ultrafilters

Definition (Baumgartner)

Let $\mathcal{I}$ be a family of subsets of a set X such that $\mathcal{I}$ contains all singletons and is closed under subsets. An ultrafilter $\mathcal{U}$ on $\mathbb{N}$ is called an $\mathcal{I}$-ultrafilter if for every $F : \mathbb{N} \to X$ there exists $A \in \mathcal{U}$ such that $F[A] \in \mathcal{I}$.

Weak $\mathcal{I}$-ultrafilters

Definition E.

Let $\mathcal{I}$ be a family of subsets of a set X such that $\mathcal{I}$ contains all singletons and is closed under subsets. An ultrafilter $\mathcal{U}$ on $\mathbb{N}$ is called an weak $\mathcal{I}$-ultrafilter if for every finite-to-one $F : \mathbb{N} \to X$ there exists $A \in \mathcal{U}$ such that $F[A] \in \mathcal{I}$.

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Definition (Hindman)

An ultrafilter $\mathcal{U}$ on $\mathbb{N}$ is called weakly summable if every $U \in \mathcal{U}$ is an IP-set.

$\mathcal{I}_{1/\mathrm{n}}$-ultrafilters

Theorem 8.

$(\mathrm{MA}_{\mathrm{ctble}})$ There exists an $\mathcal{I}_{1/n}$-ultrafilter $\mathcal{U} \in \mathbb{N}^*$ such that every $U \in \mathcal{U}$ is an ip-rich set.

$\mathcal{I}_{1/\mathrm{n}}$-ultrafilters

Theorem 8.
($\mathrm{MA}_{\mathrm{ctble}}$) There exists an $\mathcal{I}_{1/n}$-ultrafilter $\mathcal{U} \in \mathbb{N}^*$ such that every $U \in \mathcal{U}$ is an ip-rich set.

Theorem 9.
($\mathrm{MA}_{\mathrm{ctble}}$) There exists a weak $\mathcal{I}_{1/n}$-ultrafilter $\mathcal{U} \in \mathbb{N}^*$ which is weakly summable ultrafilter.

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($\mathrm{MA}_{\mathrm{ctble}}$) There exists a weak $\mathcal{I}_{1/n}$-ultrafilter $\mathcal{U} \in \mathbb{N}^*$ which is weakly summable ultrafilter.

Question
Is it consistent that there is a weakly summable $\mathcal{I}_{1/n}$-ultrafilter?

References

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