Some ideals and ultrafilters on the natural numbers

Jana Flašková

Department of Mathematics
University of West Bohemia in Pilsen

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Ultrafilters on ω

Topologically, ultrafilters on ω are points in $\beta\omega$.

Combinatorially, ultrafilters on ω are maximal downward directed upper sets in $\mathcal{P}(\omega)$.

$\mathcal{U} \subseteq \mathcal{P}(\omega)$ is an ultrafilter if

- $\mathcal{U} \neq \emptyset$ and $\emptyset \notin \mathcal{U}$
- if $U_1, U_2 \in \mathcal{U}$ then $U_1 \cap U_2 \in \mathcal{U}$
- if $U \in \mathcal{U}$ and $U \subseteq V \subseteq \omega$ then $V \in \mathcal{U}$.
- for every $M \subseteq \omega$ either M or $\omega \setminus M$ belongs to $\mathcal{U}$

Ultrafilters on ω

Example. fixed (or principal) ultrafilter $\{A \subseteq \omega : n \in A\}$

Non-principal ultrafilters exist in ZFC.

Use of ultrafilters:

ultraproducts, parameters in some forcing constructions . . .

Some classes of ultrafilters

Definition.

For $\mathcal{U}, \mathcal{V} \in \beta\omega$ we write $\mathcal{U} \leq_{RK} \mathcal{V}$ if there exists $f \in {}^{\omega}\omega$ such that $\beta f(\mathcal{V}) = \mathcal{U}$.

$\beta f(\mathcal{V}) = \mathcal{U}$ iff $(\forall V \in \mathcal{V})\ f[V] \in \mathcal{U}$ iff $(\forall U \in \mathcal{U})\ f^{-1}[U] \in \mathcal{V}$.

- The relation $\leq_{RK}$ is a quasiorder on $\beta\omega$.
- Two ultrafilters $\mathcal{U}, \mathcal{V}$ are equivalent, $\mathcal{U} \approx \mathcal{V}$, if there exists a permutation π of ω such that $\beta\pi(\mathcal{V}) = \mathcal{U}$.
- The quotient relation defined by $\leq_{RK}$ on $\beta\omega/\approx$ is Rudin-Keisler order $\leq_{RK}$.

Some classes of ultrafilters

Minimal elements in the Rudin-Keisler order on ultrafilters are selective ultrafilters.

Definition.

A free ultrafilter $\mathcal{U}$ is called a selective ultrafilter if for all partitions of ω, $\{R_i : i \in \omega\}$, either for some i, $R_i \in \mathcal{U}$, or $(\exists U \in \mathcal{U})\ (\forall i \in \omega)\ |U \cap R_i| \leq 1$.

- $\mathbb{M}_{\mathcal{U}}$ adds a dominating real if $\mathcal{U}$ is a selective ultrafilter.

Some classes of ultrafilters

Definition.

A free ultrafilter $\mathcal{U}$ is called a *P*-point if for all partitions of ω, $\{R_i : i \in \omega\}$, either for some i, $R_i \in \mathcal{U}$, or $(\exists U \in \mathcal{U})$ $(\forall i \in \omega)$ $|U \cap R_i| < \omega$.

- If $\mathcal{V}$ is a *P*-point and $\mathcal{U} \leq_{RK} \mathcal{V}$ then $\mathcal{U}$ is a *P*-point.
- *P*-points were used for the first proof of the non-homogeneity of the space ω^*. (Rudin)

Some classes of ultrafilters

Definition.

An ultrafilter $\mathcal{U}$ on ω is called a nowhere dense ultrafilter if for every $f : \omega \to 2^\omega$ there exists $U \in \mathcal{U}$ such that $f[U]$ is nowhere dense.

- If $\mathcal{V}$ is a nowhere dense ultrafilter and $\mathcal{U} \leq_{RK} \mathcal{V}$ then $\mathcal{U}$ is nowhere dense.
- $\mathbb{L}_{\mathcal{U}}$ adds no Cohen reals iff $\mathcal{U}$ is a nowhere dense ultrafilter. (Błaszczyk, Shelah)

$\mathcal{I}$-ultrafilters

Definition. (Baumgartner)

Let $\mathcal{I}$ be a family of subsets of a set X such that $\mathcal{I}$ contains all singletons and is closed under subsets.
An ultrafilter $\mathcal{U}$ on ω is called an $\mathcal{I}$-ultrafilter if for every $f : \omega \to X$ there exists $U \in \mathcal{U}$ such that $f[U] \in \mathcal{I}$.

Examples.

- nowhere dense ultrafilters ... $X = 2^\omega$ and $\mathcal{I}$ are nowhere dense sets
- P-points ... $X = 2^\omega$ and $\mathcal{I}$ are finite and converging sequences

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Examples.

- nowhere dense ultrafilters . . . $X = 2^\omega$ and $\mathcal{I}$ are nowhere dense sets
- P-points . . . $X = 2^\omega$ and $\mathcal{I}$ are finite and converging sequences
  or $X = \omega \times \omega$ and $\mathcal{I} = \text{Fin} \times \text{Fin}$
- selective ultrafilters . . . $X = \omega \times \omega$ and $\mathcal{I} = \mathcal{ED}$.

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Let $\mathcal{I}$ be a family of subsets of a set X such that $\mathcal{I}$ contains all singletons and is closed under subsets.
An ultrafilter $\mathcal{U}$ on ω is called an $\mathcal{I}$-ultrafilter if for every $f : \omega \to X$ there exists $U \in \mathcal{U}$ such that $f[U] \in \mathcal{I}$.

- wlog $\mathcal{I}$ is closed under finite unions i.e. $\mathcal{I}$ is an ideal
- $\mathcal{I}$-ultrafilters are downwards closed in $\leq_{RK}$
- if $\mathcal{I} \subseteq \mathcal{J}$ then every $\mathcal{I}$-ultrafilter is a $\mathcal{J}$-ultrafilter
- principal ultrafilters are $\mathcal{I}$-ultrafilters for every $\mathcal{I}$

$\mathcal{I}$-ultrafilters for ideals on ω

Observation.

If $\mathcal{I}$ is a maximal ideal on ω then $\mathcal{U}$ is an $\mathcal{I}$-ultrafilter if and only if $\mathcal{I}^* \not\leq_{RK} \mathcal{U}$.

Theorem.

If $\mathcal{I}$ is a maximal ideal and $\chi(\mathcal{I}) = \mathfrak{c}$ then $\mathcal{I}$-ultrafilters exist in ZFC.

Proposition.

Assume $\mathcal{I}$ is ideal on ω and $\mathcal{U}$ is an ultrafilter on ω. The following are equivalent:

- $\mathcal{U}$ is an $\mathcal{I}$-ultrafilter
- $\mathcal{V} \not\leq_{RK} \mathcal{U}$ for every ultrafilter $\mathcal{V} \supseteq \mathcal{I}^*$

$\mathcal{I}$-ultrafilters for ideals on ω

All ideals contain the ideal Fin. All ultrafilters are non-principal.

Obviously, there are no Fin-ultrafilters.

$\mathcal{I}$-ultrafilters for ideals on ω

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Obviously, there are no Fin-ultrafilters.

An ideal $\mathcal{I}$ on ω is tall if for every $A \in [\omega]^\omega$ there exists an infinite set $B \subseteq A$ such that $B \in \mathcal{I}$.

Proposition.

If $\mathcal{I}$ is not a tall ideal then there are no $\mathcal{I}$-ultrafilters.

$\mathcal{I}$-ultrafilters for ideals on ω

Theorem

($\mathrm{MA}_{\sigma\text{-centered}}$) If $\mathcal{I}$ is a tall ideal then $\mathcal{I}$-ultrafilters exist.

$\mathcal{I}$-ultrafilters for ideals on ω

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($\mathrm{MA}_{\sigma\text{-centered}}$) If $\mathcal{I}$ is a tall ideal then $\mathcal{I}$-ultrafilters exist.

Theorem

If $\mathcal{I}$ is a (tall) F_σ-ideal or analytic P-ideal then every selective ultrafilter is an $\mathcal{I}$-ultrafilter.

Some ideals on ω

Let A be a subset of ω with an increasing enumeration $A = \{a_n : n \in \omega\}$. We say that A is

thin if $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 0$

$\mathcal{I}_{1/n} = \{A \subseteq \mathbb{N} : \sum_{a\in A} \frac{1}{a} < \infty\}$

$\mathcal{Z} = \{A \subseteq \mathbb{N} : \limsup_{n\to\infty} \frac{|A\cap n|}{n} = 0\}$

$\mathcal{I}$-ultrafilters for ideals on ω

Proposition.

Every selective ultrafilter is a thin ultrafilter.

Observation (Rudin).

Every P-point is a $\mathcal{Z}$-ultrafilter.

Theorem.

($\mathrm{MA}_{\mathrm{ctble}}$) For every F_σ-ideal $\mathcal{I}$ on ω there is a P-point that is not an $\mathcal{I}$-ultrafilter.

$\mathcal{I}$-ultrafilters for ideals on ω

Theorem (Brendle).

($MA_{\sigma\text{-centered}}$) There is a nowhere dense ultrafilter which is not $\mathcal{Z}$-ultrafilter.

$\mathcal{I}$-ultrafilters for ideals on ω

Theorem (Brendle).

($MA_{\sigma\text{-centered}}$) There is a nowhere dense ultrafilter which is not $\mathcal{Z}$-ultrafilter.

Theorem.

(MA_{ctble}) There is a thin ultrafilter which is not nowhere dense.

Products of ultrafilters

Definition.

Let $\mathcal{U}$ and $\mathcal{V}_n$, $n \in \omega$, be ultrafilters on ω.
$\mathcal{U}$-sum of ultrafilters $\mathcal{V}_n$, $\sum_{\mathcal{U}}\langle \mathcal{V}_n : n \in \omega\rangle$,
is an ultrafilter on $\omega \times \omega$ defined by $M \in \sum_{\mathcal{U}}\langle \mathcal{V}_n : n \in \omega\rangle$ if and only if $\{n : \{m : \langle n, m\rangle \in A\} \in \mathcal{V}_n\} \in \mathcal{U}$.

If $\mathcal{V}_n = \mathcal{V}$ for every $n \in \omega$ then we write $\sum_{\mathcal{U}}\langle \mathcal{V}_n : n \in \omega\rangle = \mathcal{U} \cdot \mathcal{V}$
and the ultrafilter $\mathcal{U} \cdot \mathcal{V}$
is called the product of ultrafilters $\mathcal{U}$ and $\mathcal{V}$.

$\mathcal{I}$-sums

Definition. (Baumgartner)

Let $\mathcal{C}$ and $\mathcal{D}$ be two classes of ultrafilters.
We say that $\mathcal{C}$ is closed under $\mathcal{D}$-sums provided that whenever
$\{\mathcal{V}_n : n \in \omega\} \subseteq \mathcal{C}$ and $\mathcal{U} \in \mathcal{D}$
then $\sum_{\mathcal{U}} \langle \mathcal{V}_n : n \in \omega \rangle \in \mathcal{C}$.

- If $\mathcal{D}$ is a class of $\mathcal{I}$-ultrafilters then we say that $\mathcal{C}$ is closed under $\mathcal{I}$-sums.

$\mathcal{I}$-sums

No product (sum) of ultrafilters is a P-point.

Theorem (Baumgartner)

Nowhere dense ultrafilters are closed under nowhere dense sums.

$\mathcal{I}$-sums

Theorem.

Let $\mathcal{I}$ be a *P*-ideal on ω (or $\mathbb{N}$).
If $\mathcal{U}$ is an $\mathcal{I}$-ultrafilter and $\{n : \mathcal{V}_n$ is $\mathcal{I}$-ultrafilter$\} \in \mathcal{U}$ then $\sum_{\mathcal{U}} \langle \mathcal{V}_n : n \in \omega \rangle$ is an $\mathcal{I}$-ultrafilter.

In other words, if $\mathcal{I}$ is a *P*-ideal then $\mathcal{I}$-ultrafilters are closed under $\mathcal{I}$-sums.

$\mathcal{I}$-sums

Theorem.

Let $\mathcal{I}$ be a P-ideal on ω (or $\mathbb{N}$).
If $\mathcal{U}$ is an $\mathcal{I}$-ultrafilter and $\{n : \mathcal{V}_n$ is $\mathcal{I}$-ultrafilter$\} \in \mathcal{U}$ then $\sum_{\mathcal{U}} \langle \mathcal{V}_n : n \in \omega \rangle$ is an $\mathcal{I}$-ultrafilter.

In other words, if $\mathcal{I}$ is a P-ideal then $\mathcal{I}$-ultrafilters are closed under $\mathcal{I}$-sums.

Theorem.

Let $\mathcal{I}$ be a P-ideal. If there is an $\mathcal{I}$-ultrafilter then there is an $\mathcal{I}$-ultrafilter that is not a P-point.

Thin ultrafilters

Proposition.

For arbitrary $\mathcal{U} \in \omega^*$ the ultrafilter $\mathcal{U} \cdot \mathcal{U}$ is not a thin ultrafilter.

Existence of $\mathcal{I}$-ultrafilters in ZFC

Theorem (Shelah)

It is consistent with ZFC that there are no nowhere dense ultrafilters.

Proposition.

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Existence of $\mathcal{I}$-ultrafilters in ZFC

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Question.

Do $\mathcal{I}$-ultrafilters exist in ZFC for any tall analytic ideal $\mathcal{I}$?
In particular, do $\mathcal{I}_{1/n}$-ultrafilters or $\mathcal{Z}$-ultrafilters exist in ZFC?

Weak $\mathcal{I}$-ultrafilters

Definition.

Let $\mathcal{I}$ be an ideal on ω. An ultrafilter $\mathcal{U}$ on ω is called

- weak $\mathcal{I}$-ultrafilter if for each finite-to-one $f : \omega \to \omega$ there exists $U \in \mathcal{U}$ such that $f[U] \in \mathcal{I}$.
- $\mathcal{I}$-friendly ultrafilter if for each one-to-one $f : \omega \to \omega$ there exists $U \in \mathcal{U}$ such that $f[U] \in \mathcal{I}$.

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weak $\mathcal{I}$-ultrafilter if for each finite-to-one $f : \omega \to \omega$ there exists $U \in \mathcal{U}$ such that $f[U] \in \mathcal{I}$.

$\mathcal{I}$-friendly ultrafilter if for each one-to-one $f : \omega \to \omega$ there exists $U \in \mathcal{U}$ such that $f[U] \in \mathcal{I}$.

Theorem.

- (Gryzlov) $\mathcal{Z}$-friendly ultrafilters exist in ZFC.
- $\mathcal{I}_{1/n}$-friendly ultrafilters exist in ZFC.

$\mathcal{W}$-ultrafilters

The van der Waerden ideal $\mathcal{W}$ consists of subsets of ω which do not contain arbitrary long arithmetic progressions.

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The van der Waerden ideal $\mathcal{W}$ consists of subsets of ω which do not contain arbitrary long arithmetic progressions.

- Every thin ultrafilter is a $\mathcal{W}$-ultrafilter.
- Every $\mathcal{W}$-ultrafilter is $\mathcal{Z}$-ultrafilter.

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The van der Waerden ideal $\mathcal{W}$ consists of subsets of ω which do not contain arbitrary long arithmetic progressions.

- Every thin ultrafilter is a $\mathcal{W}$-ultrafilter.
- Every $\mathcal{W}$-ultrafilter is $\mathcal{Z}$-ultrafilter.

Questions.

- Do $\mathcal{W}$-ultrafilters exist in ZFC?
- Are $\mathcal{W}$-ultrafilters closed under products?