Some special points in Čech-Stone compactification of natural numbers

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Filters

Definition.

For a non-empty set X, a filter on X is a family $\mathcal{F} \subseteq \mathcal{P}(X)$ such that:

- $\mathcal{F} \neq \emptyset$ and $\emptyset \notin \mathcal{F}$
- if $F_1, F_2 \in \mathcal{F}$ then $F_1 \cap F_2 \in \mathcal{F}$
- if $F \in \mathcal{F}$ and $F \subseteq G \subseteq X$ then $G \in \mathcal{F}$.

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If moreover $\mathcal{F}$ satisfies

- for every $M \subseteq X$ either $M \in \mathcal{F}$ or $X \setminus M \in \mathcal{F}$

then $\mathcal{F}$ is called an ultrafilter.

Ideals

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For a non-empty set X, an ideal on X is a family $\mathcal{I} \subseteq \mathcal{P}(X)$ such that:

- $\mathcal{I} \neq \mathcal{P}(X)$ and $\emptyset \in \mathcal{I}$
- if $A_1, A_2 \in \mathcal{I}$ then $A_1 \cup A_2 \in \mathcal{I}$
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Examples:

$$\mathcal{Z}_0 = \{A \subseteq \mathbb{N} : \limsup_{n\to\infty} \frac{|A \cap n|}{n} = 0\}$$

$$\mathcal{I}_{1/n} = \{A \subseteq \mathbb{N} : \sum_{a \in A} \frac{1}{a} < \infty\}$$

0-points

Definition A. (Gryzlov)

An ultrafilter $\mathcal{U}$ on ω is called a 0-point if for every one-to-one function $f : \omega \to \mathbb{N}$ there exists $U \in \mathcal{U}$ such that $f[U] \in \mathcal{Z}_0$.

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Theorem (Gryzlov)
There are $2^{\mathfrak{c}}$ many distinct 0-points.

Topological consequences

Problem 235. (Hart, van Mill)

For what nowhere dense sets $A \subseteq \omega^*$ do we have $\bigcup_{\pi \in S_\omega} \pi[A] \neq \omega^*$?

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- Not for all if P-points exist.

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Examples in ZFC:

- singletons
- (Gryzlov) $A = \{\mathcal{U} \in \omega^* : \mathcal{Z}_0^* \subseteq \mathcal{U}\}$.

Summable ultrafilters

Definition B.

An ultrafilter $\mathcal{U}$ on ω is called a summable ultrafilter if for every one-to-one function $f : \omega \longrightarrow \mathbb{N}$ there exists $U \in \mathcal{U}$ such that $f[U] \in \mathcal{I}_{1/n}$.

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- Every summable ultrafilter is a 0-point.
- Every Q-point is a summable ultrafilter.

Summable ultrafilters

Theorem 1.

($\mathrm{MA}_{\mathrm{ctble}}$) There exists a P-point which is not a summable ultrafilter.

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Question

Is there a 0-point which is not a summable ultrafilter in ZFC?

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Theorem 3.

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Corollary 4.

The set $A = \{\mathcal{U} \in \omega^* : \mathcal{I}^*_{1/n} \subseteq \mathcal{U}\}$ is a nowhere dense subset of ω^* such that $\bigcup_{\pi \in S_\omega} \pi[A] \neq \omega^*$.

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Proposition 5.

There exist $2^{\mathfrak{c}}$ many distinct summable ultrafilters.

Construction

Definition.

A family $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is called

- a k-linked family if $F_0 \cap F_1 \cap \ldots \cap F_k$ is infinite whenever $F_i \in \mathcal{F}$, $i \leq k$.
- a centered system if $\mathcal{F}$ is k-linked for every k i.e., if any finite subfamily of $\mathcal{F}$ has an infinite intersection.

Construction

We say that $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is a summable family if for every one-to-one function $f : \omega \to \mathbb{N}$ there is $A \in \mathcal{F}$ such that $f[A] \in \mathcal{I}_{1/n}$.

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Proposition 6.

For every $k \in \mathbb{N}$ there exists a summable k-linked family $\mathcal{F}_k \subseteq \mathcal{P}(\omega)$.

Construction

Lemma 7.

If $\mathcal{F}_k \subseteq \mathcal{P}(\omega)$ is a k-linked family then

$$\mathcal{F} = \{F \subseteq \omega : (\forall k)(\exists U^k \in \mathcal{F}_k)\, U^k \subseteq^* F\}$$

is a centered system.

If every $\mathcal{F}_k$ is summable then $\mathcal{F}$ is summable.

More generally, if $\mathcal{I}$ is a P-ideal and for every one-to-one function $f \in {}^{\omega}\mathbb{N}$ and for every $k \in \mathbb{N}$ there exists $U^k \in \mathcal{F}_k$ such that $f[U^k] \in \mathcal{I}$ then there exists $U \in \mathcal{F}$ such that $f[U] \in \mathcal{I}$.

Problems

$$\mathcal{I}_g = \{A \subseteq \mathbb{N} : \sum_{a \in A} g(a) < \infty\}$$

Question

Does there exist an ultrafilter $\mathcal{U}$ on ω such that for every one-to-one function there exists a set $U \in \mathcal{U}$ such that $f[U] \in \mathcal{I}_g$?

In particular, for $g(n) = \frac{1}{\sqrt{n}}$?

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In particular, for $g(n) = \frac{1}{\sqrt{n}}$?

Question

Does there exist an ultrafilter $\mathcal{U}$ on ω such that for every finite-to-one function there exists a set $U \in \mathcal{U}$ such that $f[U] \in \mathcal{I}_{1/n}(\mathcal{Z}_0)$?

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