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Generic existence of ultrafilters on the natural numbers

by

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Abstract. Say that a class of ultrafilters exists generically if every filter base of size $< \mathfrak{c}$ can be extended to an ultrafilter in that class. We investigate generic existence of ultrafilters defined in terms of ideals on the natural numbers, like the summable ultrafilters and the density zero ultrafilters.

Introduction. Special classes of ultrafilters on the natural numbers ω play an important role in set theory and its applications. Recall that an ultrafilter $\mathcal{U}$ is a *P-point* if given a countable $\mathcal{A} \subseteq \mathcal{U}$, there is $B \in \mathcal{U}$ with $B \subseteq^* A$ for all $A \in \mathcal{A}$; $\mathcal{U}$ is a *Ramsey ultrafilter* if given any partition $(X_n : n \in \omega)$ of ω, either $X_n \in \mathcal{U}$ for some n or there is $X \in \mathcal{U}$ with $|X \cap X_n| \leq 1$ for all n (such an X is called a *selector*); and $\mathcal{U}$ is a *Q-point* if any partition $(X_n : n \in \omega)$ of ω into finite sets has a selector $X \in \mathcal{U}$. It is well-known (and easy to see) that $\mathcal{U}$ is Ramsey iff it is both a P-point and a Q-point. Neither P-points nor Q-points necessarily exist (see [BJ, Theorems 4.4.7 and 4.6.7]) (1). There are, however, situations when there are lots of P-points (or Q-points) in the sense that each small filter base can be extended to a P-point (Q-point, respectively).

More precisely, a classical result of Ketonen [Ke] (see also [BJ, Theorem 4.4.5]) says that every filter base of size less than $\mathfrak{c}$ can be extended to a P-point iff $\mathfrak{d} = \mathfrak{c}$ (where $\mathfrak{d}$ denotes the dominating number as usual). Similarly, Canjar [Ca] (see also [BJ, Theorem 4.5.6]) proved that every filter base of size $< \mathfrak{c}$ can be extended to a Ramsey ultrafilter iff $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$ (the *covering number* of the meager ideal, $\mathsf{cov}(\mathcal{M})$, is the smallest size of a

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(1) It is, however, a famous open problem whether it is consistent that there are neither P-points nor Q-points simultaneously.

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covering of the real line by meager sets). Similar results have been proved subsequently by several people, including the first author [Br2]. We shall say that a class of ultrafilters *exists generically* if every filter base of size $< \mathfrak{c}$ can be extended to an ultrafilter in that class. Thus generic existence of P-points is equivalent to $\mathfrak{d} = \mathfrak{c}$.

By their very definition, P-points or Ramsey ultrafilters may be thought of as containing "small" sets of integers of a certain kind. It turns out that the notions of P-point and Ramsey ultrafilter—as well as many other similar ultrafilter notions—can be reformulated in a general framework of "smallness" introduced by Baumgartner [Bau]. Let $\mathcal{I}$ be a family of subsets of a set X containing all singletons and closed under subsets. Then $\mathcal{U}$ is an *$\mathcal{I}$-ultrafilter* if for every function $f : \omega \to X$ there is $A \in \mathcal{U}$ with $f[A] \in \mathcal{I}$. In this context, P-points can be described as $\mathcal{I}$-ultrafilters where $\mathcal{I}$ is the collection of converging sequences of real numbers considered as sets (see Section 2 for more details).

While generic existence has been characterized for a number of classes of ultrafilters defined in terms of a family of "small" sets of reals (like the P-points mentioned above, but also the measure zero and nowhere dense ultrafilters, see Section 2 below), this has not been the case for $\mathcal{I}$-ultrafilters where $\mathcal{I}$ is one of the classical ideals on ω, like the summable ideal $\mathcal{I}_{1/n}$ or the density zero ideal $\mathcal{Z}$. Such ultrafilters have recently been investigated by a number of people, including the second author [Fl2].

The goal of this paper is to obtain characterizations of generic existence of such ultrafilters. It turns out that, unlike for P-points or Ramsey ultrafilters, generic existence of summable ultrafilters or density zero ultrafilters cannot be characterized by one of the classical cardinal invariants of the continuum. Rather, there are new cardinal invariants responsible for this phenomenon, for which some upper and lower bounds can be given but which can also be shown to be consistently distinct from well-known cardinals.

Outline of the paper. The paper is organized as follows. The preliminary Section 1 sets the stage by briefly reviewing the main material from the literature we need for our work. In Section 2, we first review a number of classes of ultrafilters which figure—more or less prominently—in our paper, and then prove some new implications between these classes (e.g., every measure zero ultrafilter is a density zero ultrafilter, Corollary 2.5) as well as non-implications (e.g., under a fragment of Martin's axiom MA, there is a thin ultrafilter (= hereditary Q-point) that is not a nowhere dense ultrafilter, Theorem 2.8). In Section 3, we investigate generic existence of $\mathcal{I}$-ultrafilters where $\mathcal{I}$ is an ideal on ω. We prove some general results for F_σ ideals (like the summable ideal) and for analytic P-ideals (like the density zero ideal). For example, it is consistent that P-points exist generically while $\mathcal{I}$-ultrafilters do not for any F_σ ideal $\mathcal{I}$ on ω (Corollary 3.12). We then focus on four ideals, including $\mathcal{I}_{1/n}$ and $\mathcal{Z}$. We show, for example, that in the random model, summable ultrafilters (and thus density zero ultrafilters) exist generically while the other two classes of ultrafilters do not (see Theorems 3.26 and 3.31, and Corollary 3.34(ii)). The final Section 4 collects a number of results which either are related to our work without directly dealing with the main topic, or are known results for which we could not find a reference.

1. Prologue

1.1. Ideals and ultrafilters. Let $\mathcal{I}$ be an ideal on a set X. *All ideals in this paper are proper* (*i.e.*, $X \notin \mathcal{I}$) *and contain all singletons.* We use $\mathcal{I}^+$ for the *$\mathcal{I}$-positive sets* (i.e., $\mathcal{I}^+ = \mathcal{P}(X) \setminus \mathcal{I}$) and $\mathcal{I}^*$ for the *dual filter* (i.e., $\mathcal{I}^* = \{X \setminus A : A \in \mathcal{I}\}$). The ideal $\mathcal{I}$ is *tall* if every infinite subset of X has an infinite subset belonging to $\mathcal{I}$; and $\mathcal{I}$ is a *P-ideal* if given a countable $\mathcal{A} \subseteq \mathcal{I}$, there is $B \in \mathcal{I}$ with $A \subseteq^* B$ for all $A \in \mathcal{A}$.

Dually, *all filters are proper and do not contain finite sets.* In particular, in this paper *all ultrafilters on ω are nonprincipal.* A family $\mathcal{F}$ of subsets of a set X is said to be a *filter base* on X if $\langle \mathcal{F} \rangle = \{A \subseteq X : \exists B \in \mathcal{F}\ (B \subseteq A)\}$ is a filter on X.

When considering F_σ ideals and analytic P-ideals we use their characterizations in terms of lower semicontinuous submeasures.

Recall that $\varphi : \mathcal{P}(\omega) \to [0, \infty]$ is a *submeasure* if $\varphi(\emptyset) = 0$, $\varphi(X) \leq \varphi(Y)$ for $X \subseteq Y$, $\varphi(X \cup Y) \leq \varphi(X) + \varphi(Y)$ for any X, Y, and $\varphi(\{n\}) < \infty$ for any n. A submeasure φ is called *lower semicontinuous* if $\varphi(X) = \lim_{n\to\infty} \varphi(X \cap n)$ for every $X \subseteq \omega$. Two ideals can be associated with a lower semicontinuous submeasure:

$$\mathsf{Fin}(\varphi) = \{X \subseteq \omega : \varphi(X) < \infty\},$$
$$\mathsf{Exh}(\varphi) = \Big\{X \subseteq \omega : \lim_{n\to\infty} \varphi(X \setminus n) = 0\Big\}.$$

Clearly $\mathsf{Exh}(\varphi) \subseteq \mathsf{Fin}(\varphi)$, $\mathsf{Fin}(\varphi)$ is an F_σ ideal, and $\mathsf{Exh}(\varphi)$ is an $F_{\sigma\delta}$ P-ideal. On the other hand, a classical result of Mazur [Ma] says that every F_σ ideal is of the form $\mathsf{Fin}(\varphi)$, and Solecki [So] proved that every analytic P-ideal is of the form $\mathsf{Exh}(\varphi)$.

1.2. Cardinal invariants of the continuum. In Section 3, we shall need many of the standard cardinal invariants of the continuum, and we therefore briefly review their definitions and basic relations between them. Details of everything mentioned here can be found in [Bl2] and [BJ].

First, recall

$$\mathfrak{b} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \text{ and } \forall g \in \omega^\omega \; \exists f \in \mathcal{F} \; (f \not\leq^* g)\},$$
the *unbounding number*,

$$\mathfrak{d} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \text{ and } \forall f \in \omega^\omega \; \exists g \in \mathcal{F} \; (f \leq^* g)\},$$
the *dominating number*.

Say $X \in [\omega]^\omega$ *splits* $A \in [\omega]^\omega$ if both $A \cap X$ and $A \setminus X$ are infinite. Then $\mathcal{F} \subseteq [\omega]^\omega$ is a *splitting family* if all members of $[\omega]^\omega$ are split by a member of $\mathcal{F}$; and $\mathcal{F}$ is an *unreaped family* for all $X \in [\omega]^\omega$ there is $A \in \mathcal{F}$ such that X does not split A, i.e., $A \subseteq^* X$ or $A \subseteq^* \omega \setminus X$.

Next, recall

$$\mathfrak{s} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^\omega \text{ is a splitting family}\}, \quad \text{the } \textit{splitting number},$$
$$\mathfrak{r} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^\omega \text{ is an unreaped family}\}, \quad \text{the } \textit{reaping number}.$$

Many cardinal invariants come in dual pairs, like $(\mathfrak{b}, \mathfrak{d})$ and $(\mathfrak{s}, \mathfrak{r})$. Let $R \subseteq \omega^\omega \times \omega^\omega$ be a relation on ω^ω (or a relation between different realizations of the reals) such that for all $f \in \omega^\omega$ there is $g \in \omega^\omega$ with fRg and for all $g \in \omega^\omega$ there is $f \in \omega^\omega$ with $\neg(fRg)$. Many cardinals are of the form

$$\mathfrak{b}(R) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \text{ and } \forall g \in \omega^\omega \; \exists f \in \mathcal{F} \; \neg(fRg)\}, \text{ or}$$
$$\mathfrak{d}(R) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \text{ and } \forall f \in \omega^\omega \; \exists g \in \mathcal{F} \; (fRg)\}.$$

Clearly $\mathfrak{b} = \mathfrak{b}(\leq^*)$, $\mathfrak{d} = \mathfrak{d}(\leq^*)$, $\mathfrak{s} = \mathfrak{b}(\text{does not split})$, and $\mathfrak{r} = \mathfrak{d}(\text{does not split})$. Because of duality, there is typically one proof, formalized in terms of Tukey functions, which gives two inequalities. See [Bl2, Section 4] for a detailed discussion of this. We shall present some such proofs below (Propositions 3.9 and 3.10).

When $\mathcal{I}$ is an ideal on some set X, define

$$\mathsf{add}(\mathcal{I}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{F} \notin \mathcal{I}\}, \text{ the } \textit{additivity} \text{ of } \mathcal{I},$$
$$\mathsf{cov}(\mathcal{I}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{F} = X\}, \text{ the } \textit{covering number} \text{ of } \mathcal{I},$$
$$\mathsf{non}(\mathcal{I}) = \min\{|F| : F \subseteq X \text{ and } F \notin \mathcal{I}\}, \text{ the } \textit{uniformity} \text{ of } \mathcal{I},$$
$$\mathsf{cof}(\mathcal{I}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \text{ and } \forall I \in \mathcal{I} \; \exists F \in \mathcal{F} \; (I \subseteq F)\},$$
the *cofinality* of $\mathcal{I}$.

Then $\mathsf{add}(\mathcal{I}) \leq \mathsf{cov}(\mathcal{I}) \leq \mathsf{cof}(\mathcal{I})$ and $\mathsf{add}(\mathcal{I}) \leq \mathsf{non}(\mathcal{I}) \leq \mathsf{cof}(\mathcal{I})$. If $\mathcal{I}$ is a Borel ideal on the reals, $(\mathsf{add}(\mathcal{I}), \mathsf{cof}(\mathcal{I}))$ and $(\mathsf{cov}(\mathcal{I}), \mathsf{non}(\mathcal{I}))$ can be construed as dual pairs in the sense of the preceding paragraph. We shall need (some of) these cardinals for three such ideals, the σ-ideal $\mathcal{M}$ of *meager sets*, the σ-ideal $\mathcal{N}$ of *null sets*, and the σ-ideal $\mathcal{E}$ generated by *closed null sets* (2).

(2) It is well-known that these cardinals do not depend on the particular representation of the reals.

The relationship between $\mathfrak{b}$, $\mathfrak{d}$ and the invariants of the former two is subsumed in Cichoń's diagram (see [Fr1] and [BJ, Chapter 2]) where cardinals get larger as one moves up and to the right in the diagram.

$$\begin{tikzcd}
\mathsf{cov}(\mathcal{N}) \arrow[r, dash] \arrow[dd, dash] & \mathsf{non}(\mathcal{M}) \arrow[r, dash] \arrow[d, dash] & \mathsf{cof}(\mathcal{M}) \arrow[r, dash] \arrow[d, dash] & \mathsf{cof}(\mathcal{N}) \arrow[dd, dash] \\
 & \mathfrak{b} \arrow[r, dash] \arrow[d, dash] & \mathfrak{d} \arrow[d, dash] & \\
\mathsf{add}(\mathcal{N}) \arrow[r, dash] & \mathsf{add}(\mathcal{M}) \arrow[r, dash] & \mathsf{cov}(\mathcal{M}) \arrow[r, dash] & \mathsf{non}(\mathcal{N})
\end{tikzcd}$$

Fig. 0. Cichoń's diagram

The deepest result here is the Bartoszyński–Raisonnier–Stern Theorem ([Fr1], [BJ, Section 2.3]) saying that $\mathsf{add}(\mathcal{N}) \leq \mathsf{add}(\mathcal{M})$ and, dually, $\mathsf{cof}(\mathcal{M}) \leq \mathsf{cof}(\mathcal{N})$. Also $\mathsf{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathsf{cov}(\mathcal{M})\}$ and $\mathsf{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \mathsf{non}(\mathcal{M})\}$ ([Fr1], [BJ, 2.2.9 and 2.2.11]). The cardinals we shall mainly study in Section 3 all lie between $\mathsf{cov}(\mathcal{M})$ and $\mathsf{cof}(\mathcal{N})$ (Observation 3.7), but we will also need some of the cardinals on the left-hand side of the diagram.

Further useful inequalities are $\mathfrak{s} \leq \mathfrak{d}$ and $\mathfrak{b} \leq \mathfrak{r}$ [Bl2, Theorems 3.3 and 3.8], $\mathsf{cov}(\mathcal{M}), \mathsf{cov}(\mathcal{N}) \leq \mathsf{cov}(\mathcal{E}) \leq \mathfrak{r}$ and $\mathfrak{s} \leq \mathsf{non}(\mathcal{E}) \leq \mathsf{non}(\mathcal{N}), \mathsf{non}(\mathcal{M})$ [Bl2, Theorem 5.19], as well as $\mathsf{add}(\mathcal{E}) = \mathsf{add}(\mathcal{M})$ and $\mathsf{cof}(\mathcal{E}) = \mathsf{cof}(\mathcal{M})$ (see [BaS] or [BJ, Theorem 2.6.17]).

1.3. Forcing. In Section 3, we shall present a number of independence results concerning cardinal invariants related to the ultrafilters we discuss in our work. In most cases, these results are obtained by proving ZFC-inequalities between these cardinals and classical cardinal invariants, and then appealing to known results about the values of the latter in well-known models of set theory. This allows us to "black box" a substantial part of forcing theory. There are, however, two forcing notions which we will use directly, namely, random forcing and forcings of type $\mathbb{M}(\mathcal{F})$.

Let κ be an infinite cardinal. The *measure algebra* $\mathbb{B}_\kappa$ on 2^κ (also called the *algebra for adding κ random reals*) is the quotient of Baire subsets of 2^κ by null sets [Ku2]. If $\kappa = \omega$, we write $\mathbb{B} = \mathbb{B}_\omega$ and call the latter the *random algebra*. $\mathbb{B}$ adds one *random real*, that is, a real avoiding all null sets coded in the ground model. Any $\mathbb{B}_\kappa$ is ω^ω*-bounding*, that is, every $g \in \omega^\omega$ added by $\mathbb{B}_\kappa$ is eventually dominated by $f \in \omega^\omega$ from the ground model (see [BJ, Lemma 3.1.2]). See [Ku2] and [BJ, Section 3.2] for more details about random forcing.

Let $\mathcal{F}$ be a filter on ω. We denote by $\mathbb{M}(\mathcal{F})$ *Mathias forcing* with $\mathcal{F}$. Conditions are of the form (s, A) where $s \in [\omega]^{<\omega}$ and $A \in \mathcal{F}$ with

$\max(s) < \min(A)$. The order is given by $(t, B) \leq (s, A)$ iff $B \subseteq A$ and $s \subseteq t \subseteq s \cup A$. $\mathbb{M}(\mathcal{F})$ is a σ-centered forcing which generically adds a real r which *diagonalizes* the filter $\mathcal{F}$, i.e., $r \subseteq^* A$ for all $A \in \mathcal{F}$. Furthermore, an easy genericity argument shows that $r \cap B$ is infinite for all $B \in \mathcal{F}^+$.

We shall use the following models:

- *The Cohen model.* The model obtained by adding $\kappa \geq \aleph_2$ Cohen reals over a model of CH. It satisfies $\mathsf{non}(\mathcal{M}) = \aleph_1$ and $\mathsf{cov}(\mathcal{M}) = \mathfrak{c} \geq \aleph_2$ [BJ, 3.3, 7.3.F, and 7.5.8].
- *The random model.* The model obtained by forcing with $\mathbb{B}_\kappa$ for $\kappa \geq \aleph_2$ over a model of CH. It satisfies $\mathsf{non}(\mathcal{N}) = \mathfrak{d} = \aleph_1$ and $\mathsf{cov}(\mathcal{N}) = \mathfrak{c} \geq \aleph_2$ [BJ, 3.2, 7.3.F, and 7.6.8].
- *The dual random model.* The model obtained by forcing with $\mathbb{B}_{\omega_1}$ over a model for $\mathrm{MA} + \mathfrak{c} \geq \aleph_2$. It satisfies $\mathsf{non}(\mathcal{N}) = \aleph_1$ and $\mathsf{cov}(\mathcal{N}) = \mathfrak{b} = \mathfrak{c} \geq \aleph_2$ [BJ, 3.2 and 7.6.7].
- *The dual Hechler model.* The model obtained by a finite support iteration of length ω_1 of Hechler forcing over a model of $\mathrm{MA} + \mathfrak{c} \geq \aleph_2$. It satisfies $\mathsf{cof}(\mathcal{M}) = \aleph_1$ and $\mathfrak{r} = \mathsf{non}(\mathcal{N}) = \mathfrak{c} \geq \aleph_2$ [BJ, 3.5 and 7.6.10].
- *The dual* $\mathbb{M}(\mathcal{F})$ *model.* Let $\mathcal{F}$ be a definable filter on ω, that is, $\mathcal{F}$ has a definition ϕ which interprets as a filter in all models in ZFC. For simplicity, we may as well assume that $\mathcal{F}$ is analytic without parameter. The dual $\mathbb{M}(\mathcal{F})$ model is the model obtained by a finite support iteration of length ω_1 of $\mathbb{M}(\mathcal{F})$ over a model of $\mathrm{MA} + \mathfrak{c} \geq \aleph_2$. It satisfies $\mathsf{non}(\mathcal{M}) = \aleph_1$ (because of the $\aleph_1$ many Cohen reals added at limit stages) and $\mathsf{non}(\mathcal{N}) = \mathfrak{c} \geq \aleph_2$ (this follows from the fact that $\mathbb{M}(\mathcal{F})$ is σ-centered together with [BJ, 6.5.30, 6.4.12, and 6.5.27]).
  If $\mathcal{F}^*$ is an F_σ ideal, this model also satisfies $\mathfrak{d} = \mathfrak{c}$ (this is so because $\mathbb{M}(\mathcal{F})$ does not add dominating reals in this case [Br1], see also [BJ, 6.5.2]).
- *The Sacks model.* The model obtained by a countable support iteration of length ω_2 of Sacks forcing over a model of CH. It satisfies $\mathsf{cof}(\mathcal{N}) = \mathfrak{r} = \aleph_1$ and $\mathfrak{c} = \aleph_2$ [BJ, 7.3.A].

The following models are defined in the same way, using a countable support iteration of length ω_2 over a model of CH of the corresponding forcing notion.

- *The Miller model.* It satisfies $\mathsf{non}(\mathcal{M}) = \mathsf{non}(\mathcal{N}) = \mathfrak{r} = \aleph_1$ and $\mathfrak{d} = \mathfrak{c} = \aleph_2$ [BJ, 7.3.E].
- *The Blass–Shelah model.* It satisfies $\mathfrak{r} = \aleph_1$ and $\mathfrak{s} = \mathfrak{c} = \aleph_2$ [BJ, 7.4.D] (see also [BS1]).
- *The Laver model.* It satisfies $\mathsf{cov}(\mathcal{E}) = \mathsf{non}(\mathcal{N}) = \aleph_1$ and $\mathfrak{b} = \mathfrak{c} = \aleph_2$ [BJ, 7.3.D].
- *The Mathias model.* It satisfies $\mathsf{cov}(\mathcal{E}) = \aleph_1$ and $\mathfrak{b} = \mathfrak{s} = \mathfrak{c} = \aleph_2$ [BJ, 7.4.A].

Typically, in dual models, dual cardinal invariants have opposite values, that is, if $\mathfrak{x} = \mathfrak{c}$ ($\mathfrak{x} = \aleph_1$, respectively) in a model, then $\mathfrak{y} = \aleph_1$ ($\mathfrak{y} = \mathfrak{c}$, respectively) will hold in the dual model where $\mathfrak{y}$ is the cardinal invariant dual to $\mathfrak{x}$ (as explained in Subsection 1.2). For example, $\mathfrak{d} = \aleph_1$ in the random model and $\mathfrak{b} = \mathfrak{c}$ in the dual random model. Dual models can only be built for models obtained by finite support iteration of ccc forcing or by the measure algebra. They do not exist for countable support iterations of proper forcings (the last five models above).

We sometimes use the fact that if M is a model of ZFC of size $< \mathsf{cov}(\mathcal{M})$ ($< \mathsf{cov}(\mathcal{N})$, respectively), then there is a Cohen real (random real, resp.) over M. Also, MA(countable) is equivalent to $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$ [BJ, 3.3.1 and the subsequent comment].

A forcing notion $\mathbb{P}$ has the *Laver property* [BJ, 6.3.27] if given any $p \in \mathbb{P}$, any $h \in \omega^\omega$, and any $\mathbb{P}$-name $\dot{f} \in \prod_n h(n)$ for a function strictly below h, there are $q \leq p$ and $\phi : \omega \to [\omega]^{<\omega}$ with $|\phi(n)| \leq n + 1$ for all n such that $q \Vdash \forall n\ (\dot{f}(n) \in \phi(n))$. Sacks [BJ, 7.3.2 and 6.3.38], Miller [BJ, 7.3.45], Laver [BJ, 7.3.29], and Mathias [BJ, 7.4.7] forcing and their countable support iterations [BJ, 6.3.34] all have the Laver property. If $\mathbb{P}$ is a forcing with the Laver property, then every new real added by $\mathbb{P}$ is contained in a closed measure zero set coded in the ground model [BJ, Lemma 6.3.32]. In particular, in all models obtained by a forcing with the Laver property, $\mathsf{cov}(\mathcal{E}) = \aleph_1$ holds.

2. Classes of ultrafilters

2.1. $\mathcal{I}$-ultrafilters and Katětov order. Let $\mathcal{I}$ be a family of subsets of a set X such that $\mathcal{I}$ contains all singletons and is closed under subsets. Given an ultrafilter $\mathcal{U}$ on ω, we say that $\mathcal{U}$ is a *weak $\mathcal{I}$-ultrafilter* if for every finite-to-one function $f : \omega \to X$ there exists $A \in \mathcal{U}$ such that $f[A] \in \mathcal{I}$. If the latter only holds for one-to-one functions f, we call $\mathcal{U}$ an *$\mathcal{I}$-point.* It is well-known that it suffices to work with families of sets which are tall ideals. More explicitly, if $\langle \mathcal{I} \rangle = \{\bigcup_{i<n} A_i : n \in \omega \text{ and } A_i \in \mathcal{I}\}$ is the ideal generated by $\mathcal{I}$, then the maximality property of the ultrafilter $\mathcal{U}$ implies that $\mathcal{U}$ is an $\mathcal{I}$-ultrafilter iff it is an $\langle \mathcal{I} \rangle$-ultrafilter, and similarly for "weak $\mathcal{I}$-ultrafilter" and "$\mathcal{I}$-point". Furthermore, if an ideal $\mathcal{I}$ is not tall, a nonprincipal ultrafilter cannot be an $\mathcal{I}$-ultrafilter (if $A \subseteq X$ is countable such that $\mathcal{I}{\restriction}A$ is the ideal of finite sets, then any bijection $f : \omega \to A$ witnesses that $\mathcal{U}$ is not an $\mathcal{I}$-ultrafilter).

The rationals $\mathbb{Q}(2^\omega)$ of the Cantor space 2^ω may be identified with sequences that are eventually zero. Thus, an alternative description of $\mathbb{Q}(2^\omega)$ is the collection of those $\sigma \in 2^{<\omega}$ such that either $\sigma = \langle\rangle$ is the empty sequence or $|\sigma| \geq 1$ and $\sigma(|\sigma| - 1) = 1$ (simply cut an eventually zero sequence at the last place where it assumes the value 1). Since the exact representation of $\mathbb{Q}(2^\omega)$ does not matter for most proofs, we shall be rather sloppy with this and often identify $\mathbb{Q}(2^\omega)$ with $2^{<\omega}$ though formally this is not correct ([3]). A similar comment applies to the rationals $\mathbb{Q}(\omega^\omega)$ of the Baire space ω^ω.

Consider the following ideals:

- conv, the ideal on $\mathbb{Q}(2^\omega)$ generated by converging (to reals) sequences of rational numbers,
- count, the ideal of sets with countable closure, on $\mathbb{Q}(2^\omega)$ (or $\mathbb{Q}(\omega^\omega)$),
- mz, the ideal of sets with closure of measure zero, on $\mathbb{Q}(2^\omega)$,
- nwd, the ideal of nowhere dense sets, on $\mathbb{Q}(2^\omega)$ (or $\mathbb{Q}(\omega^\omega)$),
- disc, the ideal generated by discrete sets, on $\mathbb{Q}(2^\omega)$,
- scat, the ideal generated by scattered sets, on $\mathbb{Q}(2^\omega)$,
- K_σ, the ideal of sets with σ-compact closure, on $\mathbb{Q}(\omega^\omega)$,
- $\mathcal{R}$, the ideal generated by homogeneous sets in some fixed instance of the random graph on ω,
- $\mathsf{Fin} \times \mathsf{Fin}$, the ideal generated by vertical sections and sets bounded by functions, on $\omega \times \omega$,
- $\mathcal{ED}$, the F_σ ideal generated by vertical sections and graphs of functions on $\omega \times \omega$,
- $\mathcal{ED}_{\mathsf{fin}}$, the F_σ ideal generated by graphs of functions bounded by the identity function, on $\Delta = \{(i,j) \in \omega \times \omega : j \leq i\}$,
- thin, the ideal generated by thin sets on ω,
- $\mathcal{SC}$, the ideal generated by SC-sets on ω,
- $\mathcal{I}_{1/n}$, the F_σ P-ideal of summable sets on ω,
- $\mathcal{Z}$, the P-ideal of sets of density zero, on ω.

In this context recall that $A = \{a_n \in \omega : n \in \omega\}$ is *thin* if $\lim a_n/a_{n+1} = 0$, and an *SC-set* if $\lim(a_{n+1} - a_n) = \infty$. A set $A \subseteq \omega$ is *summable* if $\sum_{n\in A} 1/n+1 < \infty$, and *of density zero* if $\lim |A \cap n|/n = 0$.

The corresponding classes of ultrafilters have been investigated by a number of authors (see, in particular, [Bau], [Sh], [Br2], [Bar], [Fl1], [Fl2], and [HZ2]). The following establishes a close connection with the better known classes of ultrafilters mentioned in the introduction.

Observation 2.1 ([Fl4]).

(i) ([Bau]) *The following are equivalent for an ultrafilter $\mathcal{U}$ on ω:*

(1) *$\mathcal{U}$ is a P-point,*
(2) *$\mathcal{U}$ is a (weak)* conv*-ultrafilter* (conv*-point*),
(3) *$\mathcal{U}$ is a (weak)* $\mathsf{Fin} \times \mathsf{Fin}$*-ultrafilter* ($\mathsf{Fin} \times \mathsf{Fin}$*-point*).

([3]) The only place where this matters is in the proof of Proposition 2.6 below, and there we shall be careful.

(ii) *The following are equivalent for an ultrafilter $\mathcal{U}$ on ω:*

(1) *$\mathcal{U}$ is a Ramsey ultrafilter,*
(2) *$\mathcal{U}$ is an (a weak) $\mathcal{R}$-ultrafilter ($\mathcal{R}$-point),*
(3) *$\mathcal{U}$ is an (a weak) $\mathcal{ED}$-ultrafilter ($\mathcal{ED}$-point).*

(iii) *The following are equivalent for an ultrafilter $\mathcal{U}$ on ω:*

(1) *$\mathcal{U}$ is a Q-point,*
(2) *$\mathcal{U}$ is a weakly* thin*-ultrafilter* (thin*-point*),
(3) *$\mathcal{U}$ is a weak $\mathcal{ED}_{\mathsf{fin}}$-ultrafilter (an $\mathcal{ED}_{\mathsf{fin}}$-point).*

Here, the statements without parentheses may be replaced by those in parentheses. E.g., in (i)(2), "$\mathcal{U}$ is a conv-ultrafilter", "$\mathcal{U}$ is a weak conv-ultrafilter", and "$\mathcal{U}$ is a conv-point" are all equivalent, and similarly for the other statements.

To understand the connection between $\mathcal{I}$-ultrafilters for different choices of $\mathcal{I}$, the *Katětov order* $\leq_{\mathrm{K}}$ plays a crucial role. For ideals $\mathcal{I}$ and $\mathcal{J}$ on countable sets X and Y we write $\mathcal{I} \leq_{\mathrm{K}} \mathcal{J}$ if there is a function $f : Y \to X$ such that $f^{-1}(I) \in \mathcal{J}$ for all $I \in \mathcal{I}$. The *Katětov–Blass order* is defined by $\mathcal{I} \leq_{\mathrm{KB}} \mathcal{J}$ if there is a finite-to-one function $f : Y \to X$ such that $f^{-1}(I) \in \mathcal{J}$ for all $I \in \mathcal{I}$.

Observation 2.2 ([Fl4]). *Let $\mathcal{I}$ and $\mathcal{J}$ be tall ideals.*

(i) (see also [Hr1, Proposition 4.5]) *An ultrafilter $\mathcal{U}$ on ω is an $\mathcal{I}$-ultrafilter iff $\mathcal{I} \not\leq_{\mathrm{K}} \mathcal{U}^*$.*
(ii) *An ultrafilter $\mathcal{U}$ on ω is a weak $\mathcal{I}$-ultrafilter iff $\mathcal{I} \not\leq_{\mathrm{KB}} \mathcal{U}^*$.*
(iii) *If $\mathcal{I} \leq_{\mathrm{K}} \mathcal{J}$ and $\mathcal{U}$ is an $\mathcal{I}$-ultrafilter, then it is also a $\mathcal{J}$-ultrafilter.*
(iv) *If $\mathcal{I} \leq_{\mathrm{KB}} \mathcal{J}$ and $\mathcal{U}$ is a weak $\mathcal{I}$-ultrafilter, then it is also a weak $\mathcal{J}$-ultrafilter.*

It would be tempting to conjecture that some converse of this holds, namely, that $\mathcal{I} \not\leq_{\mathrm{K}} \mathcal{J}$ implies that, under some assumption like MA(σ-centered), an $\mathcal{I}$-ultrafilter that is not a $\mathcal{J}$-ultrafilter can be constructed. This, however, is false: e.g., $\mathsf{Fin} \times \mathsf{Fin} \not\leq_{\mathrm{K}} \mathsf{conv}$ ($\mathcal{ED} \not\leq_{\mathrm{K}} \mathcal{R}$, resp.), yet both ideals describe the P-points (Ramsey ultrafilters, resp.). The following stronger fact is well-known:

Observation 2.3.

(i) $\mathsf{Fin} \times \mathsf{Fin} \not\leq_{\mathrm{K}} \mathcal{Z}$. *More generally,* $\mathsf{Fin} \times \mathsf{Fin} \not\leq_{\mathrm{K}} \mathcal{I}$ *for any P-ideal $\mathcal{I}$.*
(ii) $\mathcal{ED} \not\leq_{\mathrm{K}} \mathsf{nwd}$. *In particular,* $\mathsf{Fin} \times \mathsf{Fin} \not\leq_{\mathrm{K}} \mathsf{nwd}$.

Proof. (i) Suppose that $f : \omega \to \omega \times \omega$ witnesses that $\mathsf{Fin} \times \mathsf{Fin} \leq_{\mathrm{K}} \mathcal{I}$. We may assume that the inverse images of the vertical sections belong to $\mathcal{I}$. Since $\mathcal{I}$ is a P-ideal, there is a set $A \subseteq \omega \times \omega$ with finite vertical sections such that $f^{-1}(A) \in \mathcal{I}^*$, a contradiction.

(ii) Again let $f : 2^{<\omega} \to \omega \times \omega$ be such that the inverse images of the vertical sections are nowhere dense. Then it is easy to construct a graph of a (partial) function $A \subseteq \omega \times \omega$ such that $f^{-1}(A)$ is dense. ∎

The Katětov–Blass order on the ideals introduced above can be summarized in the following diagram where ideals get KB-larger as one moves up. (In fact, as witnesses for the reduction one can always choose one-to-one functions.) For a similar diagram, see [Hr2] and the recent [BFV].

This diagram is complete in the sense that no further line can be added. That is, if $\mathcal{I}$ is not below $\mathcal{J}$ in the diagram, then $\mathcal{I} \not\leq_{\mathrm{K}} \mathcal{J}$. This follows, on the one hand, from Observation 2.3, and on the other hand from the fact that, for many pairs $\mathcal{I}$, $\mathcal{J}$, there are consistently $\mathcal{I}$-ultrafilters that are not $\mathcal{J}$-ultrafilters (4) so that $\mathcal{I} \not\leq_{\mathrm{K}} \mathcal{J}$ follows by Observation 2.2(iii).

Some of the ideals (e.g., count or nwd) may be considered on different underlying sets like the rationals of 2^ω and the rationals of ω^ω. However, these different versions are easily seen to be KB-equivalent.

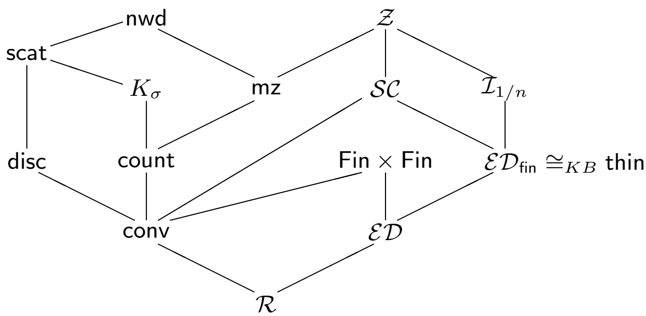

Fig. 1

Almost all of the lines are trivial, and most are inclusion relations. The nontrivial lines are the ones between $\mathcal{R}$ and conv, $\mathcal{R}$ and $\mathcal{ED}$, conv and $\mathsf{Fin} \times \mathsf{Fin}$, conv and $\mathcal{SC}$, mz and $\mathcal{Z}$, K_σ and scat, as well as the equivalence $\mathcal{ED}_{\mathsf{fin}} \cong_{KB} \mathsf{thin}$.

For $\mathcal{R} \leq_{\mathrm{KB}} \mathsf{conv}$, see [Me, Lemma 3.3.3]; for $\mathcal{R} \leq_{\mathrm{KB}} \mathcal{ED}$, partition a subset of ω into countably many countable 0-homogeneous sets A_n such that the color between any two elements of different A_n is 1 (this is possible by the universality of the random graph), and then let $f : \omega \times \omega \to \omega$ map the vertical sections to the A_n; for $\mathcal{ED}_{\mathsf{fin}} \cong_{KB} \mathsf{thin}$, see [Fl4, Lemma 3.3]; for $\mathsf{conv} \leq_{\mathrm{KB}} \mathsf{Fin} \times \mathsf{Fin}$, take countably many sequences A_n converging to different limits and let $f : \omega \times \omega \to 2^{<\omega}$ map the vertical sections to

(4) This is explained in detail in the discussion following Figure 2 below. Note that for Borel ideals $\mathcal{I}$ and $\mathcal{J}$, $\mathcal{I} \leq_{\mathrm{K}} \mathcal{J}$ is a $\mathbf{\Sigma}^1_2$ statement and therefore absolute. Thus the consistency of $\mathcal{I}$-ultrafilters that are not $\mathcal{J}$-ultrafilters implies that $\mathcal{I} \not\leq_{\mathrm{K}} \mathcal{J}$ holds in ZFC.

the A_n; for $\mathsf{conv} \leq_{\mathrm{KB}} \mathcal{SC}$, let $f : \omega \to 2^{<\omega}$ be defined by $f(m) = \sigma_m$ where $\{\sigma_{2^n-1}, \sigma_{2^n}, \ldots, \sigma_{2^{n+1}-2}\}$ lists 2^n in backward lexicographic order for all $n \in \omega$ (if $A \subseteq 2^{<\omega}$ is a converging sequence then, for each k, almost all elements of A have the same initial segment of length k, and thus, by definition of the backward lexicographic order, almost any two distinct elements of $f^{-1}(A)$ have difference at least 2^k).

We show:

Proposition 2.4. $\mathsf{mz} \leq_{\mathrm{KB}} \mathcal{Z}$.

Proof. Enumerate all sequences in $2^{<\omega}$ of length n as

$$2^n = \{\sigma_{2^n-1}, \sigma_{2^n}, \ldots, \sigma_{2^{n+1}-2}\}.$$

Then $f : \omega \to 2^{<\omega} : m \mapsto \sigma_m$ is a bijection.

Assume $A \subseteq 2^{<\omega}$ has closure of measure zero. Then there are $\tau_i \in 2^{<\omega}$, $i \in \omega$, such that $\mu(\bigcup_{i\in\omega}[\tau_i]) = 1$ and $\bigcup_{i\in\omega}[\tau_i] \cap A = \emptyset$, that is, $\sigma \not\supseteq \tau_i$ for all $\sigma \in A$ and all $i \in \omega$.

Fix $\varepsilon > 0$. There exists i_0 such that $\mu(\bigcup_{i<i_0}[\tau_i]) > 1 - \varepsilon$. Consider $n_0 \in \omega$ such that $|\tau_i| \leq n_0$ for every $i < i_0$. For every $n \geq n_0$ we get $|f^{-1}(A) \cap [2^n - 1, 2^{n+1} - 2]|/2^n < \varepsilon$. Unfixing ε, we see that $f^{-1}(A)$ has density zero. ∎

Corollary 2.5. *Every measure zero ultrafilter is a $\mathcal{Z}$-ultrafilter.*

Proposition 2.6. $K_\sigma \leq_{\mathrm{KB}} \mathsf{scat}$.

Proof. Define $f : \mathbb{Q}(2^\omega) \to \mathbb{Q}(\omega^\omega)$ as follows: $f(s) = (n_0, \ldots, n_{k-1})$ where $k = |s^{-1}(\{1\})|$ and $s^{-1}(\{1\}) = \{n_0 < n_1 < \cdots < n_{k-1}\}$. (Since $s \in \mathbb{Q}(2^\omega)$, this means $|s| = n_{k-1} + 1$, and f is indeed one-to-one.)

Let $A \subseteq \omega^\omega$ be closed. Define the rank $\mathrm{rk}_A(\sigma)$ for $\sigma \in \omega^{<\omega}$ by

$$\mathrm{rk}_A(\sigma) = \begin{cases} 0 & \text{if } [\sigma] \cap A \text{ is compact,} \\ \min\{\beta : \forall^\infty k\ (\sup\{\mathrm{rk}_A(\sigma\hat{\ }k\hat{\ }\tau) : \tau \in \omega^{<\omega}\} < \beta)\} & \text{otherwise.} \end{cases}$$

(So, in the noncompact case, the rank is always at least 1, and it is exactly 1 iff almost all $[\sigma\hat{\ }k] \cap A$ are compact.)

A standard and well-known argument shows that $A \cap [\sigma]$ is σ-compact iff $\mathrm{rk}_A(\tau) < \omega_1$ for all $\tau \supseteq \sigma$. (Indeed, if $\mathrm{rk}_A(\tau) = \infty$ for some $\tau \supseteq \sigma$, then we can recursively construct a superperfect tree T with $[T] \subseteq A$ with stem τ and splitting nodes the nodes of rank ∞. On the other hand, by induction on α we can show that if $\mathrm{rk}_A(\tau) \leq \alpha$ for all $\tau \supseteq \sigma$, then $A \cap [\sigma]$ is σ-compact: for $\alpha = 0$, this is obvious; for $\alpha > 0$, note that $T = \{\tau : \tau \supseteq \sigma \text{ and } \mathrm{rk}_A(\tau) = \alpha\}$ generates a finitely branching tree T and that $A \cap [\sigma] = [T] \cup \bigcup\{A \cap [\tau] : \tau \supseteq \sigma$ is such that $\rho_A(\rho) < \alpha$ for all $\rho \supseteq \tau\}$ is a union of a compact set and (by induction hypothesis) countably many σ-compact sets.)

For $B \subseteq \mathbb{Q}(2^\omega)$ and $s \in B$ define the rank ρ_B by $\rho_B(s) = \alpha$ if $s \in B^\alpha \setminus B^{\alpha+1}$, where B^α denotes the αth Cantor–Bendixson derivative of B, as usual. Note that B is scattered iff all $s \in B$ have rank $< \omega_1$.

Let $A \subseteq \mathbb{Q}(\omega^\omega)$. We claim that $\rho_{f^{-1}(A)}(f^{-1}(\sigma)) \leq \mathrm{rk}_{\bar{A}}(\sigma)$ for all $\sigma \in A \cap \mathrm{ran}(f)$. This implies in particular that if $\bar{A}$ is σ-compact, then $f^{-1}(A)$ is scattered (and if $\bar{A}$ is compact, then $f^{-1}(A)$ is discrete). Hence f witnesses $K_\sigma \leq_{\mathrm{KB}} \mathsf{scat}$.

We use induction on rank. If $\mathrm{rk}_{\bar{A}}(\sigma) = 0$, then $\bar{A} \cap [\sigma\hat{\ }k] = \emptyset$ for almost all k (by compactness). So there is k_0 such that $A \cap [\sigma\hat{\ }k] = \emptyset$ for all $k \geq k_0$. Hence $f^{-1}(A) \cap [f^{-1}(\sigma){\restriction}k_0] = \{f^{-1}(\sigma)\}$. This shows that $f^{-1}(\sigma)$ is isolated in $f^{-1}(A)$ and so $\rho_{f^{-1}(A)}(f^{-1}(\sigma)) = 0$. If $\mathrm{rk}_{\bar{A}}(\sigma) = \alpha > 0$, then for almost all k and all τ, $\mathrm{rk}_{\bar{A}}(\sigma\hat{\ }k\hat{\ }\tau) < \alpha$. Therefore, by induction hypothesis, for almost all k and all τ, $f^{-1}(\sigma\hat{\ }k\hat{\ }\tau)$ does not belong to $(f^{-1}(A))^\alpha$. Hence $f^{-1}(\sigma)$ is isolated in $(f^{-1}(A))^\alpha$ (or does not belong to it). Thus $\rho_{f^{-1}(A)}(f^{-1}(\sigma)) \leq \alpha$. ∎

COROLLARY 2.7. *Every K_σ-ultrafilter is a scattered ultrafilter.*

Using Observation 2.1 we now obtain the following diagram of inclusion relations between the ultrafilter classes where a line means that the class below is included in the class above:

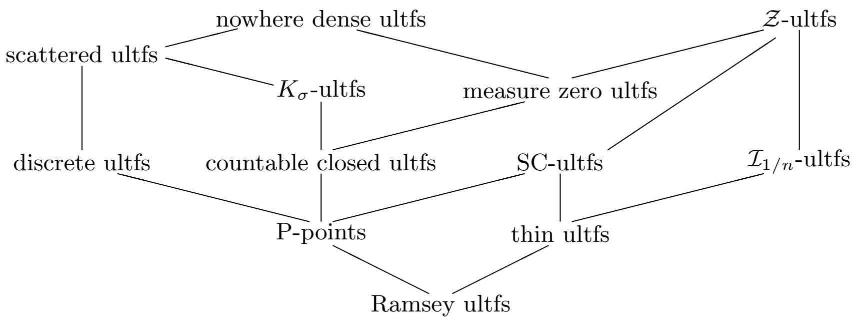

Fig. 2

No line can be added in ZFC. That is, if the ultrafilter class $\mathcal{C}$ is not below the ultrafilter class $\mathcal{D}$ in the diagram, then, *consistently*, there is an ultrafilter in $\mathcal{C}$ which does not belong to $\mathcal{D}$. This follows from Theorems 2.8 and 2.10 which will be proved below and from a number of known results which we now recall.

Baumgartner [Bau] proved that (under MA(σ-centered)) there is a measure zero ultrafilter that is not scattered. Assuming there is a P-point there is a countable closed ultrafilter that is neither discrete nor SC; in fact, if $\mathcal{U}$ is a P-point, then $\mathcal{U}^\omega$ is countable closed [Br2], the $\mathcal{U}^n$ are discrete but $\mathcal{U}^\omega$ is not [Bau], and $\mathcal{U}^2$ is not SC [Fl2, Proposition 3.2.2]. By results of Barney [Bar], it is consistent there are measure zero ultrafilters that are not K_σ-ultrafilters (in fact, by our Proposition 2.6, this is a consequence of Baumgartner's result cited at the beginning of this paragraph). The second author proved (under MA(countable)) that there is a P-point that is not a summable ultrafilter [Fl3] and that there is a summable ultrafilter that is not an SC-ultrafilter [Fl2, Corollary 2.3.7]. Hong and Zhang [HZ2] and (independently) the first author [Br4] proved that (under MA(σ-centered)) there is a discrete ultrafilter that is not density zero. In [HZ2], it is also proved that (under CH) there is a K_σ-ultrafilter that is not density zero.

We complete this cycle of results by showing, in the next two subsections, that under a fragment of MA there is a thin ultrafilter that is not nowhere dense and a discrete ultrafilter that is not K_σ.

2.2. A thin ultrafilter

THEOREM 2.8. *Assume MA(countable). There exists a thin ultrafilter which is not a nowhere dense ultrafilter.*

We use:

LEMMA 2.9. *Assume MA(countable). Assume $\mathcal{F}$ is a filter base on $\mathbb{Q}$ such that $|\mathcal{F}| < \mathfrak{c}$ and every $F \in \mathcal{F}$ is somewhere dense. Also assume $f : \mathbb{Q} \to \omega$ is a function. Then there exists $G \subseteq \mathbb{Q}$ such that $f[G]$ is thin and $G \cap F$ is somewhere dense for every $F \in \mathcal{F}$.*

Proof. If there is a set $F_0 \in \mathcal{F}$ such that $f[F_0]$ is thin then $G = F_0$ has the required property. So we may assume that $f[F]$ is not thin for any $F \in \mathcal{F}$. If there exists $K \in [\omega]^{<\omega}$ such that $f^{-1}(K) \cap F$ is somewhere dense for every $F \in \mathcal{F}$ then let $G = f^{-1}(K)$.

In the following we will assume that no such set exists, i.e.,

(♣) for every $K \in [\omega]^{<\omega}$ there is $F_K \in \mathcal{F}$ such that $f^{-1}(K) \cap F_K$ is nowhere dense in $\mathbb{Q}$.

CASE I: $\forall F \in \mathcal{F}\ \exists^\infty n$ *($f^{-1}\{n\} \cap F$ is somewhere dense).*

By assumption the set $S_F = \{n \in \omega : f^{-1}\{n\} \cap F \text{ is somewhere dense}\}$ is infinite for every $F \in \mathcal{F}$. Define a poset $\mathbb{P} = \{K \in [\omega]^{<\omega} : \forall u, v \in K \text{ (if } u < v \text{ then } u^2 < v)\}$ where $K \le_{\mathbb{P}} L$ iff $K = L$ or $K \supset L$ and $\min(K \setminus L) > \max L$. For every $F \in \mathcal{F}$ define $D_F = \{K \in \mathbb{P} : K \cap S_F \neq \emptyset\}$.

CLAIM 2.9.1. *D_F is dense in $\mathbb{P}$ for every $F \in \mathcal{F}$.*

Proof. Let $L \in \mathbb{P}$. Since S_F is infinite there exists $n \in S_F$ such that $n > (\max L)^2$. Put $K = L \cup \{n\}$. Obviously, $K \le_{\mathbb{P}} L$ and $K \in D_F$. ■

The family $\mathcal{D} = \{D_F : F \in \mathcal{F}\}$ consists of less than $\mathfrak{c}$ many dense sets in $\mathbb{P}$. By Martin's Axiom for countable posets there exists a $\mathcal{D}$-generic filter $\mathcal{G}$. Let $A = \bigcup\{K : K \in \mathcal{G}\}$.

Now it is easy to verify the following:

- $\forall F \in \mathcal{F}$ ($f^{-1}(A) \cap F$ is somewhere dense). (Given $F \in \mathcal{F}$ there is $K \in \mathcal{G}$ such that $K \cap S_F \neq \emptyset$. Thus also $A \cap S_F$ is non-empty and $f^{-1}(A) \cap F$ is somewhere dense.)
- A is thin. (Whenever a_i, a_{i+1} are two successive elements in A there exists $K \in \mathcal{G}$ such that $a_i, a_{i+1} \in K$. So we have $a_i^2 < a_{i+1}$ and $\lim_{i\to\infty} a_i/a_{i+1} \leq \lim_{i\to\infty} 1/a_i = 0$.)

To complete the proof in Case I, set $G = f^{-1}(A)$.

Case II: *$\exists F_0 \in \mathcal{F}\ \forall^\infty n$ ($f^{-1}\{n\} \cap F_0$ is nowhere dense).*

It follows from (♣) that we may actually assume that $f^{-1}\{n\} \cap F_0$ is nowhere dense for every $n \in \omega$. For every $F \in \mathcal{F}$ fix an open set $U_F \subseteq \mathbb{Q}$ such that $F_0 \cap F$ is dense in U_F (such a set exists because $F_0 \cap F$ is somewhere dense in $\mathbb{Q}$) and for every U_F fix its countable (clopen) base $\mathcal{B}_F = \{B_{F,i} : i \in \omega\}$.

Define a poset $\mathbb{P} = \{K \in [\mathbb{Q}]^{<\omega} : \forall u, v \in f[K]$ (if $u < v$ then $u^2 < v$)$\}$ with $K \leq_{\mathbb{P}} L$ iff $K \supseteq L$. For every $F \in \mathcal{F}$ and $i \in \omega$ define $D_{F,i} = \{K \in \mathbb{P} : K \cap F \cap B_{F,i} \neq \emptyset\}$.

Claim 2.9.2. *$D_{F,i}$ is dense in $\mathbb{P}$ for all $F \in \mathcal{F}$ and $i \in \omega$.*

Proof. Fix $L \in \mathbb{P}$. The set $f^{-1}[0, (\max f[L])^2] \cap F_0 \cap F$ is nowhere dense because it is a union of finitely many nowhere dense sets. So there is an open set $B' \subseteq B_{F,i}$ such that $B' \cap f^{-1}[0, (\max f[L])^2] \cap F_0 \cap F = \emptyset$. Since $F_0 \cap F$ is dense in U_F and B' is an open subset of U_F, the intersection $B' \cap F_0 \cap F$ is nonempty. Choose $q \in B' \cap F_0 \cap F$ and set $K = L \cup \{q\}$. Obviously, $K \leq_{\mathbb{P}} L$ and $K \in D_{F,i}$. ∎

The family $\mathcal{D} = \{D_{F,i} : F \in \mathcal{F},\ i \in \omega\}$ consists of less than $\mathfrak{c}$ many dense sets in $\mathbb{P}$. By Martin's Axiom for countable posets there exists a $\mathcal{D}$-generic filter $\mathcal{G}$. Let $G = \bigcup\{K : K \in \mathcal{G}\}$.

Now it is easy to verify that G has the required properties:

- $\forall F \in \mathcal{F}$ ($G \cap F$ is somewhere dense). (The set $G \cap F$ is dense in U_F because $G \cap F \cap B_{F,i} \neq \emptyset$ for every $i \in \omega$.)
- $f[G]$ is thin. (Let $f[G] = \{a_k : k \in \omega\}$ be an increasing enumeration of $f[G]$. For every a_k, a_{k+1} there exists $K \in \mathcal{G}$ such that $a_k, a_{k+1} \in f[K]$. So we have $a_k^2 < a_{k+1}$ and $\lim_{k\to\infty} a_k/a_{k+1} \leq \lim_{k\to\infty} 1/a_k = 0$.)

This completes the proof of the lemma. ∎

Proof of Theorem 2.8. Fix a bijection $b : \omega \to \mathbb{Q}$. Enumerate $\omega^\omega = \{f_\alpha : \alpha < \mathfrak{c}\}$. By transfinite induction on $\alpha < \mathfrak{c}$ we will construct filter bases $\mathcal{F}_\alpha$, $\alpha < \mathfrak{c}$, (on ω) so that the following conditions are satisfied:

(i) $\mathcal{F}_0$ is the Fréchet filter,
(ii) $\mathcal{F}_\alpha \subseteq \mathcal{F}_\beta$ whenever $\alpha \le \beta$,
(iii) $\mathcal{F}_\gamma = \bigcup_{\alpha<\gamma} \mathcal{F}_\alpha$ for γ limit,
(iv) $\forall \alpha\ (|\mathcal{F}_\alpha| \le |\alpha| \cdot \omega)$,
(v) $\forall \alpha\ \forall F \in \mathcal{F}_\alpha$ ($b[F]$ is somewhere dense in $\mathbb{Q}$),
(vi) $\forall \alpha\ \exists F \in \mathcal{F}_{\alpha+1}$ ($f_\alpha[F]$ is thin).

Suppose we already know $\mathcal{F}_\alpha$. If there is a set $F \in \mathcal{F}_\alpha$ such that $f_\alpha[F]$ is thin then set $\mathcal{F}_{\alpha+1} = \mathcal{F}_\alpha$. If $f_\alpha[F]$ is not thin for any $F \in \mathcal{F}_\alpha$ then apply Lemma 2.9 with the filter base $\{b[F] : F \in \mathcal{F}_\alpha\}$ and the function $f_\alpha \circ b^{-1}$. Let $\mathcal{F}_{\alpha+1}$ be the filter base generated by $\mathcal{F}_\alpha$ and $b^{-1}(G)$ where G is the set obtained in Lemma 2.9.

Finally, let $\mathcal{F} = \bigcup_{\alpha<\mathfrak{c}} \mathcal{F}_\alpha$. Since $b[F]$ is a somewhere dense subset of $\mathbb{Q}$ for every $F \in \mathcal{F}$, the filter base $\mathcal{F}$ can be extended to an ultrafilter which is not a nowhere dense ultrafilter. Every ultrafilter extending $\mathcal{F}$, however, is a thin ultrafilter because of condition (vi). ■

2.3. A discrete ultrafilter

THEOREM 2.10. *Assume MA(σ-centered). There is a discrete ultrafilter which is not a K_σ ultrafilter.*

Since it does not matter which representation of the rationals in 2^ω and ω^ω we use, for simplicity we shall work with $2^{<\omega}$ and $\omega^{<\omega}$ instead of $\mathbb{Q}(2^\omega)$ and $\mathbb{Q}(\omega^\omega)$, for discrete and K_σ sets, respectively.

Using an argument similar to the proof of Theorem 2.8 from Lemma 2.9, we see that it suffices to prove the following:

LEMMA 2.11. *Assume MA(σ-centered). Let $\mathcal{F} \subseteq K_\sigma^+$ be a filter base on $\omega^{<\omega}$ with $|\mathcal{F}| < \mathfrak{c}$ and let $f : \omega^{<\omega} \to 2^{<\omega}$. Then there is $B \subseteq \omega^{<\omega}$ such that $f[B]$ is discrete and $\mathcal{F} \cup \{B\}$ still generates a filter base in K_σ^+.*

Proof. If,

($*$) for some $s \in 2^{<\omega}$, $f^{-1}(\{s\}) \cap A \in K_\sigma^+$ for all $A \in \mathcal{F}$,

then $B = f^{-1}(\{s\})$ works. So assume this is not the case.

Next assume

($**$) for all $A \in \mathcal{F}$ there is $s \in 2^{<\omega}$ such that $f^{-1}(\{s\}) \cap A \in K_\sigma^+$.

Since $\mathcal{F}$ is a filter base and since ($*$) fails, for all $n \in \omega$ there is $s_n \in 2^n$ such that for all $A \in \mathcal{F}$ there is $s \in 2^{<\omega}$ with $s_n \subseteq s$ and $f^{-1}(\{s\}) \cap A \in K_\sigma^+$. Hence, by König's Lemma, there is $z \in 2^\omega$ such that for all $A \in \mathcal{F}$ and n there is $s \supseteq z{\restriction}n$ with $f^{-1}(\{s\}) \cap A \in K_\sigma^+$. Let $X = \{x : \omega \to 2^{<\omega} : x(n) \supseteq z{\restriction}n \text{ for all } n\}$ and, for $A \in \mathcal{F}$, set $X_A = \{x \in X : \forall n\ (f^{-1}(\{x(n)\}) \cap A \in K_\sigma)\}$. Then X_A is nowhere dense in X. Since $|\mathcal{F}| < \mathsf{cov}(\mathcal{M}) = \mathfrak{c}$, there is $c \in X$ with $c \notin X_A$ for all $A \in \mathcal{F}$. Hence, for all

$A \in \mathcal{F}$, there is n with $f^{-1}(\{c(n)\}) \cap A \in K_\sigma^+$, $\mathrm{ran}(c)$ is discrete, and so $B = f^{-1}(\mathrm{ran}(c))$ works. So assume that $(**)$ also fails.

Hence there is $A \in \mathcal{F}$ such that $f^{-1}(\{s\}) \cap A \in K_\sigma$ for all s. Since we may assume the set B we want to construct is a subset of A, it is irrelevant what f does outside of A and we may as well assume $f{\restriction}(\omega^{<\omega} \setminus A)$ is one-to-one. That is, without loss of generality, we may suppose that $A = \omega^{<\omega}$ and $f^{-1}(\{s\}) \in K_\sigma$ for all s. Say $F \subseteq \omega^{<\omega}$ is a *canonical finite tree* (*cft* for short) if F is a finite tree and $t \in F$, $|t'| \le |t|$ and $t' \le t$ (i.e., $t'(i) \le t(i)$ for all $i \in \mathrm{dom}(t')$) imply $t' \in F$. Let $Z := \{((\sigma_t, \tau_t) \in (\omega^{<\omega})^2 : t \in F) : F$ is a cft$\}$ be the collection of all sequences of pairs of elements of $\omega^{<\omega}$, indexed by elements of cfts (i.e., the collection of all functions with domain some cft and range contained in $(\omega^{<\omega})^2$). Clearly Z is a countable set. Consider the following p.o. $\mathbb{P}$. Conditions are pairs $p = (g^p, \Phi^p) = (g, \Phi)$ such that

(i) $g : \mathrm{ran}(f) \to \omega$ is a finite partial function,
(ii) for $s \ne r$ from $\mathrm{dom}(g)$, $[s{\restriction}g(s)] \cap [r{\restriction}g(r)] = \emptyset$,
(iii) for $A \in \mathcal{F}$, $A \setminus \bigcup\{f^{-1}[s{\restriction}g(s)] : s \in \mathrm{dom}(g)\} \in K_\sigma^+$,
(iv) $\Phi : \mathcal{F} \to Z$ is a finite partial function,
(v) for all $A \in \mathrm{dom}(\Phi)$, $\Phi(A) = ((\sigma_t^A, \tau_t^A) : t \in F_A)$ where $F_A = F_A^p$ is a cft, and for all $t \in F_A$:
   - the τ_t^A are the splitting nodes of an initial segment of a superperfect tree (this means that if $t\hat{\ }k \in F_A$, then $\tau_t^A \subset \tau_{t\hat{\ }k}^A$ and $\tau_{t\hat{\ }k'}^A(|\tau_t^A|) < \tau_{t\hat{\ }k}^A(|\tau_t^A|)$ for $k' < k$),
   - $\tau_t^A \subset \sigma_{t\hat{\ }k}^A$ and $\sigma_{t\hat{\ }k}^A{\restriction}|\tau_t^A| + 1 = \tau_{t\hat{\ }k}^A{\restriction}|\tau_t^A| + 1$,
   - $\sigma_t^A \in A \cap f^{-1}(\mathrm{dom}(g))$, and
   - $\exists^\infty n \; (A \cap [\tau_t^A\hat{\ }n] \setminus \bigcup\{f^{-1}[s{\restriction}g(s)] : s \in \mathrm{dom}(g)\} \in K_\sigma^+)$.

The order $q \le p$ is given by

- $g^q \supseteq g^p$,
- $\mathrm{dom}(\Phi^q) \supseteq \mathrm{dom}(\Phi^p)$,
- $\Phi^q(A) \supseteq \Phi^p(A)$ for all $A \in \mathrm{dom}(\Phi^p)$ (more explicitly, $F_A^q \supseteq F_A^p$ and $\sigma_t^{A,q} = \sigma_t^{A,p}$, $\tau_t^{A,q} = \tau_t^{A,p}$ for $t \in F_A^p$ and $A \in \mathrm{dom}(\Phi^p)$).

Claim 2.11.1. *$\mathbb{P}$ is σ-centered.*

Proof. Let $\{h_A : A \in \mathcal{F}\}$ be a family of functions from ω to Z such that for all finite partial functions $\Psi : \mathcal{F} \to Z$ there are infinitely many n with $h_A(n) = \Psi(A)$ for all $A \in \mathrm{dom}(\Psi)$.

(The construction of such h_A is a standard argument: let $Z = \{z_n : n \in \omega\}$. Construct a strictly increasing sequence $(k_n \in \omega : n \in \omega)$ and sequences $(\rho_v \in Z^{k_n} : v \in 2^n)$ for all n such that $v \subseteq w$ implies $\rho_v \subseteq \rho_w$ and for all n and all functions $\psi : 2^{n+1} \to \{z_j : j < n\}$ there is $i \in [k_n, k_{n+1})$ such that $\rho_v(i) = \psi(v)$ for all $v \in 2^{n+1}$. Let $\chi : \mathcal{F} \to 2^\omega$ be a one-to-one function and set $h_A = \bigcup\{\rho_{\chi(A){\restriction}n} : n \in \omega\} \in Z^\omega$. To see that this works fix a finite partial function $\Psi : \mathcal{F} \to Z$. Let n be large enough so that the $\chi(A){\restriction}n$ are all distinct for $A \in \mathrm{dom}(\Psi)$, and $\mathrm{ran}(\Psi) \subseteq \{z_j : j < n\}$. There is $\psi : 2^{n+1} \to \{z_j : j < n\}$ such that $\psi(\chi(A){\restriction}n+1) = \Psi(A)$ for $A \in \mathrm{dom}(\Psi)$. If $i \in [k_n, k_{n+1})$ is such that $\rho_v(i) = \psi(v)$ for all $v \in 2^{n+1}$, then $h_A(i) = \rho_{\chi(A){\restriction}n+1}(i) = \psi(\chi(A){\restriction}n+1) = \Psi(A)$ for all $A \in \mathrm{dom}(\Psi)$.)

For $n \in \omega$ and a finite partial function $g : \mathrm{ran}(f) \to \omega$, let $P_{n,g} = \{p \in \mathbb{P} : g^p = g \text{ and } \Phi^p(A) = h_A(n) \text{ for all } A \in \mathrm{dom}(\Phi^p)\}$. Any set R of finitely many conditions in $P_{n,g}$ clearly has a common extension, namely $(g, \bigcup\{\Phi^p : p \in R\})$. So the $P_{n,g}$ are centered. On the other hand, given $p \in \mathbb{P}$, the property of the h_A implies that $p \in P_{n,g^p}$ for infinitely many n. ∎

Claim 2.11.2. *For all k, $D_k = \{p : |\mathrm{dom}(g^p)| \ge k\}$ is dense.*

Proof. It suffices to show that given $p \in \mathbb{P}$ there is $q \le p$ with $|\mathrm{dom}(g^q)| = |\mathrm{dom}(g^p)| + 1$. Let $\ell := \sum\{|F_A| : A \in \mathrm{dom}(\Phi^p)\} + 2$. Choose distinct $s_i \in \mathrm{ran}(f) \setminus \bigcup\{[s{\restriction}g^p(s)] : s \in \mathrm{dom}(g^p)\}$, $i < \ell$. This choice is possible by condition (iii) and because $f^{-1}(\{s\}) \in K_\sigma$ for all s. Next choose the values $g(s_i)$ such that the $[s_i{\restriction}g(s_i)]$ are pairwise disjoint and also disjoint from the $[s{\restriction}g^p(s)]$.

Notice that for each $A \in \mathrm{dom}(\Phi^p)$ and each $t \in F_A$, there is at most one i such that the set

$$\Big\{n : A \cap [\tau_t^{A}{}^\frown n] \setminus \Big(\bigcup\{f^{-1}[s{\restriction}g^p(s)] : s \in \mathrm{dom}(g^p)\} \cup f^{-1}[s_i{\restriction}g(s_i)]\Big) \in K_\sigma^+\Big\}$$

is finite. (For suppose for $i = i_0, i_1$ and some $m \in \omega$, we had $A \cap [\tau_t^{A}{}^\frown n] \setminus (\bigcup\{f^{-1}[s{\restriction}g^p(s)] : s \in \mathrm{dom}(g^p)\} \cup f^{-1}[s_i{\restriction}g(s_i)]) \in K_\sigma$ for all $n > m$. Then we would also have $A \cap [\tau_t^{A}{}^\frown n] \setminus \bigcup\{f^{-1}[s{\restriction}g^p(s)] : s \in \mathrm{dom}(g^p)\} \in K_\sigma$ for all $n > m$, contradicting the last part of clause (v) for the condition p.) Hence there are at least two i's such that for all A and t the latter set is infinite, that is, adding either of the corresponding s_i preserves clause (v).

For at most one such i, adding s_i can violate clause (iii). (For suppose for $i = i_0, i_1$ we had $A_i \setminus (\bigcup\{f^{-1}[s{\restriction}g^p(s)] : s \in \mathrm{dom}(g^p)\} \cup f^{-1}[s_i{\restriction}g(s_i)]) \in K_\sigma$ for some $A_i \in \mathcal{F}$. Then $(A_{i_0} \cap A_{i_1}) \setminus \bigcup\{f^{-1}[s{\restriction}g^p(s)] : s \in \mathrm{dom}(g^p)\} \in K_\sigma$, contradicting (iii) for p.)

Hence there is i such that if we let $\Phi^q = \Phi^p$, $\mathrm{dom}(g^q) = \mathrm{dom}(g^p) \cup \{s_i\}$, and $g^q(s_i) = g(s_i)$, then q is a condition extending p. ∎

Since we may set $\Phi(A) = \emptyset$, it is easy to see that we may add any A to the domain of Φ. However, we may also extend the set $\Phi(A)$:

Claim 2.11.3. *Given any $A \in \mathcal{F}$ and any cft F, $E_{A,F} = \{p : A \in \mathrm{dom}(\Phi^p)$ and $F \subseteq F_A^p\}$ is dense.*

Proof. It suffices to prove that given $p \in \mathbb{P}$ with $A \in \mathrm{dom}(\Phi^p)$ and $t \in \omega^{<\omega}$ there is $q \le p$ with $t \in F_A^q$. We may assume that $t \notin F_A^p$ and either $t = \langle\rangle$ or, letting $t = u^\frown k$, $u \in F_A^p$ and $u^\frown k' \in F_A^p$ for all $k' < k$. For this is the single step for recursively building up the required q in finitely many steps, according to the definition of "cft". Since both cases are similar, we only treat the (somewhat more general) second case.

First, using the last part of clause (v), choose n such that $n > \tau^A_{u\hat{}k'}(|\tau^A_u|)$ for all $k' < k$ and

$$A \cap [\tau^A_u\hat{}n] \setminus \bigcup\{f^{-1}[s\restriction g^p(s)] : s \in \mathrm{dom}(g^p)\} \in K^+_\sigma.$$

Let $\ell := \sum\{|F_{A'}| : A' \in \mathrm{dom}(\Phi^p)\} + 3$. Choose $\sigma_i \in A \setminus \bigcup\{f^{-1}[s\restriction g^p(s)] : s \in \mathrm{dom}(g^p)\}$, $i < \ell$, with $\tau^A_u\hat{}n \subseteq \sigma_i$ and $s_i = f(\sigma_i)$ all distinct. This is clearly possible by choice of n and because $f^{-1}(\{s\}) \in K_\sigma$ for all s. By the argument in the second and third paragraphs of the proof of the previous claim we can find $i < \ell$ and extend p to a condition p' such that $s_i \in \mathrm{dom}(g^{p'})$ and additionally

$$A \cap [\tau^A_u\hat{}n] \setminus \bigcup\{f^{-1}[s\restriction g^{p'}(s)] : s \in \mathrm{dom}(g^{p'})\} \in K^+_\sigma.$$

By definition of K_σ, this means we can find $\tau^A_t \supseteq \tau^A_u\hat{}n$ such that

$$\exists^\infty n \left(A \cap [\tau^A_t\hat{}n] \setminus \bigcup\{f^{-1}[s\restriction g^{p'}(s)] : s \in \mathrm{dom}(g^{p'})\} \in K^+_\sigma\right).$$

Let $\sigma^A_t = \sigma_i$, and set $g^q = g^{p'}$, $\Phi^q(A) = \Phi^p(A) \cup \{(\sigma^A_t, \tau^A_t)\}$ and $\Phi^q(A') = \Phi^p(A')$ for $A' \neq A$ from $\mathrm{dom}(\Phi^p)$. Clearly q is a condition extending (p' and thus also) p. ∎

By MA(σ-centered), there is a filter $\mathcal{G} \subseteq \mathbb{P}$ meeting all D_k and all $E_{A,F}$. Let $g = \bigcup\{g^p : p \in \mathcal{G}\}$. By Claim 2.11.2, g is an infinite partial function from $\mathrm{ran}(f)$ to ω, and by (ii), $\mathrm{dom}(g)$ is a discrete set. By (v) and Claim 2.11.3, for all $A \in \mathcal{F}$, the $\{\tau^A_t : t \in F^p_A \text{ and } p \in \mathcal{G}\}$ are splitting nodes of a superperfect tree T_A such that its set of branches, $[T_A]$, is contained in the closure of $A \cap f^{-1}(\mathrm{dom}(g))$. Thus $B = f^{-1}(\mathrm{dom}(g))$ is as required. ∎

3. Generic existence

3.1. Basic results. Let $\mathcal{I}$ be a tall ideal on a countable set X. To simplify definitions and proofs below, we shall work with $X = \omega$. To characterize generic existence of $\mathcal{I}$-ultrafilters, we introduce the cardinal invariant $\mathfrak{ge}(\mathcal{I})$, called the *generic existence number* (5),

$$\mathfrak{ge}(\mathcal{I}) = \min\{|\mathcal{F}| : \mathcal{F} \text{ is a filter base, } \mathcal{F} \subseteq \mathcal{I}^+, \text{ and } \forall I \in \mathcal{I}\ \exists F \in \mathcal{F}\ (|I \cap F| < \aleph_0)\}.$$

This cardinal has been introduced independently by Hong and Zhang [HZ1] and called $\mathsf{non}^{**}(\mathcal{I})$ there. They also remarked independently the following:

(5) The stipulation $\mathcal{F} \subseteq \mathcal{I}^+$ in this definition is in fact redundant, for if there is $I \in \mathcal{I} \cap \mathcal{F}$, then $|I \cap F| = \aleph_0$ for all $F \in \mathcal{F}$. Our reason for keeping $\mathcal{F} \subseteq \mathcal{I}^+$ is that it makes the definition more natural.

Observation 3.1.

(i) *If* $\mathfrak{ge}(\mathcal{I}) = \mathfrak{c}$, *then any filter base of size* $< \mathfrak{c}$ *can be extended to an* $\mathcal{I}$*-ultrafilter.*
(ii) *There is a filter base of size* $\mathfrak{ge}(\mathcal{I})$ *which cannot be extended to an* $\mathcal{I}$*-point.*
(iii) *Every ultrafilter generated by less than* $\mathfrak{ge}(\mathcal{I})$ *many sets is an* $\mathcal{I}$*-ultrafilter.*
(iv) *The following are equivalent:*

(1) $\mathfrak{ge}(\mathcal{I}) = \mathfrak{c}$.
(2) *Generic existence of* $\mathcal{I}$*-ultrafilters.*
(3) *Generic existence of weak* $\mathcal{I}$*-ultrafilters.*
(4) *Generic existence of* $\mathcal{I}$*-points.*

Proof. (i) Once we prove that given a filter base $\mathcal{F}$ of size $< \mathfrak{c}$ and a function $f : \omega \to \omega$, we can find $G \in [\omega]^\omega$ such that $\mathcal{F} \cup \{F \cap G : F \in \mathcal{F}\}$ is still a filter base and $f(G) \in \mathcal{I}$, then a straightforward recursive construction of length $\mathfrak{c}$ produces the required $\mathcal{I}$-ultrafilter.

So assume $\mathcal{F}$ and f are given. If $f(F) \in \mathcal{I}$ for some $F \in \mathcal{F}$, we are done. Hence suppose that $f(\mathcal{F}) \subseteq \mathcal{I}^+$. Then, by assumption $\mathfrak{ge}(\mathcal{I}) = \mathfrak{c}$, there is $I \in \mathcal{I}$ such that $|I \cap f(F)| = \aleph_0$ for all $F \in \mathcal{F}$. Set $G = f^{-1}(I)$. Then $G \cap F$ is infinite for all $F \in \mathcal{F}$ and $f(G) = I \in \mathcal{I}$, so G is as required.

(ii) Let $\mathcal{F} \subseteq \mathcal{I}^+$ be a filter base of size $\mathfrak{ge}(\mathcal{I})$ such that for all $I \in \mathcal{I}$ there is $F \in \mathcal{F}$ with $I \cap F$ being finite. Let $f = \mathrm{id}$ be the identity function. Clearly there is no ultrafilter $\mathcal{U}$ containing $\mathcal{F}$ such that $U = f(U) \in \mathcal{I}$ for some $U \in \mathcal{U}$.

(iii) If $\mathcal{F}$ is a filter base of size $< \mathfrak{ge}(\mathcal{I})$ which generates an ultrafilter $\mathcal{U}$, and $f \in \omega^\omega$, then, by the argument of (i), $f(F) \in \mathcal{I}$ for some $F \in \mathcal{F}$ (the case $f(\mathcal{F}) \subseteq \mathcal{I}^+$ cannot happen because $\mathcal{U}$ is an ultrafilter).

(iv) follows from (i) and (ii). ∎

Observation 3.2. *For tall ideals* $\mathcal{I}$, $\mathfrak{ge}(\mathcal{I}) \geq \aleph_1$.

Proof. Assume $\mathcal{F}$ is a countable filter base. Then $\mathcal{F}$ has a pseudointersection $F \in [\omega]^\omega$. By tallness, there is a countable $I \subseteq F$ belonging to $\mathcal{I}$, and I has infinite intersection with all members of $\mathcal{F}$. Thus $\mathcal{F}$ cannot be a witness for the value of $\mathfrak{ge}(\mathcal{I})$. ∎

This cardinal is closely related to two other cardinal invariants of the ideal $\mathcal{I}$ which have been introduced by Hernández and Hrušák [HH], the *uniformity* and *cofinality* of $\mathcal{I}$:

$$\mathsf{non}^*(\mathcal{I}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^\omega \text{ and } \forall I \in \mathcal{I}\, \exists F \in \mathcal{F}\, (|I \cap F| < \aleph_0)\},$$
$$\mathsf{cof}^*(\mathcal{I}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \text{ and } \forall I \in \mathcal{I}\, \exists F \in \mathcal{F}\, (I \subseteq^* F)\}.$$

In fact, for the latter definition, the star is irrelevant for tall ideals:

$$\mathsf{cof}^*(\mathcal{I}) = \mathsf{cof}(\mathcal{I}) := \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \text{ and } \forall I \in \mathcal{I}\, \exists F \in \mathcal{F}\ (I \subseteq F)\}.$$

The point is that tallness implies that $\mathsf{cof}^*(\mathcal{I})$ is infinite, and then $\mathsf{cof}^*(\mathcal{I}) = \mathsf{cof}(\mathcal{I})$ follows. See also Subsection 1.2 for $\mathsf{cof}(\mathcal{I})$.

Observation 3.3. *For all ideals $\mathcal{I}$, $\mathsf{non}^*(\mathcal{I}) \le \mathfrak{ge}(\mathcal{I}) \le \mathsf{cof}(\mathcal{I})$.*

In fact, using a two-ideal version of the cofinality ([6]), we obtain a characterization of $\mathfrak{ge}(\mathcal{I})$. Let $\mathcal{I} \subseteq \mathcal{J}$ be ideals and set

$$\mathsf{cof}(\mathcal{I}, \mathcal{J}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{J} \text{ and } \forall I \in \mathcal{I}\, \exists F \in \mathcal{F}\ (I \subseteq F)\}.$$

Then $\mathsf{cof}(\mathcal{I}, \mathcal{I}) = \mathsf{cof}(\mathcal{I})$, and obviously $\mathsf{cof}(\mathcal{I}, \mathcal{J}) \le \mathsf{cof}(\mathcal{I})$ and $\mathsf{cof}(\mathcal{I}, \mathcal{J}) \le \mathsf{cof}(\mathcal{J})$. We have:

Observation 3.4. *For all ideals $\mathcal{I}$, $\mathfrak{ge}(\mathcal{I}) = \min\{\mathsf{cof}(\mathcal{I}, \mathcal{J}) : \mathcal{I} \subseteq \mathcal{J}\} = \min\{\mathsf{cof}(\mathcal{J}) : \mathcal{I} \subseteq \mathcal{J}\}$.*

Proof. If $\mathcal{F}$ witnesses the value of $\mathsf{cof}(\mathcal{I}, \mathcal{J})$, then clearly $\mathcal{F}^* = \{\omega \setminus F : F \in \mathcal{F}\} \subseteq \mathcal{I}^+$ is a filter base as in the definition of $\mathfrak{ge}(\mathcal{I})$. On the other hand, if $\mathcal{F} \subseteq \mathcal{I}^+$ is a filter base witnessing the value of $\mathfrak{ge}(\mathcal{I})$, then $\mathcal{F}^* = \{\omega \setminus F : F \in \mathcal{F}\}$ is a base of an ideal $\mathcal{J}$ which contains $\mathcal{I}$ (because, if $I \in \mathcal{I}$, then there is $F \in \mathcal{F}$ with $|I \cap F| < \aleph_0$, i.e. $I \subseteq^* \omega \setminus F$). ∎

It is well-known that the Katětov–Blass order is connected with uniformity non^* of ideals:

Observation 3.5 ([HH, Proposition 3.1] or [Hr1, Theorem 1.2]). *For all ideals $\mathcal{I} \le_{\mathrm{KB}} \mathcal{J}$, $\mathsf{non}^*(\mathcal{I}) \le \mathsf{non}^*(\mathcal{J})$.*

For generic existence, the Katětov order is enough:

Observation 3.6. *For all ideals $\mathcal{I} \le_{\mathrm{K}} \mathcal{J}$, $\mathfrak{ge}(\mathcal{I}) \le \mathfrak{ge}(\mathcal{J})$.*

Proof. Let $\mathcal{F}$ be a witness for the value of $\mathfrak{ge}(\mathcal{J})$, and let f be a Katětov reduction. We claim that $f(\mathcal{F})$ is a witness for the value of $\mathfrak{ge}(\mathcal{I})$. Clearly, if $F \in \mathcal{F}$, then $f(F) \in \mathcal{I}^+$, in particular $f(F)$ is infinite, and $f(\mathcal{F})$ forms a filter base. If $I \in \mathcal{I}$, then $f^{-1}(I) \in \mathcal{J}$, thus there is $F \in \mathcal{F}$ such that $f^{-1}(I) \cap F$ is finite, whence $I \cap f(F)$ is finite. ∎

Using the cardinal $\mathfrak{ge}(\mathcal{I})$ as well as the characterizations of some of the classical classes of ultrafilters in the language of $\mathcal{I}$-ultrafilters, as explained in Section 2 (see, in particular, Observation 2.1), we now obtain reformulations of the classical theorems about generic existence mentioned in the Introduction.

For example, in view of the characterization of P-points as conv-ultrafilters and $\mathsf{Fin} \times \mathsf{Fin}$-ultrafilters (Observation 2.1(i)), Ketonen's theorem [Ke]

([6]) Unlike the other definitions of cardinal invariants here, this definition will be used for ideals on arbitrary sets.

quoted in the Introduction may be strengthened to $\mathfrak{d} = \mathfrak{ge}(\mathsf{conv}) = \mathfrak{ge}(\mathsf{Fin} \times \mathsf{Fin})$. Hong and Zhang [HZ1, Theorem 3.6] give a direct proof of $\mathfrak{d} = \mathfrak{ge}(\mathsf{Fin} \times \mathsf{Fin})$, and we provide a direct proof of $\mathfrak{d} = \mathfrak{ge}(\mathsf{conv})$ in the appendix (Proposition 4.4) because we could not find a reference. Concerning the other cardinals mentioned above, it is known that $\mathsf{non}^*(\mathsf{conv}) = \mathsf{non}^*(\mathsf{Fin} \times \mathsf{Fin}) = \aleph_0$, $\mathsf{cof}(\mathsf{Fin} \times \mathsf{Fin}) = \mathfrak{d}$, and $\mathsf{cof}(\mathsf{conv}) = \mathfrak{c}$ (see [Me, Theorems 1.6.15 and 1.6.19]).

Similarly, since Ramsey ultrafilters are $\mathcal{ED}$-ultrafilters and $\mathcal{R}$-ultrafilters (Observation 2.1(ii)), Canjar's theorem [Ca] from the Introduction becomes $\mathsf{cov}(\mathcal{M}) = \mathfrak{ge}(\mathcal{ED}) = \mathfrak{ge}(\mathcal{R})$. See [HZ1, Theorem 3.7] for a direct proof for $\mathcal{ED}$, and Proposition 4.5 for a direct proof for $\mathcal{R}$. We have $\mathsf{non}^*(\mathcal{ED}) = \mathsf{non}^*(\mathcal{R}) = \aleph_0$ and $\mathsf{cof}(\mathcal{ED}) = \mathsf{cof}(\mathcal{R}) = \mathfrak{c}$ (see [Me, Theorems 1.6.4 and 1.6.30]).

Also, results by the first author characterizing generic existence of nowhere dense and measure zero ultrafilters [Br2, Theorems C, D, and F] say that $\mathfrak{ge}(\mathsf{nwd}) = \mathsf{cof}(\mathsf{nwd}) = \mathsf{cof}(\mathcal{M}) = \max\{\mathsf{non}(\mathcal{M}), \mathfrak{d}\}$ and $\mathfrak{ge}(\mathsf{mz}) = \mathsf{cof}(\mathcal{E}, \mathcal{M}) = \max\{\mathsf{non}(\mathcal{E}), \mathfrak{d}\}$. Furthermore $\mathsf{non}^*(\mathsf{mz}) = \mathsf{non}^*(\mathsf{nwd}) = \aleph_0$ (see Proposition 4.1 below for a stronger result). Also $\mathsf{cof}(\mathsf{mz}) = \mathsf{cof}(\mathcal{E}) = \mathsf{cof}(\mathcal{M})$ (see Subsection 1.2 and [Br2, Subsection 1.3]). In fact, $\mathfrak{ge}(\mathcal{I})$ can be characterized in terms of classical cardinal invariants, for all $\mathcal{I}$ with $\mathcal{I} \leq_{\mathrm{K}} \mathsf{nwd}$ in Figure 1 except for disc and scat (7).

We shall investigate generic existence of thin ultrafilters, SC-ultrafilters, summable ultrafilters, and density zero ultrafilters. Unfortunately, unlike the results in the previous paragraphs, we do not have nice characterizations of the cardinal $\mathfrak{ge}$ for these ultrafilters. There is, however, a reason for this. Using several ZFC-provable inequalities and some independence results, we will see that $\mathfrak{ge}(\mathcal{I})$ cannot be characterized as any of the classical cardinal invariants, for $\mathcal{I}$ being either thin or $\mathcal{SC}$ or $\mathcal{I}_{1/n}$ or $\mathcal{Z}$. Furthermore, we shall see that the $\mathfrak{ge}$-numbers are consistently distinct for these four ideals (see Corollary 3.34), and that, except for thin (see Conjecture 3.20), they are consistently larger than the corresponding non^*-numbers (Corollaries 3.24, 3.33, 3.38).

By $\mathfrak{ge}(\mathcal{ED}) = \mathsf{cov}(\mathcal{M})$ mentioned above, by Figure 1 (Subsection 2.1) and its discussion, and by Observation 3.6, we see:

Observation 3.7. $\mathsf{cov}(\mathcal{M}) \leq \mathfrak{ge}(\mathsf{thin}) \leq \mathfrak{ge}(\mathcal{SC}) \leq \mathfrak{ge}(\mathcal{Z}) \leq \mathsf{cof}(\mathcal{N})$ *and* $\mathfrak{ge}(\mathsf{thin}) \leq \mathfrak{ge}(\mathcal{I}_{1/n}) \leq \mathfrak{ge}(\mathcal{Z})$.

The last inequality of the first line uses Fremlin's $\mathsf{cof}(\mathcal{Z}) = \mathsf{cof}(\mathcal{N})$ [Fr2] and Observation 3.3.

(7) About the latter, we only know $\mathfrak{d} \leq \mathfrak{ge}(\mathsf{disc}) \leq \mathfrak{ge}(\mathsf{scat}) \leq \mathsf{cof}(\mathcal{M})$ and the consistency of $\mathfrak{ge}(\mathsf{scat}) < \mathsf{cof}(\mathcal{M})$ in the random model [Bau, Theorem 3.5]. See [Br2, Problem 4.4.(1)].

There are some results which can be proved in general for analytic ideals, in particular for ideals which have an easy description in terms of lower semicontinuous submeasures, like F_σ ideals and analytic P-ideals (see Subsection 1.1 for such ideals).

Recall that a function $\phi : \mathrm{dom}(\phi) \to [\omega]^{<\omega}$ is a *partial slalom* if $\mathrm{dom}(\phi) \subseteq \omega$ is infinite and $|\phi(n)| \le n$ for all $n \in \mathrm{dom}(\phi)$. If $f(n) \in \phi(n)$ for almost all $n \in \mathrm{dom}(\phi)$, we say the partial slalom ϕ *localizes* the function $f \in \omega^\omega$. Let pLoc denote the collection of partial slaloms. Similarly, Loc denotes the collections of total slaloms, i.e., those $\phi \in \mathsf{pLoc}$ with $\mathrm{dom}(\phi) = \omega$. Let

$$\begin{aligned}
\mathfrak{d}(\mathsf{pLoc}) &= \min\{|\Phi| : \Phi \subseteq \mathsf{pLoc} \text{ and} \\
&\qquad \forall g \in \omega^\omega\ \exists \phi \in \Phi\ \forall^\infty n \in \mathrm{dom}(\phi)\ (g(n) \in \phi(n))\}, \\
\mathfrak{d}(\mathsf{Loc}) &= \min\{|\Phi| : \Phi \subseteq \mathsf{Loc} \text{ and } \forall g \in \omega^\omega\ \exists \phi \in \Phi\ \forall^\infty n\ (g(n) \in \phi(n))\}, \\
\mathfrak{d}(\mathsf{pbdLoc}) &= \min\Big\{\kappa : \forall h \in \omega^\omega\ \exists \Phi \subseteq \mathsf{pLoc} \text{ with } |\Phi| \le \kappa \text{ and} \\
&\qquad \forall g \in \prod_n h(n)\ \exists \phi \in \Phi\ \forall^\infty n \in \mathrm{dom}(\phi)\ (g(n) \in \phi(n))\Big\}, \\
\mathfrak{d}(\mathsf{bdLoc}) &= \min\Big\{\kappa : \forall h \in \omega^\omega\ \exists \Phi \subseteq \mathsf{Loc} \text{ with } |\Phi| \le \kappa \text{ and} \\
&\qquad \forall g \in \prod_n h(n)\ \exists \phi \in \Phi\ \forall^\infty n\ (g(n) \in \phi(n))\Big\}.
\end{aligned}$$

Here p stands for "partial" and bd stands for "bounded". For later reference, we state what is known about these cardinals.

Fact 3.8.

(a) $\mathfrak{d}(\mathsf{Loc}) = \max\{\mathfrak{d}, \mathfrak{d}(\mathsf{bdLoc})\}$ *and* $\mathfrak{d}(\mathsf{pLoc}) = \max\{\mathfrak{d}, \mathfrak{d}(\mathsf{pbdLoc})\}$.
(b) (Bartoszyński) $\mathsf{cof}(\mathcal{M}) \le \mathfrak{d}(\mathsf{pLoc}) \le \mathfrak{d}(\mathsf{Loc}) = \mathsf{cof}(\mathcal{N})$.
(c) $\mathsf{cov}(\mathcal{N}), \mathsf{cov}(\mathcal{M}) \le \mathsf{cov}(\mathcal{E}) \le \mathfrak{d}(\mathsf{pbdLoc}) \le \mathfrak{d}(\mathsf{bdLoc})$.
(d) $\mathfrak{d}(\mathsf{pbdLoc})$ (*and thus any of the four cardinals defined here*) *is consistently larger than* $\mathfrak{d}$ *and* $\mathsf{non}(\mathcal{N})$.
(e) $\mathfrak{d}(\mathsf{bdLoc})$ (*and thus also* $\mathfrak{d}(\mathsf{pbdLoc})$) *is consistently smaller than* $\mathfrak{b}$ *and* $\mathfrak{s}$ (*and thus also smaller than* $\mathsf{non}(\mathcal{N})$ *and* $\mathsf{non}(\mathcal{M})$).

Proof. (a) is obvious. The equality in (b) is a classical result of Bartoszyński [Ba1], while the standard proof of the Bartoszyński–Raisonnier–Stern Theorem (see also [BJ, Section 2.3], [Fr1], and Subsection 1.2) in fact yields $\mathsf{cof}(\mathcal{M}) \le \mathfrak{d}(\mathsf{pLoc}) \le \mathsf{cof}(\mathcal{N})$ ([8]). In (c), only $\mathsf{cov}(\mathcal{E}) \le \mathfrak{d}(\mathsf{pbdLoc})$ needs proof (see the end of Subsection 1.2 for the cardinals related to $\mathcal{E}$).

([8]) In the literature, $\mathfrak{d}(\mathsf{pLoc})$ is also known as linear prediction number $\mathfrak{v}_\ell$, but since we work with slaloms and localization here and not with predictors, we prefer the notation $\mathfrak{d}(\mathsf{pLoc})$.

To see the latter, note that if $h \in \omega^\omega$ increases fast enough, say $h(n) \geq n2^n$ for all $n \in \omega$, then, for $\phi \in \mathsf{pLoc}$,

$$A_\phi = \Big\{ g \in \prod_n h(n) : \forall^\infty n \in \mathrm{dom}(\phi)\ (g(n) \in \phi(n)) \Big\}$$

is an F_σ null set in the space $\prod_n h(n)$ and thus belongs to the ideal $\mathcal{E}$. Hence $\mathsf{cov}(\mathcal{E}) \leq \mathfrak{d}(\mathsf{pbdLoc})$.

(d) holds in the random model (Subsection 1.3), by (c). Preservation of $\mathfrak{d}(\mathsf{bdLoc}) = \aleph_1$ by forcings with the Laver property (see [Ka] or [HMM]) implies that (e) holds in the Mathias model (Subsection 1.3) ([9]). ∎

Let φ be a lower semicontinuous submeasure, $f \in \omega^\omega$ a strictly increasing function, and $\bar{\epsilon} = (\epsilon_n > 0 : n \in \omega)$. Define

$$\begin{aligned}
\mathcal{X}_{f,\bar{\epsilon}} &= \{g : \omega \to [\omega]^{<\omega} : \forall n\ (g(n) \subseteq [f(n), f(n+1)) \text{ and } \varphi(g(n)) < \epsilon_n)\},\\
\mathcal{H}_{f,\bar{\epsilon}} &= \{h : \omega \to [[\omega]^{<\omega}]^{<\omega} \text{ partial with infinite domain} : \forall n \in \mathrm{dom}(h)\\
&\qquad (|h(n)| \leq n \text{ and } \forall a \in h(n)\ (a \subseteq [f(n), f(n+1)) \text{ and } \varphi(a) < \epsilon_n))\},\\
\mathcal{H}^0_{f,\bar{\epsilon}} &= \{h \in \mathcal{H}_{f,\bar{\epsilon}} : h \text{ is total}\}.
\end{aligned}$$

First assume $\mathcal{I} = \mathsf{Fin}(\varphi)$. Fix f such that $\varphi([f(n), f(n+1))) > n^3$ for all $n \in \omega$. Let $\bar{\omega} = (n : n \in \omega)$.

PROPOSITION 3.9. *There are functions*

$$\mathcal{H}_{f,\bar{\omega}} \to \mathcal{I}^+ \ (h \mapsto A_h) \quad \textit{and} \quad \mathcal{I} \to \mathcal{X}_{f,n}\ (B \mapsto g_B)$$

such that for all $h \in \mathcal{H}_{f,\bar{\omega}}$ and all $B \in \mathcal{I}$, if $g_B(n) \in h(n)$ for almost all $n \in \mathrm{dom}(h)$ then $|B \cap A_h| < \aleph_0$. Furthermore the A_h for $h \in \mathcal{H}^0_{f,\bar{\omega}}$ form a filter base in $\mathcal{I}^+$.

Proof. Let $A_h = \bigcup_{n \in \mathrm{dom}(h)} ([f(n), f(n+1)) \setminus \bigcup h(n))$. Notice that for $n \in \mathrm{dom}(h)$, $\varphi([f(n), f(n+1)) \setminus \bigcup h(n)) > n^3 - n^2$. So $\varphi(A_h) = \infty$ and $A_h \in \mathcal{I}^+$. The same argument shows that for finitely many $h \in \mathcal{H}^0_{f,\bar{\omega}}$, the intersection of the A_h still belongs to $\mathcal{I}^+$.

For $n > \varphi(B)$, let $g_B(n) = B \cap [f(n), f(n+1))$. For $n \leq \varphi(B)$, let $g_B(n) = \emptyset$.

Now assume that for almost all $n \in \mathrm{dom}(h)$, $g_B(n) \in h(n)$. Then for almost all n, $B \cap A_h \cap [f(n), f(n+1)) = \emptyset$. Thus $B \cap A_h$ is finite as required. ∎

Now assume $\mathcal{I} = \mathsf{Exh}(\varphi)$. Let $\delta_0 = \lim_n \varphi(\omega \setminus n)$. By definition of $\mathsf{Exh}(\varphi)$ and because $\omega \notin \mathcal{I}$, we have $\delta_0 > 0$ and possibly $\delta_0 = \infty$. This time $f \in \omega^\omega$ will be treated as an additional variable, but we only consider f such that $\varphi([f(n), f(n+1))) > \delta$ for all $n \in \omega$ where $\delta = \delta_0/2$ in case $\delta_0 < \infty$ and $\delta = 1$ otherwise. By lower semicontinuity any f increasing fast enough has

([9]) In [Ka] and [HMM], $\mathfrak{d}(\mathsf{bdLoc})$ is called $\mathfrak{l}$ because of its connections with the Laver property.

this property (for, given any $m \in \omega$, $\varphi(\omega \setminus m) \geq \delta_0$ implies that for every large enough ℓ, $\varphi([m, \ell)) > \delta$). Let $\epsilon_n = \delta/n^2$ for all $n \in \omega$.

Proposition 3.10. *There are functions*

$$\mathcal{H}_{f,\bar{\epsilon}} \to \mathcal{I}^+ \ (h_f \mapsto A_{h_f}), \qquad \mathcal{I} \to \omega^\omega \ (B \mapsto f_B), \qquad \mathcal{I} \to \mathcal{X}_{f,\epsilon_n} \ (B \mapsto g_{B,f})$$

such that for all $f \in \omega^\omega$, all $h_f \in \mathcal{H}_{f,\bar{\epsilon}}$ and all $B \in \mathcal{I}$, if $f_B \leq^* f$ and $g_{B,f}(n) \in h_f(n)$ for almost all $n \in \mathrm{dom}(h_f)$ then $|B \cap A_{h_f}| < \aleph_0$. Furthermore the A_{h_f} for $h_f \in \mathcal{H}^0_{f,\bar{\epsilon}}$ (for possibly distinct f) form a filter base in $\mathcal{I}^+$.

Proof. As in the previous proof, we let

$$A_{h_f} = \bigcup_{n \in \mathrm{dom}(h_f)} ([f(n), f(n+1)) \setminus \bigcup h_f(n)).$$

This time we have $\varphi([f(n), f(n+1)) \setminus \bigcup h_f(n)) > \delta - \delta/n$. So $\varphi(A_{h_f}) \geq \delta$ and $A_{h_f} \in \mathcal{I}^+$. If $h_{f_i} \in \mathcal{H}^0_{f_i,\bar{\epsilon}}$ for $i < k$, then, choosing large enough appropriate n_i, we can arrange that $\varphi(\bigcap_{i<k}[f_i(n_i), f_i(n_i + 1))) > \delta/2^k$, while $\varphi(\bigcup_{i<k} \bigcup h_{f_i}(n_i))$ is bounded by $\sum_{i<k} \epsilon_{n_i} n_i = \delta \sum_{i<k} 1/n_i$. Thus $\varphi(\bigcap_{i<k} A_{h_{f_i}}) \geq \delta/2^k$.

Next choose f_B such that $\varphi(B \setminus f_B(n)) < \epsilon_n$ for all n. Finally, define

$$g_{B,f}(n) = \begin{cases} B \cap [f(n), f(n+1)) & \text{if } \varphi(B \cap [f(n), f(n+1))) < \epsilon_n, \\ \emptyset & \text{otherwise.} \end{cases}$$

Notice that if $f_B \leq^* f$, then $g_{B,f}(n) = B \cap [f(n), f(n+1))$ for almost all n.

If we also assume that $g_{B,f}(n) \in h_f(n)$ for almost all $n \in \mathrm{dom}(h_f)$, then for almost all n, $B \cap A_{h_f} \cap [f(n), f(n+1)) = \emptyset$. Thus $B \cap A_{f_h}$ is finite as required. ■

Corollary 3.11.

(i) *Let $\mathcal{I}$ be an F_σ ideal. Then* $\mathsf{non}^*(\mathcal{I}) \leq \mathfrak{d}(\mathsf{pbdLoc})$ *and* $\mathfrak{ge}(\mathcal{I}) \leq \mathfrak{d}(\mathsf{bdLoc})$.

(ii) *Let $\mathcal{I}$ be an analytic P-ideal. Then* $\mathsf{non}^*(\mathcal{ED}_{\mathsf{fin}}) \leq \mathsf{non}^*(\mathcal{I}) \leq \mathfrak{d}(\mathsf{pLoc})$ *and* $\mathfrak{ge}(\mathcal{I}) \leq \mathsf{cof}(\mathcal{N})$.

Proof. (i) Since $\mathcal{I}$ is an F_σ ideal, by Mazur's Theorem there is a lower semicontinuous submeasure φ such that $\mathcal{I} = \mathsf{Fin}(\varphi)$. Let f and $\bar{\omega}$ be as in the paragraph before Proposition 3.9. Since $\mathcal{X}_{f,\bar{\omega}}$ can be identified with the space of functions in ω^ω bounded by $\bar{h} \in \omega^\omega$ where $\bar{h}(n) = |\{a \subseteq [f(n), f(n+1)) : \varphi(a) < n\}|$, the definition of $\mathfrak{d}(\mathsf{pbdLoc})$ gives us $\mathcal{B} \subseteq \mathcal{H}_{f,\bar{\omega}}$ of size $\mathfrak{d}(\mathsf{pbdLoc})$ such that for all $g \in \mathcal{X}_{f,\bar{\omega}}$ there is $h \in \mathcal{B}$ such that $g(n) \in h(n)$ for almost all $n \in \mathrm{dom}(h)$, Let $\mathcal{A} = \{A_h : h \in \mathcal{B}\}$. Then $|\mathcal{A}| \leq \mathfrak{d}(\mathsf{pbdLoc})$ and $\mathcal{A}$ is a witness for $\mathsf{non}^*(\mathcal{I})$ by 3.9 ([10]).

For the second part of the statement, use the last clause of 3.9 to obtain a witness for $\mathfrak{ge}(\mathcal{I})$ of size $\leq \mathfrak{d}(\mathsf{bdLoc})$.

(ii) The first inequality follows from [HH, Proposition 3.2].

Since $\mathcal{I}$ is an analytic P-ideal, by Solecki's Theorem there is a lower semicontinuous submeasure φ such that $\mathcal{I} = \mathsf{Exh}(\varphi)$. Let $\bar{\epsilon} = (\epsilon_n : n \in \omega)$ be as in the paragraph before Proposition 3.10. Let $\mathcal{F}$ be a dominating family of size $\mathfrak{d}$. Fix $f \in \mathcal{F}$. Since $\mathcal{X}_{f,\epsilon}$ can be identified with the space of functions in ω^ω bounded by $\bar{h} \in \omega^\omega$ where $\bar{h}(n) = |\{a \subseteq [f(n), f(n+1)) : \varphi(a) < \epsilon_n\}|$, the definition of $\mathfrak{d}(\mathsf{pbdLoc})$ gives us $\mathcal{B}_f \subseteq \mathcal{H}_{f,\bar{\epsilon}}$ of size $\leq \mathfrak{d}(\mathsf{pbdLoc})$ such that for all $g \in \mathcal{X}_{f,\bar{\epsilon}}$ there is $h_f \in \mathcal{B}_f$ such that $g(n) \in h_f(n)$ for almost all $n \in \mathrm{dom}(h_f)$. Let $\mathcal{A} = \{A_{h_f} : h_f \in \mathcal{B}_f$ and $f \in \mathcal{F}\}$. Then $|\mathcal{A}| \leq \max\{\mathfrak{d}, \mathfrak{d}(\mathsf{pbdLoc})\} = \mathfrak{d}(\mathsf{pLoc})$ and $\mathcal{A}$ is a witness for $\mathsf{non}^*(\mathcal{I})$: given $B \in \mathcal{I}$, first find $f \in \mathcal{F}$ dominating f_B and then $h_f \in \mathcal{B}_f$ such that $g_{B,f}(n) \in h_f(n)$ for almost all $n \in \mathrm{dom}(h_f)$; thus $B \cap A_{h_f}$ is finite by 3.10.

For the second part of the statement, use the last clause of 3.10 as well as $\mathfrak{d}(\mathsf{Loc}) = \mathsf{cof}(\mathcal{N})$ to obtain a witness for $\mathfrak{ge}(\mathcal{I})$. ∎

Of course, the second part of (ii) is not a new result because it also follows from Todorčević's $\mathsf{cof}(\mathcal{I}) \leq \mathsf{cof}(\mathcal{N})$ ([To], see also [Ba2, Theorem 4.23]). (i) strengthens a result of Hrušák et al. [HMM, Corollary 4.6] who proved $\mathsf{non}^*(\mathcal{I}) \leq \mathfrak{d}(\mathsf{bdLoc})$ for F_σ ideals $\mathcal{I}$. See Subsection 4.2 for results dual to Corollary 3.11.

COROLLARY 3.12.

(i) $\mathfrak{ge}(\mathcal{I}) < \min\{\mathfrak{b}, \mathfrak{s}\}$ *is consistent for all* F_σ *ideals* $\mathcal{I}$ *(simultaneously).*
(ii) *It is consistent that P-points generically exist and for any* F_σ *ideal,* $\mathcal{I}$*-ultrafilters do not.*

Both statements hold in the Mathias model.

Proof. (i) This follows from Fact 3.8(e) and Corollary 3.11(i).

(ii) This is immediate from (i). Recall here that P-points exist generically (by Ketonen's Theorem [Ke]) iff $\mathfrak{d} = \mathfrak{c}$, and that $\max\{\mathfrak{b}, \mathfrak{s}\} \leq \mathfrak{d}$ (see Subsection 1.2).

An alternative model for (ii) with continuum arbitrarily large can be obtained by starting with a model of MA and going through all forcings of type $\mathbb{M}(\mathcal{I}^*)$, where $\mathcal{I}$ is an F_σ ideal which arises in some intermediate extension, cofinally often in a finite support iteration whose length has co-

([10]) Since $\mathcal{A} \subseteq \mathcal{I}^+$, this proof really shows $\mathsf{non}^+(\mathcal{I}) \leq \mathfrak{d}(\mathsf{pbdLoc})$—see Proposition 4.1(v) below. A similar comment applies to the proof of (ii) and Proposition 4.1(vi).

finality ω_1 ([11]). See Subsection 1.3 for this forcing, and see below in Theorem 3.14 through Corollary 3.16 for why the iteration forces $\mathfrak{ge}(\mathcal{I}) = \aleph_1$. To see the iteration preserves $\mathfrak{d} = \mathfrak{c}$, use the fact that $\mathbb{M}(\mathcal{I}^*)$ does not add dominating reals [Br1] (for this, see again Subsection 1.3). ∎

Conjecture 3.13. *If $\mathcal{I}$ is an analytic P-ideal, then $\mathfrak{ge}(\mathcal{I}) \leq \mathfrak{d}(\mathsf{pLoc})$, and if $\mathcal{I}$ is an F_σ ideal, then $\mathfrak{ge}(\mathcal{I}) \leq \mathfrak{d}(\mathsf{pbdLoc})$.*

Theorem 3.14. *Let $\mathcal{I}$ be an analytic ideal such that $\mathbb{M}(\mathcal{I}^*)$ generically adds an $\mathcal{I}$-positive set which has $\mathcal{I}$-positive intersection with all ground model $\mathcal{I}$-positive sets. Then $\mathfrak{ge}(\mathcal{I}) < \mathsf{non}(\mathcal{N})$ is consistent and holds in the dual $\mathbb{M}(\mathcal{I}^*)$ model.*

Proof. $\mathbb{M}(\mathcal{I}^*)$ adds a real r that is almost disjoint from all $I \in \mathcal{I}$ in the ground model (see Subsection 1.3 for the properties of $\mathbb{M}(\mathcal{I}^*)$). Furthermore $r \cap B$ is infinite for all $B \in \mathcal{I}^+$ from the ground model. We assume additionally that $r \cap B \in \mathcal{I}^+$ for all $B \in \mathcal{I}^+$ from the ground model.

Now start with a model of MA + $\mathfrak{c} \geq \aleph_2$ and perform a finite support iteration of $\mathbb{M}(\mathcal{I}^*)$ of length ω_1 to obtain the dual $\mathbb{M}(\mathcal{I}^*)$ model. Let $(r_\alpha : \alpha < \omega_1)$ be the sequence of generics. By applying the assumption (see the previous paragraph) in appropriate intermediate models, we see that any finite intersection of the r_α is still $\mathcal{I}$-positive. In particular, they form a filter base. On the other hand, any $I \in \mathcal{I}$ lies in some intermediate extension, and therefore there is an α such that $r_\alpha \cap I$ is finite. Thus the r_α are a witness for $\mathfrak{ge}(\mathcal{I}) = \aleph_1$ in the final model.

On the other hand, the dual $\mathbb{M}(\mathcal{I}^*)$ model satisfies $\mathsf{non}(\mathcal{N}) = \mathfrak{c}$ (see Subsection 1.3) ([12]). ∎

Lemma 3.15. *Let $\mathcal{I}$ be either an F_σ ideal or an analytic P-ideal. Then $\mathbb{M}(\mathcal{I}^*)$ generically adds an $\mathcal{I}$-positive set which has $\mathcal{I}$-positive intersection with all ground model $\mathcal{I}$-positive sets.*

Proof. We again use the characterization of such ideals via lower semicontinuous submeasures φ. Consider first the case $\mathcal{I} = \mathsf{Fin}(\varphi)$. Then, given any $n \in \omega$, any $A \in \mathcal{I}^+$ and any condition $(s, B) \in \mathbb{M}(\mathcal{I}^*)$, there is t with $s \subseteq t \subseteq s \cup B$ such that $\varphi(t \cap A) > n$ (because $\varphi(B \cap A) = \infty$). Hence $\varphi(r \cap A) = \infty$ where r is the generic.

Next let $\mathcal{I} = \mathsf{Exh}(\varphi)$. Let $A \in \mathcal{I}^+$. Hence $\delta = \lim_n \varphi(A \setminus n) > 0$. Note that $\lim_n \varphi((A \cap B) \setminus n) = \delta$ for $B \in \mathcal{I}^*$. Thus, given any $n \in \omega$ and any condition

([11]) Note that this is not a dual $\mathbb{M}(\mathcal{I}^*)$ model even though all iterands are of the form $\mathbb{M}(\mathcal{I}^*)$.

([12]) In Proposition 3.21 below, we will show that $\mathfrak{d}(\mathsf{bd} \neq) = \mathfrak{c}$ (see the beginning of Subsection 3.2 for the definition of this cardinal) in the dual $\mathbb{M}(\mathcal{I}^*)$ model. By $\mathfrak{d}(\mathsf{bd} \neq) \leq \mathsf{non}(\mathcal{N})$ (see Fact 3.17(d)), this is even a stronger result.

$(s, B) \in \mathbb{M}(\mathcal{I}^*)$, there is t with $s \subseteq t \subseteq s \cup B$ such that $\varphi((A \cap t) \setminus n) > \delta/2$. Therefore we see again that $r \cap A \in \mathcal{I}^+$ where r is the generic. ∎

Notice that this is not true for analytic ideals in general. Consider for example $\mathcal{I} = \mathsf{Fin} \times \mathsf{Fin}$. Since the vertical sections are in the ideal, the Mathias generic for $\mathcal{I}^*$ has finite intersection with all vertical sections. This means, however, that it belongs to $\mathcal{I}$. Of course, since $\mathfrak{ge}(\mathcal{I}) = \mathfrak{d}$ (see the discussion after Observation 3.6), $\mathfrak{ge}(\mathcal{I}) < \mathsf{non}(\mathcal{N})$ is still consistent.

Corollary 3.16. *Let $\mathcal{I}$ be either an F_σ ideal or an analytic P-ideal. Then $\mathfrak{ge}(\mathcal{I}) < \mathsf{non}(\mathcal{N})$ is consistent and holds in the dual $\mathbb{M}(\mathcal{I}^*)$ model.*

For F_σ ideals, this alternatively follows from 3.12(i) because $\mathfrak{s} \leq \mathsf{non}(\mathcal{N})$ in ZFC (see Subsection 1.2). Also, if Conjecture 3.13 were true, Corollary 3.16 would follow from the known consistency of $\mathfrak{d}(\mathsf{pLoc}) < \mathsf{non}(\mathcal{N})$. The latter holds in the dual $\mathbb{PLOC}$ model where $\mathbb{PLOC}$ is the standard forcing for adding a generic partial slalom localizing all ground model functions (see, e.g., [Br3, p. 47] for the definition of this forcing). A finite support iteration of $\mathbb{PLOC}$ over a model of $\mathrm{MA} + \mathfrak{c} \geq \aleph_2$ of length ω_1 generically adds a witness for $\mathfrak{d}(\mathsf{pLoc}) = \aleph_1$, while $\mathbb{PLOC}$ is σ-centered and the preservation of $\mathsf{non}(\mathcal{N}) = \mathfrak{c}$ is proved like for dual $\mathbb{M}(\mathcal{I}^*)$ type models (see Subsection 1.3).

3.2. Thin ultrafilters. We already know (see Section 2) that thin ultrafilters and $\mathcal{ED}_{\mathsf{fin}}$-ultrafilters are the same, and that the Q-points are exactly the weakly thin ultrafilters and the weak $\mathcal{ED}_{\mathsf{fin}}$-ultrafilters. Define the following cardinals.

$$\mathfrak{d}(\neq) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \text{ and } \forall g \in \omega^\omega\ \exists f \in \mathcal{F}\ \forall^\infty n\ (f(n) \neq g(n))\},$$
$$\mathfrak{d}(\mathsf{p}\neq) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \text{ consists of partial functions with infinite domain and } \forall g \in \omega^\omega\ \exists f \in \mathcal{F}\ \forall^\infty n \in \mathrm{dom}(f)\ (f(n) \neq g(n))\},$$
$$\mathfrak{d}(\mathsf{bd}\neq) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \text{ is bounded and } \forall g \in \omega^\omega\ \exists f \in \mathcal{F}\ \forall^\infty n\ (f(n) \neq g(n))\},$$
$$\mathfrak{d}(\mathsf{pbd}\neq) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \text{ is bounded, consists of partial functions with infinite domain, and } \forall g \in \omega^\omega\ \exists f \in \mathcal{F}\ \forall^\infty n \in \mathrm{dom}(f)\ (f(n) \neq g(n))\}.$$

For later reference, we state what is known about these cardinals.

Fact 3.17.

(a) $\mathfrak{d}(\mathsf{p}\neq) = \min\{\mathfrak{d}, \mathfrak{d}(\mathsf{pbd}\neq)\}$ *and* $\mathfrak{d}(\mathsf{p}\neq) \leq \mathfrak{d}(\mathsf{pbd}\neq) \leq \mathfrak{d}(\mathsf{bd}\neq)$ [13].
(b) (Bartoszyński and Miller) $\mathsf{cov}(\mathcal{M}) = \mathfrak{d}(\neq) = \mathfrak{d}(\mathsf{p}\neq)$.

[13] It is also known that the value of $\mathfrak{d}(\mathsf{pbd}\neq)$ does not depend on the bound h as long as $h(n)$ goes to infinity (see, e.g., [BrG, Lemma 14]). On the other hand, using the methods of [KO], one can prove that for different bounds $h \in \omega^\omega$, the cardinals $\mathfrak{d}_h(\mathsf{bd}\neq)$ may simultaneously assume many different values.

(c) $\mathfrak{d}(\mathsf{pbd}\neq) \leq \mathfrak{d}(\mathsf{pbdLoc})$ *and* $\mathfrak{d}(\mathsf{bd}\neq) \leq \mathfrak{d}(\mathsf{bdLoc})$.
(d) $\mathfrak{d}(\mathsf{bd}\neq) \leq \mathsf{non}(\mathcal{N})$.
(e) $\mathfrak{d}(\neq) < \min\{\mathfrak{d}, \mathfrak{d}(\mathsf{bd}\neq)\}$ *is consistent* (14).
(f) $\mathfrak{d}(\mathsf{pbd}\neq) > \mathsf{cof}(\mathcal{M}) = \mathfrak{d} = \mathsf{cov}(\mathcal{M})$ *is consistent.*

Proof. (a) To see the equality, let $\mathcal{F}$ be a witness for the value of $\mathfrak{d}(\mathsf{p}\neq)$. We may assume $\mathfrak{d} > |\mathcal{F}|$ (otherwise we are done). Hence there is $h \in \omega^\omega$ such that for all $f \in \mathcal{F}$, $f(n) < h(n)$ for infinitely many $n \in \mathrm{dom}(f)$. Thus, for $f \in \mathcal{F}$, letting g_f be the restriction of f to $\{n \in \mathrm{dom}(f) : f(n) < h(n)\}$, we see that $\mathcal{G} = \{g_f : f \in \mathcal{F}\}$ is a family of partial functions bounded by h and witnessing the value of $\mathfrak{d}(\mathsf{pbd}\neq)$.

(b) is the Bartoszyński–Miller characterization of $\mathsf{cov}(\mathcal{M})$ [BJ, Section 2.4] (see also [Ba1]), and (c) is obvious.

(d) Note that if $h \in \omega^\omega$ increases fast enough, say $h(n) \geq 2^n$ for all $n \in \omega$, and $g \in \prod_n h(n)$, then

$$A_g = \Big\{x \in \prod_n h(n) : \exists^\infty n\ (x(n) = g(n))\Big\}$$

defines a G_δ null set. Thus if $B \subseteq \prod_n h(n)$ is nonnull, for all $g \in \prod_n h(n)$ there is $f \in B$ with $f \notin A_g$, i.e., $f(n) \neq g(n)$ for almost all n. Hence B is a witness for $\mathfrak{d}(\mathsf{bd}\neq)$.

For (e) see [GJS, Theorems 0.16 and 3.8] (an alternative model is the dual eventually different reals model), while (f) holds in the dual Hechler model (see [Me, Theorem 1.6.12]). ∎

Concerning cardinal invariants related to the ideals $\mathcal{ED}_{\mathsf{fin}}$ and thin, we have $\mathsf{cof}(\mathcal{ED}_{\mathsf{fin}}) = \mathsf{cof}(\mathsf{thin}) = \mathfrak{c}$ (see [Me, Theorem 1.6.6] for $\mathcal{ED}_{\mathsf{fin}}$ and Proposition 4.6 below for thin). Furthermore:

Proposition 3.18.

(i) (Hrušák et al. [HMM]) $\mathsf{non}^*(\mathcal{ED}_{\mathsf{fin}}) \leq \mathfrak{r}$.
(ii) (Hrušák et al. [HMM]) $\mathsf{non}^*(\mathcal{ED}_{\mathsf{fin}}) = \mathfrak{d}(\mathsf{pbd}\neq)$.
(iii) $\mathsf{non}^*(\mathsf{thin}) = \mathfrak{d}(\mathsf{pbd}\neq)$.
(iv) $\mathfrak{d}(\mathsf{pbd}\neq) \leq \mathfrak{ge}(\mathcal{ED}_{\mathsf{fin}}) = \mathfrak{ge}(\mathsf{thin}) \leq \mathfrak{d}(\mathsf{bd}\neq)$.

Proof. (iii) is immediate by (ii), $\mathcal{ED}_{\mathsf{fin}} \cong_{KB} \mathsf{thin}$, and Observation 3.5.

(iv) The first inequality is obvious by (ii), (iii), and Observation 3.3, so let us prove the second. Let $g \in \omega^\omega$. Clearly it suffices to show that there are functions $\prod_n g(n) \to \mathcal{ED}^+_{\mathsf{fin}} : f \mapsto X_f$ and $\mathcal{ED}_{\mathsf{fin}} \to \prod_n g(n) : Y \mapsto h_Y$ such

(14) In an earlier version of this paper we erroneously also claimed that $\mathsf{cov}(\mathcal{M}) = \min\{\mathfrak{d}, \mathfrak{d}(\mathsf{bd}\neq)\}$. We thank Michael Hrušák for pointing out that this is false and that $\mathsf{cov}(\mathcal{M}) < \min\{\mathfrak{d}, \mathfrak{d}(\mathsf{bd}\neq)\}$ in the model of [GJS]. He also pointed out that $\mathsf{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathfrak{d}(\mathsf{pbd}\neq)\} = \min\{\mathfrak{b}, \mathfrak{d}(\mathsf{bd}\neq)\}$. The proof is like the one of (a), using additionally $\mathsf{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathsf{cov}(\mathcal{M})\}$ (see Subsection 1.2).

that the X_f for $f \in \prod_n g(n)$ form a filter base and whenever $f(n) \neq h_Y(n)$ for almost all n then $Y \cap X_f$ is finite.

To this end, partition ω into intervals I_n, $n \in \omega$, of length n^2, and let each I_n be a union of intervals J_n^j, $j < n$, of length n. Let k_n be the number of sequences of the form $s{\restriction}I_n$ with $s(i) < g(i)$ for every $i \in I_n$, and let ψ_n be a bijection between $\prod_{i \in I_n} g(i)$ and k_n. For $f \in \prod_n g(n)$, let

$$X_f = \Big\{(k_n, \psi_n(s)) : n \in \omega,\ s \in \prod_{i \in I_n} g(i) \text{ and } \exists j < n\ (s{\restriction}J_n^j = f{\restriction}J_n^j)\Big\}.$$

If f_ℓ, $\ell < m$, from $\prod_n g(n)$ are given, for any $n \geq m$ we can find $s \in \prod_{i \in I_n} g(i)$ with $s{\restriction}J_n^\ell = f_\ell{\restriction}J_n^\ell$ for all $\ell < m$. Hence $\bigcap_{\ell<m} X_{f_\ell} \neq \emptyset$, and in fact, since the number of such s goes to infinity as $n \to \infty$, it follows that $\bigcap_{\ell<m} X_{f_\ell} \in \mathcal{ED}_{\mathsf{fin}}^+$.

Given $Y \in \mathcal{ED}_{\mathsf{fin}}$ find functions h_ℓ, $\ell < m$, below the identity such that $Y \subseteq \bigcup_{\ell<m} h_\ell$. For $n \geq m$, define $h_Y{\restriction}I_n$ such that h_Y agrees with $\psi_n^{-1}(h_\ell(k_n))$ at the ℓth element of J_n^j for each $\ell < m$ and $j < n$. We need to check that X_f and h_Y are as required.

Suppose that $f(i) \neq h_Y(i)$ for almost all i, say, for all $i \geq i_0$. We may assume that $i_0 = \min(I_{n_0})$ for some $n_0 \geq m$. Let $n \geq n_0$. Assume $(k_n, \psi_n(s)) \in X_f$. Then $s{\restriction}J_n^j = f{\restriction}J_n^j$ for some $j < n$. On the other hand, for each $\ell < m$, $\psi_n^{-1}(h_\ell(k_n))$ agrees with h_Y at the ℓth element of J_n^j. Since f and h_Y disagree everywhere on this interval, $s{\restriction}J_n^j \neq \psi_n^{-1}(h_\ell(k_n)){\restriction}J_n^j$. Thus $\psi_n(s) \neq h_\ell(k_n)$. Since this is true for every $\ell < m$, we have $(k_n, \psi_n(s)) \notin Y$. Therefore $Y \cap X_f$ is finite. ∎

Corollary 3.19. $\mathfrak{ge}(\mathsf{thin}) > \mathsf{cof}(\mathcal{M})$ *is consistent. In particular, it is consistent that thin ultrafilters generically exist while nowhere dense ultrafilters do not. In fact, this holds in the dual Hechler model.*

The first statement follows from the proposition and Fact 3.17(f). For the second statement, recall that $\mathfrak{ge}(\mathsf{nwd}) = \mathsf{cof}(\mathcal{M})$ (see the discussion after Observation 3.6).

This should be compared with Theorem 2.8. It is an alternative method for (consistently) obtaining thin ultrafilters that are not nowhere dense.

Conjecture 3.20. $\mathfrak{d}(\mathsf{pbd}\neq) = \mathfrak{ge}(\mathcal{ED}_{\mathsf{fin}}) = \mathfrak{ge}(\mathsf{thin})$.

The second inequality in Proposition 3.18(iv) above is consistently strict:

Proposition 3.21. $\mathfrak{ge}(\mathcal{ED}_{\mathsf{fin}}) < \mathfrak{d}(\mathsf{bd}\neq)$ *is consistent. In fact, this holds in the model of Theorem* 3.14.

Proof. Since $\mathcal{ED}_{\mathsf{fin}}$ is an F_σ ideal, $\mathfrak{ge}(\mathcal{ED}_{\mathsf{fin}}) = \aleph_1$ in the model of Theorem 3.14 by Lemma 3.15. Therefore it suffices to show that $\mathfrak{d}(\mathsf{bd}\neq)$ remains of size $\mathfrak{c}$ if σ-centered forcing is iterated over a model of MA. This is a standard argument which we include for the sake of completeness.

Let $\mathbb{P}$ be a σ-centered forcing notion, $\mathbb{P} = \bigcup_k \mathbb{P}_k$ with each $\mathbb{P}_k$ being centered. Let $g \in \omega^\omega$. Let $\dot{f}$ be a $\mathbb{P}$-name for a function in $\prod_n g(n)$. Then

($*$) there is a set $\{f_k : k \in \omega\}$ of ground model functions in $\prod_n g(n)$ such that whenever $h \in \prod_n g(n)$ agrees with all f_k infinitely often, then $\exists^\infty n\ (h(n) = \dot{f}(n))$ is forced.

To see this, for each k and n, find $f_k(n) \in g(n)$ such that no condition of $\mathbb{P}_k$ forces that $\dot{f}(n) \neq f_k(n)$. Such $f_k(n)$ exists by the centeredness of $\mathbb{P}_k$ and the fact that there are only finitely many possibilities for $\dot{f}(n)$. Now let $p \in \mathbb{P}$ and $n_0 \in \omega$. There is k such that $p \in \mathbb{P}_k$. Also there is $n \geq n_0$ such that $f_k(n) = h(n)$. By the choice of f_k there is $q \leq p$ such that q forces that $\dot{f}(n) = f_k(n) = h(n)$. Hence it is forced that $\dot{f}$ agrees with h infinitely often.

Now consider the iteration in the proof of Theorem 3.14. It is well-known that the whole iteration is σ-centered (since it has length $\omega_1 < \mathfrak{c}$; see, e.g., [Bag, Lemma 3.3]). Let $\kappa < \mathfrak{c}$ and let $\dot{F} = \{\dot{f}^\alpha : \alpha < \kappa\}$ be a name for a family of functions in $\prod_n g(n)$. Consider $\mathcal{F} = \{f^\alpha_k : \alpha < \kappa \text{ and } k \in \omega\}$ as in ($*$). Clearly $|\mathcal{F}| < \mathfrak{c}$. Thus, by MA in the ground model, there is $h \in \prod_n g(n)$ such that h agrees with all f^α_k infinitely often. By ($*$) this means that h agrees with all $\dot{f}^\alpha$ infinitely often in the extension. Hence $\dot{\mathcal{F}}$ is not a witness for $\mathfrak{d}(\mathsf{bd}\neq)$. This proves $\mathfrak{d}(\mathsf{bd}\neq) = \mathfrak{c}$ in the generic extension. ∎

3.3. SC-ultrafilters. We turn to SC-ultrafilters. We have $\mathsf{cof}(\mathcal{SC}) = \mathfrak{c}$ (see Proposition 4.6). Part (i) of 3.18 can be improved as follows:

PROPOSITION 3.22. $\mathsf{non}^*(\mathcal{SC}) \leq \mathfrak{r}$.

Proof. A family $\mathcal{R} \subseteq [\omega]^\omega$ is a *hereditarily unreaped family* if for any $R \in \mathcal{R}$, $\{S \in \mathcal{R} : S \subseteq R\}$ is unreaped. It is well-known and easy to see that there is a hereditarily unreaped family $\mathcal{R}$ of size $\mathfrak{r}$. Fix such an $\mathcal{R}$. For $R \in \mathcal{R}$ and $n \in \omega$, let $X_{R,n} = \{m+n : m \in R\}$. We claim that $\mathcal{X} = \{X_{R,n} : R \in \mathcal{R}$ and $n \in \omega\}$ is a witness for $\mathsf{non}^*(\mathcal{SC})$.

Let $A \in \mathcal{SC}$. Suppose $A = \bigcup_{i<k} A_i$ where each A_i is an SC-set. This means that

($\star$) there is an ℓ_0 such that for all $i < k$ and all $a \neq b \in A_i$ larger than ℓ_0, we have $|a-b| > 2k$.

Let $B_0 = A$. There is $R_0 \in \mathcal{R}$ such that either $R_0 \cap B_0$ is finite or $R_0 \subseteq^* B_0$. In the former case, $X_{R_0,0} \cap A$ is finite and we are done. In the latter case we proceed. Assume for the moment $k \geq 2$, let $i < k-1$ and assume B_i and R_i have been constructed so that $R_i \subseteq^* B_i$. Let $B_{i+1} = \{m \in R_i : m+(i+1) \in A\}$. Since $\mathcal{R}$ is hereditarily unreaped, there is $R_{i+1} \in \mathcal{R}$ with $R_{i+1} \subseteq R_i$ such that either $R_{i+1} \cap B_{i+1}$ is finite (this holds in particular when B_{i+1} is finite in which case we can take $R_{i+1} = R_i$), or $R_{i+1} \subseteq^* B_{i+1}$. In the former case, $X_{R_{i+1},i+1} \cap A$ is finite and we are again done. Otherwise we proceed.

Now suppose B_{k-1} and R_{k-1}, $k \geq 1$, have been constructed and $R_{k-1} \subseteq^* B_{k-1}$. We claim that $X_{R_{k-1},k} \cap A$ is finite. Indeed, by construction, there is an $\ell_1 \geq \ell_0$ such that for all $m \in R_{k-1}$ larger than ℓ_1 and all $i < k$, $m+i \in A$. By ($\star$), for fixed $m \geq \ell_0$, each A_j can contain at most one number of the form $m + i$, $i < k$. Hence, if $m \geq \ell_1$ and $m \in R_{k-1}$, then each A_j must contain exactly one such $m + i$. Therefore no A_j can contain $m + k$. Thus $X_{R_{k-1},k} \cap A$ is bounded by ℓ_1. ■

Observation 3.23. $\mathfrak{ge}(\mathcal{SC}) \geq \mathfrak{d}$.

This follows from $\mathsf{conv} \leq_{\mathrm{K}} \mathcal{SC}$ (see the discussion after Figure 1 in Subsection 2.1), $\mathfrak{ge}(\mathsf{conv}) = \mathfrak{d}$ (Proposition 4.4), and Observation 3.6.

Corollary 3.24. $\mathsf{non}^*(\mathcal{SC}) < \mathfrak{ge}(\mathcal{SC})$ *is consistent.*

This holds in any model for $\mathfrak{d} > \mathfrak{r}$, like the Miller model or the Blass–Shelah model (see Subsection 1.3). For a model of $\mathfrak{d} > \mathfrak{r}$ where $\mathfrak{d}$ and $\mathfrak{r}$ can assume arbitrary values see [BS2] (this is a model of dual $\mathbb{M}(\mathcal{U})$ type).

To be able to compare the cardinals of the SC-sets and those of the summable ideal (see the next subsection), we prove the following two theorems.

Theorem 3.25. $\mathsf{non}^*(\mathcal{SC}) > \mathsf{non}(\mathcal{N})$ *is consistent. In fact, this holds in the dual random model.*

Theorem 3.26. $\mathfrak{ge}(\mathcal{SC}) < \mathsf{cov}(\mathcal{N})$ *is consistent. In fact, this holds in the random model.*

We first show 3.25 and then 3.26. Before beginning the proof of Theorem 3.25, we prove the following two lemmata.

Lemma 3.27. *Let $\dot{X}$ be a $\mathbb{B}$-name for an infinite subset of ω, and let $B \in \mathbb{B}$, $\epsilon > 0$, and $n_0, k \in \omega$. Then there is a finite set a such that*

(i) $\min a \geq n_0$,
(ii) $|n - m| \geq k$ *for all* $n \neq m \in a$,
(iii) $\mu([\![a \cap \dot{X} \neq \emptyset]\!] \cap B) \geq (1 - \epsilon)\mu(B)$.

Here $\mathbb{B}$ denotes the random algebra (see Subsection 1.3) and μ is Lebesgue measure.

Proof. We first claim that we can find a and $C \subseteq B$ with $\mu(C) \geq \frac{1}{2k}\mu(B)$ satisfying (i) and (ii) and such that $C \Vdash a \cap \dot{X} \neq \emptyset$.

Indeed, choose $n_1 > n_0$ such that $\mu([\![[n_0, n_1) \cap \dot{X} \neq \emptyset]\!] \cap B) \geq \frac{1}{2}\mu(B)$. Then split $[n_0, n_1)$ into k (possibly empty) pieces a_i, $i < k$, such that distinct elements of each a_i have distance at least k. Clearly, for one $i < k$, we must have $\mu([\![a_i \cap \dot{X} \neq \emptyset]\!] \cap B) \geq \frac{1}{2k}\mu(B)$.

Using this claim, recursively construct finite sets a_i and conditions B_i, $i \in \omega$, such that

- $B_0 \subseteq B$ and $B_{i+1} \subseteq B \setminus \bigcup_{j \leq i} B_j$,
- $\mu(B_0) = \frac{1}{2k}\mu(B)$ and $\mu(B_{i+1}) = \frac{1}{2k}\mu(B \setminus \bigcup_{j \leq i} B_j)$,
- $n_0 \leq \min(a_0)$ and $\max(a_i) + k \leq \min(a_{i+1})$,
- (ii) holds for all a_i,
- $B_i \Vdash a_i \cap \dot{X} \neq \emptyset$.

Now notice that, by induction,

$$\mu\Big(B \setminus \bigcup_{j<i} B_j\Big) = \frac{(2k-1)^i}{(2k)^i}\mu(B) \quad \text{and} \quad \mu(B_i) = \frac{1}{2k}\,\frac{(2k-1)^i}{(2k)^i}\mu(B)$$

Therefore

$$\mu\Big(\bigcup_{i\in\omega} B_i\Big) = \sum_{i\in\omega} \mu(B_i) = \frac{1}{2k}\sum_{i\in\omega}\Big(\frac{2k-1}{2k}\Big)^i \mu(B) = \mu(B)$$

Hence, by choosing N large enough, $a = \bigcup_{i<N} a_i$ is as required. ∎

Lemma 3.28.

(i) *Let $\dot{X}$ be a $\mathbb{B}$-name for an infinite subset of ω and let $B \in \mathbb{B}$. Then there is an SC-set $A \in V$ such that B forces $\dot{X} \cap A$ is infinite.*

(ii) *Let $\kappa < \mathfrak{c}$ and λ be arbitrary. Assume* MA_κ*(σ-centered). Let $\dot{X}_\alpha$, $\alpha < \kappa$, be $\mathbb{B}_\lambda$-names for infinite subsets of ω and let $B \in \mathbb{B}_\lambda$. Then there is an SC-set $A \in V$ such that B forces $\dot{X}_\alpha \cap A$ is infinite for all $\alpha < \kappa$.*

Here $\mathbb{B}_\lambda$ denotes the measure algebra for adding λ random reals (see Subsection 1.3).

Proof. (i) is a special case of (ii) for $\kappa = \lambda = \omega$. We prove (ii).

Let $\mathbb{P}$ be the following forcing notion: conditions p are quadruples $(a^p, k^p, \epsilon^p, F^p)$ such that

(A) $a^p \subseteq \omega$ is finite (approximations of A),
(B) $k^p \in \omega$,
(C) $\epsilon^p \in \mathbb{Q}$, $\epsilon^p > 0$,
(D) $F^p \subseteq \kappa$ is finite,
(E) $\mu([\![|\dot{X}_\alpha \cap a^p| \geq k^p]\!] \cap B) \geq (1 - \epsilon^p)\mu(B)$ for all $\alpha \in F^p$,

and the order $q \leq_{\mathbb{P}} p$ is given by

- $a^q \supseteq a^p$ and $\min(a^q \setminus a^p) \geq \max(a^p) + k^p$,
- $|n - m| \geq k^p$ for all $n \neq m$ in $a^q \setminus a^p$,

- $k^q \geq k^p$,
- $\epsilon^q \leq \epsilon^p$,
- $F^q \supseteq F^p$.

We need to verify the following four claims:

CLAIM 3.28.1. *$\mathbb{P}$ is σ-centered.*

CLAIM 3.28.2. *$D_\alpha = \{p : \alpha \in F^p\}$ is dense for $\alpha < \kappa$.*

CLAIM 3.28.3. *$E_k = \{p : k^p \geq k\}$ is dense for $k \in \omega$.*

CLAIM 3.28.4. *$F_\epsilon = \{p : \epsilon^p \leq \epsilon\}$ is dense for $\epsilon \in \mathbb{Q}$ with $\epsilon > 0$.*

Proof of Claim 3.28.1. Fix $a \subseteq \omega$ finite, $k \in \omega$ and $\epsilon \in \mathbb{Q}$ with $\epsilon > 0$. Then $P_{a,k,\epsilon} = \{p \in \mathbb{P} : a^p = a,\ k^p = k,\ \epsilon^p = \epsilon\}$ is clearly centered, and $\mathbb{P}$ is the countable union of the $P_{a,k,\epsilon}$. ∎

Proof of Claim 3.28.2. Let $\alpha \in \kappa$ and $p \in \mathbb{P}$. We may assume $\alpha \notin F^p$. Applying the previous lemma k^p times with $\dot{X} = \dot{X}_\alpha$, $\epsilon = \epsilon^p/k^p$, $k = k^p$, we find sets a_i, $i < k^p$, such that

- $\min(a_0) \geq \max(a^p) + k^p$, $\min(a_{i+1}) \geq \max(a_i) + k^p$ for all $i < k^p - 1$,
- $|n - m| \geq k^p$ for all $n \neq m$ in a_i and all $i < k^p$,
- $\mu(\llbracket a_i \cap \dot{X}_\alpha \neq \emptyset \rrbracket \cap B) \geq (1 - \epsilon)\mu(B)$.

Define q by letting $a^q = a^p \cup \bigcup\{a_i : i < k^p\}$, $k^q = k^p$, $\epsilon^q = \epsilon^p$, and $F^q = F^p \cup \{\alpha\}$. Clearly $\mu(\llbracket |\dot{X}_\alpha \cap a^q| \geq k^q \rrbracket \cap B) \geq (1 - \epsilon^q)\mu(B)$. Hence $q \in \mathbb{P}$, $q \leq_{\mathbb{P}} p$ and $q \in D_\alpha$, as required. ∎

Proof of Claim 3.28.3. Let $k \in \omega$ and $p \in \mathbb{P}$. We may assume $k > k^p$. Let $k^q = k$, $\epsilon^q = \epsilon^p$, and $F^q = F^p$. Apply Lemma 3.27 $k \cdot |F^p|$ times to obtain $a^q \supseteq a^p$ such that $\mu(\llbracket |\dot{X}_\alpha \cap (a^q \setminus a^p)| \geq k^q \rrbracket \cap B) \geq (1 - \epsilon^q)\mu(B)$ for all $\alpha \in F^q$, $q \leq_{\mathbb{P}} p$, and $q \in E_k$. ∎

Proof of Claim 3.28.4. Let $\epsilon \in \mathbb{Q}$ with $\epsilon > 0$ and $p \in \mathbb{P}$. We may assume $\epsilon < \epsilon^p$. Let $k^q = k^p$, $\epsilon^q = \epsilon$, $F^q = F^p$, and again apply Lemma 3.27 $k^p \cdot |F^p|$ times as in the proof of the previous claim to obtain the required $q \leq_{\mathbb{P}} p$ with $q \in F_\epsilon$. ∎

We continue with the proof of Lemma 3.28. Using MA_κ(σ-centered), we find a filter $G \subseteq \mathbb{P}$ such that $G \cap D_\alpha \neq \emptyset$ and $G \cap E_k \neq \emptyset$ and $G \cap F_\epsilon \neq \emptyset$ for all $\alpha < \kappa$, all $k \in \omega$, and all $\epsilon \in \mathbb{Q}$ with $\epsilon > 0$. Let $A = \bigcup\{a^p : p \in G\}$. By definition of $\leq_{\mathbb{P}}$ and since G meets all E_k, we deduce that A is an SC-set. Since G meets all D_α, E_k, and F_ϵ, we see that for all $\alpha < \kappa$, $\mu(\llbracket |\dot{X}_\alpha \cap A| = \aleph_0 \rrbracket \cap B) = \mu(B)$, that is, $B \Vdash |\dot{X}_\alpha \cap A| = \aleph_0$, as required. ∎

Proof of Theorem 3.25. Now assume MA and $\mathfrak{c} \geq \aleph_2$. Force with $\mathbb{B}_{\omega_1}$. The resulting model, the dual random model, satisfies $\mathsf{non}(\mathcal{N}) = \aleph_1$ (see Subsection 1.3). To see that $\mathsf{non}^*(\mathcal{SC}) = \mathfrak{c}$, apply the previous lemma. This completes the proof of Theorem 3.25. ∎

Before beginning the proof of Theorem 3.26, we prove the following two lemmata.

LEMMA 3.29. *Assume that $F \subseteq \omega$ contains arbitrarily long intervals. Let $\dot{X}$ be a $\mathbb{B}$-name for an SC-set, and assume $B \in \mathbb{B}$. Then there are $C \leq B$ and $G \subseteq F$ containing arbitrarily long intervals such that C forces $\dot{X} \cap G = \emptyset$.*

Proof. Since $\mathbb{B}$ is ω^ω-bounding (see Subsection 1.3), by strengthening B if necessary we may assume there is $g \in \omega^\omega$ such that

$$B \Vdash \forall k\, \forall n \neq m \in \dot{X}\ (n, m \geq g(k) \Rightarrow |n - m| \geq k).$$

Let $F_k \subseteq F$, $k \in \omega$, be intervals of length 2^{2k} with $\min(F_k) \geq g(2^{2k})$. Then $B \Vdash \forall k\ (|\dot{X} \cap F_k| \leq 1)$. Let F_k be a union of 2^k intervals F_k^i, $i < k$, of length 2^k. Then, for each k, there is $i_k < 2^k$ such that

$$\mu([\![\dot{X} \cap F_k^{i_k} \neq \emptyset]\!] \cap B) \leq \frac{1}{2^k}\mu(B).$$

Let $G = \bigcup_{k \geq 2} F_k^{i_k}$. Then clearly

$$\mu([\![\dot{X} \cap G \neq \emptyset]\!] \cap B) \leq \mu(B)/2.$$

Therefore, G and $C = B \setminus [\![\dot{X} \cap G \neq \emptyset]\!]$ are as required. ∎

LEMMA 3.30. *Assume CH. There is a $\subseteq^*$-decreasing chain $(F_\alpha : \alpha < \omega_1)$ of sets containing arbitrarily long intervals such that, whenever $\dot{X}$ is a $\mathbb{B}$-name for an SC-set and $B \in \mathbb{B}$, then there are $C \leq B$ and $\alpha < \omega_1$ such that C forces $\dot{X} \cap F_\alpha = \emptyset$.*

Proof. Let $(\dot{X}_\alpha, B_\alpha)$, $\alpha < \omega_1$, and list all pairs $(\dot{X}, B)$ such that $\dot{X}$ is a nice $\mathbb{B}$-name (in the sense of Kunen [Ku1, Definition VII.5.11]) for an SC-set and $B \in \mathbb{B}$. Recursively construct F_α such that

- $F_0 = \omega$,
- F_α contains arbitrarily long intervals,
- if α is limit then F_α is a pseudointersection of the F_β for $\beta < \alpha$,
- if $\alpha = \beta + 1$ is successor, then $F_\alpha \subseteq F_\beta$ and there is $C \leq B_\beta$ such that $C \Vdash \dot{X}_\beta \cap F_\alpha = \emptyset$.

It is clear that this can be done by the previous lemma. ∎

Proof of Theorem 3.26. In the random model, the F_α from the previous lemma witness $\mathfrak{ge}(\mathcal{SC}) = \aleph_1$: let $\dot{X}$ be a $\mathbb{B}_\lambda$-name for an SC-set; there is $A \subseteq \lambda$ countable such that $\dot{X}$ is a $\mathbb{B}_A$-name, i.e., we may construe $\dot{X}$ as a $\mathbb{B}$-name; by 3.30, there is α such that some condition in the generic forces $\dot{X} \cap F_\alpha = \emptyset$. This completes the proof of Theorem 3.26. ∎

3.4. Summable ultrafilters. It is well-known that $\mathsf{cof}(\mathcal{I}_{1/n}) = \mathsf{cof}(\mathcal{N})$ [Fr2]. A lower bound for $\mathfrak{ge}(\mathcal{I}_{1/n})$ is given by

THEOREM 3.31. $\mathfrak{ge}(\mathcal{I}_{1/n}) \geq \mathsf{cov}(\mathcal{E})$. *In particular,* $\mathfrak{ge}(\mathcal{I}_{1/n}) \geq \mathsf{cov}(\mathcal{N})$.

Proof. It suffices to show that there are functions $\prod_n 2^n \to \mathcal{I}_{1/n} : f \mapsto Z_f$ and $\mathcal{I}^+_{1/n} \to \mathcal{E} : X \mapsto B_X$ such that $Z_f \cap X$ is infinite whenever $f \notin B_X$. For then, given $\mathcal{F} \subseteq \mathcal{I}^+_{1/n}$ of size $< \mathsf{cov}(\mathcal{E})$, there is $f \in \prod_n 2^n \setminus \bigcup\{B_X : X \in \mathcal{F}\}$. Hence $Z_f \cap X$ is infinite for all $X \in \mathcal{F}$ and $\mathcal{F}$ cannot be a witness for $\mathfrak{ge}(\mathcal{I}_{1/n})$.

Consider a partition of ω into consecutive intervals I_n where $|I_n| = 2^n$ for every n. There exists a one-to-one correspondence between functions in $\prod_n 2^n$ and selectors on $\{I_n : n \in \omega\}$: one can assign to every $f \in \prod_n 2^n$ a set $Z_f \subseteq \omega$ in such a way that $|Z_f \cap I_n| = 1$ for every $n \in \omega$ by defining $Z_f = \{\min(I_n) + f(n) : n \in \omega\}$. Clearly each Z_f belongs to $\mathcal{I}_{1/n}$.

Given $X \in \mathcal{I}^+_{1/n}$, define $\varphi_X : \omega \to [\omega]^{<\omega}$ by $\varphi_X(n) = \{m - \min I_n : m \in I_n \cap X\}$.

For the closed set $B_{X,m} = \{f \in \prod_n 2^n : \forall n \geq m\ (f(n) \notin \varphi_X(n))\}$ one obtains the following upper bound of $\mu(B_{X,m})$:

$$\begin{aligned}\mu(B_{X,m}) &= \prod_{n\geq m} \frac{|I_n \setminus X|}{2^n} = \prod_{n\geq m} \left(1 - \frac{|X \cap I_n|}{2^n}\right) \leq \prod_{n\geq m} e^{-|X\cap I_n|/2^n} \\ &= e^{\sum_{n\geq m} -|X\cap I_n|/2^n}.\end{aligned}$$

Hence $\mu(B_{X,m}) = 0$ because $\sum_{n\geq m} |X \cap I_n|/2^n = \infty$ for every $X \notin \mathcal{I}_{1/n}$.

Consequently, $B_X = \{f \in \prod_n 2^n : \forall^\infty n\ (f(n) \notin \varphi_X(n))\} = \bigcup_{m\in\omega} B_{X,m}$ is an F_σ null set and thus belongs to $\mathcal{E}$. If $f \notin B_X$ then $f(n) \in \varphi_X(n)$ for infinitely many $n \in \omega$. It follows that $Z_f \cap X$ is infinite ([15]). ∎

Proposition 3.32 (Hernández and Hrušák [HH, Theorem 3.7]).

$$\mathsf{non}^*(\mathcal{I}_{1/n}) \leq \mathsf{non}(\mathcal{N}).$$

Corollary 3.33. *$\mathsf{non}^*(\mathcal{I}_{1/n}) < \mathfrak{ge}(\mathcal{I}_{1/n})$ is consistent.*

This holds in any model for $\mathsf{non}(\mathcal{N}) < \mathsf{cov}(\mathcal{N})$, like the random and dual random models.

Furthermore we obtain:

Corollary 3.34.

(i) *$\mathsf{non}^*(\mathcal{I}_{1/n}) < \mathsf{non}^*(\mathcal{SC})$ is consistent. In particular, $\mathsf{non}^*(\mathsf{thin}) < \mathsf{non}^*(\mathcal{SC})$ and $\mathsf{non}^*(\mathcal{I}_{1/n}) < \mathsf{non}^*(\mathcal{Z})$ are consistent.*
(ii) *$\mathfrak{ge}(\mathcal{SC}) < \mathfrak{ge}(\mathcal{I}_{1/n})$ is consistent. In particular, $\mathfrak{ge}(\mathsf{thin}) < \mathfrak{ge}(\mathcal{I}_{1/n})$ and $\mathfrak{ge}(\mathcal{SC}) < \mathfrak{ge}(\mathcal{Z})$ are consistent.*
(iii) *$\mathfrak{ge}(\mathcal{I}_{1/n}) < \mathfrak{ge}(\mathcal{SC})$ is consistent. In particular, $\mathfrak{ge}(\mathsf{thin}) < \mathfrak{ge}(\mathcal{SC})$ and $\mathfrak{ge}(\mathcal{I}_{1/n}) < \mathfrak{ge}(\mathcal{Z})$ are consistent.*

([15]) Since we did not use the fact that $\mathcal{F}$ is a filter base, this proof really shows $\mathsf{non}^+(\mathcal{I}_{1/n}) \geq \mathsf{cov}(\mathcal{E})$—see Proposition 4.1(vii) below.

Proof. By Figure 1 and Observations 3.5 and 3.6, it suffices to prove the first statement of each of the three items.

(i) This holds in the dual random model, by 3.25 and 3.32.

(ii) This holds in the random model, by 3.26 and 3.31.

(iii) Since $\mathfrak{ge}(\mathcal{SC}) \geq \mathfrak{d}$ by Observation 3.23, this holds in any model with $\mathfrak{ge}(\mathcal{I}_{1/n}) < \mathfrak{d}$, like the models for Corollary 3.12. More concretely, this is true in the dual $\mathbb{M}(\mathcal{I}^*_{1/n})$ model or, alternatively, in the Mathias, Laver, or Miller models. ∎

Conjecture 3.35. $\mathfrak{r} < \mathsf{non}^*(\mathcal{I}_{1/n})$ *is consistent.*

This would also imply the consistency of $\mathsf{non}^*(\mathcal{SC}) < \mathsf{non}^*(\mathcal{I}_{1/n})$, by Proposition 3.22, and thus of $\mathsf{non}^*(\mathsf{thin}) < \mathsf{non}^*(\mathcal{I}_{1/n})$ and of $\mathsf{non}^*(\mathcal{SC}) < \mathsf{non}^*(\mathcal{Z})$ (by Figure 1 and Observation 3.5). There is strong evidence that 3.35 holds because the dual inequality is indeed consistent (Proposition 4.3 below).

3.5. Density zero ultrafilters. We already mentioned Fremlin's $\mathsf{cof}(\mathcal{Z}) = \mathsf{cof}(\mathcal{N})$ [Fr2]. Lower and upper bounds for $\mathsf{non}^*(\mathcal{Z})$ are as follows:

Theorem 3.36.

(i) (Hernández and Hrušák [HH, Theorems 3.10 and 3.12]) *We have* $\min\{\mathfrak{d}, \mathsf{cov}(\mathcal{N})\} \leq \mathsf{non}^*(\mathcal{Z}) \leq \max\{\mathfrak{d}, \mathsf{non}(\mathcal{N})\}$.

(ii) (Raghavan and Shelah [RS]) $\mathfrak{b} \leq \mathsf{non}^*(\mathcal{Z})$ ([16]).

Observation 3.37. $\mathfrak{ge}(\mathcal{Z}) \geq \max\{\mathsf{non}^*(\mathcal{Z}), \mathfrak{d}, \mathsf{cov}(\mathcal{E}), \mathsf{non}(\mathcal{E})\}$.

This follows from Observation 3.3, from $\mathcal{I}_{1/n} \leq_{\mathrm{K}} \mathcal{Z}$ and $\mathfrak{ge}(\mathcal{I}_{1/n}) \geq \mathsf{cov}(\mathcal{E})$ (Theorem 3.31), as well as from $\mathsf{mz} \leq_{\mathrm{K}} \mathcal{Z}$ (Proposition 2.4) and $\mathfrak{ge}(\mathsf{mz}) = \mathsf{cof}(\mathcal{E}, \mathcal{M}) = \max\{\mathfrak{d}, \mathsf{non}(\mathcal{E})\}$ [Br2]. (For the definition of $\mathsf{cof}(\mathcal{I}, \mathcal{J})$, see before Observation 3.4.)

Corollary 3.38. $\mathsf{non}^*(\mathcal{Z}) < \mathfrak{ge}(\mathcal{Z})$ *is consistent.*

By Theorem 3.36(i), Observation 3.37 and $\mathsf{cov}(\mathcal{N}) \leq \mathsf{cov}(\mathcal{E})$, this holds in any model for $\max\{\mathfrak{d}, \mathsf{non}(\mathcal{N})\} < \mathsf{cov}(\mathcal{N})$, like the random model.

Conjecture 3.39. $\mathfrak{ge}(\mathcal{Z}) < \mathsf{non}(\mathcal{M})$ *is consistent. In particular, it is consistent that nowhere dense ultrafilters generically exist, while density zero ultrafilters do not.*

3.6. Some models. The following table summarizes the values of the cardinal invariants of this section in some standard models of ZFC. For completeness, we include $\mathsf{cov}(\mathcal{M})$ as lower bound and $\mathsf{cof}(\mathcal{N})$ as upper bound.

(16) This improves an earlier result of Hernández, Hrušák, and Zapletal ([HH, Theorem 3.4] and [HrZ, Proposition 4.1]) saying that $\min\{\mathfrak{b}, \mathsf{non}(\mathcal{N})\} \leq \mathsf{non}^*(\mathcal{Z})$.

All results here follow from the values of classical cardinal invariants in these models (see Subsection 1.3), and the ZFC-inequalities as well as the consistency results discussed in the preceding subsections.

	cov $(\mathcal{M})$	non^* $(\mathcal{ED}_{\mathsf{fin}})$	$\mathfrak{ge}$ (thin)	non^* $(\mathcal{SC})$	$\mathfrak{ge}$ $(\mathcal{SC})$	non^* $(\mathcal{I}_{1/n})$	$\mathfrak{ge}$ $(\mathcal{I}_{1/n})$	non^* $(\mathcal{Z})$	$\mathfrak{ge}$ $(\mathcal{Z})$	cof $(\mathcal{N})$
$\mathbb{C}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$
$\mathbb{B}$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\mathfrak{c}$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$
dual $\mathbb{B}$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$
dual $\mathbb{D}$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$
$\mathbb{M}$	$\aleph_1$	$\aleph_1$	$\aleph_1$	?	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$
$\mathbb{L}$	$\aleph_1$	$\aleph_1$	$\aleph_1$	?	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$
$\mathbb{MI}$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	?	$\mathfrak{c}$	$\mathfrak{c}$
$\mathbb{BS}$	$\aleph_1$	$\aleph_1$	?	$\aleph_1$	$\mathfrak{c}$	?	?	?	$\mathfrak{c}$	$\mathfrak{c}$
$\mathbb{S}$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$
dual $\mathbb{M}(\mathcal{I}^*_{1/n})$	$\aleph_1$	$\aleph_1$	$\aleph_1$	?	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	?	$\mathfrak{c}$	$\mathfrak{c}$
dual $\mathbb{M}(\mathcal{Z}^*)$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\mathfrak{c}$

Fig. 3

Here $\mathbb{C}$ denotes the Cohen model, $\mathbb{B}$ the random model, "dual $\mathbb{B}$" the dual random model, "dual $\mathbb{D}$" the dual Hechler model, $\mathbb{M}$ the Mathias model, $\mathbb{L}$ the Laver model, $\mathbb{MI}$ the Miller model, $\mathbb{BS}$ the Blass–Shelah model, $\mathbb{S}$ the Sacks model, "dual $\mathbb{M}(\mathcal{I}^*_{1/n})$" the finite support iteration of $\mathbb{M}(\mathcal{I}^*_{1/n})$ of length ω_1 over a model of MA + $\neg$CH, and "dual $\mathbb{M}(\mathcal{Z}^*)$" the analogous model for $\mathbb{M}(\mathcal{Z}^*)$ (see Subsection 1.3 for more details about these models).

4. Epilogue

4.1. A variant of the uniformity. If we compare the definitions of $\mathsf{non}^*(\mathcal{I})$ and $\mathfrak{ge}(\mathcal{I})$, it is very natural to consider the following cardinal invariant:

$$\mathsf{non}^+(\mathcal{I}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I}^+ \text{ and } \forall I \in \mathcal{I}\, \exists F \in \mathcal{F}\, (|I \cap F| < \aleph_0)\}.$$

Clearly, $\mathsf{non}^*(\mathcal{I}) \le \mathsf{non}^+(\mathcal{I}) \le \mathfrak{ge}(\mathcal{I})$. Also $\mathcal{I} \le_{\mathrm{K}} \mathcal{J}$ implies $\mathsf{non}^+(\mathcal{I}) \le \mathsf{non}^+(\mathcal{J})$. We summarize what we know about this new cardinal.

PROPOSITION 4.1.

(i) $\mathsf{non}^+(\mathcal{I}) = \mathsf{non}^*(\mathcal{I}) = \aleph_0$ *for all* $\mathcal{I} \le_{\mathrm{K}} \mathsf{nwd}$. *In particular, for tall ideals* $\mathcal{I}$ *Katětov below* nwd, *we have* $\mathsf{non}^+(\mathcal{I}) < \mathfrak{ge}(\mathcal{I})$.

(ii) $\mathsf{non}^+(\mathsf{Fin} \times \mathsf{Fin}) = \mathfrak{d} = \mathfrak{ge}(\mathsf{Fin} \times \mathsf{Fin}) > \aleph_0 = \mathsf{non}^*(\mathsf{Fin} \times \mathsf{Fin})$.

(iii) $\mathsf{non}^+(\mathcal{ED}) = \mathsf{cov}(\mathcal{M}) = \mathfrak{ge}(\mathcal{ED}) > \aleph_0 = \mathsf{non}^*(\mathcal{ED})$.

(iv) $\mathsf{non}^+(\mathcal{ED}_{\mathsf{fin}}) = \mathsf{non}^*(\mathcal{ED}_{\mathsf{fin}}) = \mathfrak{d}(\mathsf{pbd}\neq)$.

(v) *If* $\mathcal{I}$ *is* F_σ, *then* $\mathsf{non}^+(\mathcal{I}) \le \mathfrak{d}(\mathsf{pbdLoc})$.

(vi) *If $\mathcal{I}$ is an analytic P-ideal, then* $\mathfrak{d}(\mathsf{pbd}\neq) \leq \mathsf{non}^+(\mathcal{I}) \leq \mathfrak{d}(\mathsf{pLoc})$.

(vii) $\mathsf{cov}(\mathcal{E}) \leq \mathsf{non}^+(\mathcal{I}_{1/n})$. *In particular,* $\mathsf{non}^+(\mathcal{I}_{1/n}) > \mathsf{non}^*(\mathcal{I}_{1/n})$ *and* $\mathsf{non}^+(\mathcal{Z}) > \mathsf{non}^*(\mathcal{Z})$ *are both consistent.*

Proof. (i) The basic open neighborhoods are in nwd^+, and thus witness $\mathsf{non}^+(\mathsf{nwd}) = \aleph_0$.

(ii)&(iii) For non^* and $\mathfrak{ge}$ see the discussion after Observation 3.6. Since $\mathsf{non}^+(\mathcal{I}) \leq \mathfrak{ge}(\mathcal{I})$, it suffices to show $\mathsf{non}^+(\mathsf{Fin}\times\mathsf{Fin}) \geq \mathfrak{d}$ and $\mathsf{non}^+(\mathcal{ED}) \geq \mathsf{cov}(\mathcal{M})$, respectively.

For the former, fix $\mathcal{F} \subseteq (\mathsf{Fin}\times\mathsf{Fin})^+$ with $|\mathcal{F}| < \mathfrak{d}$. For $A \in \mathcal{F}$ define

$$f_A(n) = \min\{k \geq n : (m,k) \in A \text{ for some } m \geq n\}.$$

This is well-defined because A is positive. By assumption, there is $g \in \omega^\omega$ with $g \not\leq^* f_A$ for all $A \in \mathcal{F}$. Without loss of generality, g is strictly increasing. Set $B = \{(n,k) : k \leq g(n) \text{ and } n \in \omega\}$. Then $B \in \mathsf{Fin}\times\mathsf{Fin}$ and $|B \cap A| = \aleph_0$ for all $A \in \mathcal{F}$, as required.

For the latter, fix $\mathcal{F} \subseteq \mathcal{ED}^+$ with $|\mathcal{F}| < \mathsf{cov}(\mathcal{M})$. For $A \in \mathcal{F}$ define a partial function $g_A : \omega \to \omega$ by stipulating $n \in \mathrm{dom}(g_A)$ if $(n,k) \in A$ for some $k \in \omega$ and then setting $g_A(n) = \min\{k : (n,k) \in A\}$ for $n \in \mathrm{dom}(g_A)$. Since $A \in \mathcal{ED}^+$, $\mathrm{dom}(g_A)$ must be infinite. Given $f : \omega \to \omega$ partial with infinite domain, let $B_f = \{x \in \omega^\omega : \forall^\infty n \in \mathrm{dom}(f)\ (x(n) \neq f(n))\}$. It is well-known and easy to see that B_f is meager.

By assumption on $\mathcal{F}$, there is $c \in \omega^\omega$ such that $c \notin B_{g_A}$ for all $A \in \mathcal{F}$. This means that for all $A \in \mathcal{F}$, $c(n) = g_A(n)$ holds for infinitely many $n \in \mathrm{dom}(g_A)$, that is, $(n, c(n)) \in A$ for infinitely many n. Identifying c with its graph, we see that $|c \cap A| = \aleph_0$ for all $A \in \mathcal{F}$. Since c is a function, $c \in \mathcal{ED}$ is immediate, and the proof of $\mathsf{non}^+(\mathcal{ED}) \geq \mathsf{cov}(\mathcal{M})$ is complete.

(iv) By Proposition 3.18(ii), it suffices to show $\mathsf{non}^+(\mathcal{ED}_{\mathsf{fin}}) \leq \mathfrak{d}(\mathsf{pbd}\neq)$. The proof is similar to the corresponding proof for $\mathsf{non}^*(\mathcal{ED}_{\mathsf{fin}})$ [HMM, Proposition 3.6 and Lemma 3.9], but there are some subtle differences, and therefore we include the argument.

In the first step we show $\mathsf{non}^+(\mathcal{ED}_{\mathsf{fin}}) \leq \mathfrak{r}$. Let $\mathcal{R}$ be a hereditarily unreaped family of size $\mathfrak{r}$ (see the proof of Proposition 3.22 for this notion and a similar argument). For $R \in \mathcal{R}$ and $n \in \omega$, let

$$X_{R,n} = \{(m,i) : m \in R \text{ and } n\sqrt{m} \leq i < (n+1)\sqrt{m} \leq m\}.$$

Clearly $X_{R,n} \in \mathcal{ED}_{\mathsf{fin}}^+$. We claim that $\mathcal{X} = \{X_{R,n} : R \in \mathcal{R}, n \in \omega\}$ is a witness for $\mathsf{non}^+(\mathcal{ED}_{\mathsf{fin}})$. To see this, fix $A \in \mathcal{ED}_{\mathsf{fin}}$. There are functions f_j, $j < k$, below the identity such that $A \subseteq \bigcup_{j<k} f_j$. Set

$$A_n = \{m : \exists j < k\ (n\sqrt{m} \leq f_j(m) < (n+1)\sqrt{m})\}$$

for $n \in \omega$. Since $\mathcal{R}$ is unreaped, we may find $R_0 \in \mathcal{R}$ such that either $R_0 \cap A_0$ is finite or $R_0 \subseteq^* A_0$. More generally, if R_n, $n < k-1$, has been constructed, we may find $R_{n+1} \in \mathcal{R}$ with $R_{n+1} \subseteq R_n$ such that either $R_{n+1} \cap A_{n+1}$ is finite or $R_{n+1} \subseteq^* A_{n+1}$. Assume first that $R_n \cap A_n$ is finite for some $n < k$, and let n be minimal with this property. Then clearly $X_{R_n,n} \cap A$ is finite. If, on the other hand, $R_n \subseteq^* A_n$ for all $n < k$, then $X_{R_{k-1},k} \cap A$ is finite. This completes the first step of the proof.

In the second step we show $\mathsf{non}^+(\mathcal{ED}_{\mathsf{fin}}) \leq \mathfrak{d}(\mathsf{pbd}\neq)$. Fix $g \in \omega^\omega$. Let $\kappa < \mathsf{non}^+(\mathcal{ED}_{\mathsf{fin}})$, and let $\mathcal{F}$ be a family of partial functions below g of size κ. We need to find a total function $h \in \omega^\omega$ such that for all $f \in \mathcal{F}$ there are infinitely many $n \in \mathrm{dom}(f)$ with $f(n) = h(n)$.

Partition ω into intervals I_n, $n \in \omega$, of length n. Let k_n be the number of sequences of the form $s \restriction I_n$ with $s(i) < g(i)$ for all $i \in I_n$. Let $\psi_n : \prod_{i\in I_n} g(i) \to k_n$ be a bijection. For $f \in \mathcal{F}$ let

$$X_f = \Big\{(k_n, \psi_n(s)) : I_n \cap \mathrm{dom}(f) \neq \emptyset,\ s \in \prod_{i\in I_n} g(i) \text{ and }$$
$$\exists i \in I_n \cap \mathrm{dom}(f)\ (s(i) = f(i))\Big\}.$$

Clearly $X_f \in \mathcal{ED}^+_{\mathsf{fin}}$. By assumption there is $Y \in \mathcal{ED}_{\mathsf{fin}}$ such that $Y \cap X_f$ is infinite for all $f \in \mathcal{F}$. Assume $Y \subseteq \bigcup_{\ell<m} h_\ell$. For $f \in \mathcal{F}$ fix $\ell_f = \ell < m$ such that $Z_f = \{k_n : I_n \cap \mathrm{dom}(f) \neq \emptyset \text{ and } (k_n, h_\ell(k_n)) \in X_f\}$ is infinite. By the first step, we know that $\{Z_f : f \in \mathcal{F}\}$ is reaped by some $A \in [\omega]^\omega$, that is, $A \cap Z_f$ and $Z_f \setminus A$ are both infinite for all $f \in \mathcal{F}$. Iterating this argument, we obtain a partition $(A_\ell : \ell < m)$ of ω such that for all $f \in \mathcal{F}$ and all $\ell < m$, $A_\ell \cap Z_f$ is infinite. Define $h \in \omega^\omega$ by stipulating $h \restriction I_n = \psi_n^{-1}(h_\ell(k_n))$ if $k_n \in A_\ell$. To see that h is as required, fix $f \in \mathcal{F}$. Let $\ell = \ell_f$. If $k_n \in A_\ell \cap Z_f$, then $(k_n, h_\ell(k_n)) \in X_f$ and $h \restriction I_n = \psi_n^{-1}(h_\ell(k_n))$, that is, for some $i \in I_n \cap \mathrm{dom}(f)$ we have $h(i) = \psi_n^{-1}(h_\ell(k_n))(i) = f(i)$. Since $A_\ell \cap Z_f$ is infinite, there are infinitely many such $i \in \mathrm{dom}(f)$, as required.

(v)&(vi) The proof of Corollary 3.11 shows $\mathsf{non}^+(\mathcal{I}) \leq \mathfrak{d}(\mathsf{pbdLoc})$ for F_σ ideals $\mathcal{I}$ and $\mathsf{non}^+(\mathcal{I}) \leq \mathfrak{d}(\mathsf{pLoc})$ for analytic P-ideals $\mathcal{I}$. For the lower bound in (vi), use Corollary 3.11(ii) and Proposition 3.18(ii).

(vii) The proof of Theorem 3.31 shows $\mathsf{cov}(\mathcal{E}) \leq \mathsf{non}^+(\mathcal{I}_{1/n})$. The strict inequalities hold in the random model (cf. Corollaries 3.33 and 3.38). ∎

We do not know whether $\mathsf{non}^+(\mathcal{SC}) \geq \mathfrak{d}$ or $\mathsf{non}^+(\mathcal{SC}) \leq \mathfrak{r}$ (cf. Observation 3.23 and Proposition 3.22). Also, we do not know of any ideal $\mathcal{I}$ for which $\mathsf{non}^*(\mathcal{I}) < \mathsf{non}^+(\mathcal{I})$ and $\mathsf{non}^+(\mathcal{I}) < \mathfrak{ge}(\mathcal{I})$ are both consistent.

4.2. Duality. For a tall ideal $\mathcal{I}$ on ω define

$$\mathsf{add}^*(\mathcal{I}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \text{ and } \forall X \in \mathcal{I}\ \exists I \in \mathcal{F}\ (|I \setminus X| = \aleph_0)\},$$
$$\mathsf{cov}^*(\mathcal{I}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \text{ and } \forall X \in [\omega]^\omega\ \exists I \in \mathcal{F}\ (|X \cap I| = \aleph_0)\}.$$

Then add^* is dual to cof^*, and cov^* is dual to non^*. Also $\mathsf{add}(\mathcal{I}) \geq \aleph_1$ is equivalent to $\mathcal{I}$ being a P-ideal. Furthermore $\mathsf{add}^*(\mathcal{I}) \leq \mathsf{cov}^*(\mathcal{I}) \leq \mathsf{cof}^*(\mathcal{I})$ and $\mathsf{add}^*(\mathcal{I}) \leq \mathsf{non}^*(\mathcal{I})$. Moreover, $\mathcal{I} \leq_{\mathrm{K}} \mathcal{J}$ implies $\mathsf{cov}^*(\mathcal{I}) \geq \mathsf{cov}^*(\mathcal{J})$ ([HH, Proposition 3.1] or [Hr1, Theorem 1.2]). See [HH] for more on these cardinals.

The dual versions of some of the cardinals in Subsection 3.1 are given by:

$$\begin{aligned}
\mathfrak{b}(\mathsf{pLoc}) &= \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \text{ and} \\
&\qquad \forall \phi \in \mathsf{pLoc}\ \exists g \in \mathcal{F}\ \exists^\infty n \in \mathrm{dom}(\phi)\ (g(n) \notin \phi(n))\}, \\
\mathfrak{b}(\mathsf{Loc}) &= \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \text{ and } \forall \phi \in \mathsf{Loc}\ \exists g \in \mathcal{F}\ \exists^\infty n\ (g(n) \notin \phi(n))\}, \\
\mathfrak{b}(\mathsf{pbdLoc}) &= \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \text{ is bounded and} \\
&\qquad \forall \phi \in \mathsf{pLoc}\ \exists g \in \mathcal{F}\ \exists^\infty n \in \mathrm{dom}(\phi)\ (g(n) \notin \phi(n))\}, \\
\mathfrak{b}(\mathsf{bdLoc}) &= \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \text{ is bounded and} \\
&\qquad \forall \phi \in \mathsf{Loc}\ \exists g \in \mathcal{F}\ \exists^\infty n\ (g(n) \notin \phi(n))\}.
\end{aligned}$$

Then $\mathfrak{b}(\mathsf{Loc}) = \mathsf{add}(\mathcal{N})$ [Ba1] (see also [BJ, Section 2.3]), and a number of results dual to those for the $\mathfrak{d}$-cardinals hold as well ([17]). Here we only explain what Propositions 3.9 and 3.10 say for the dual cardinals.

Corollary 4.2.

(i) *Let $\mathcal{I}$ be an F_σ ideal. Then* $\mathsf{cov}^*(\mathcal{I}) \geq \mathfrak{b}(\mathsf{pbdLoc})$.

(ii) *Let $\mathcal{I}$ be an analytic P-ideal. Then* $\mathsf{cov}^*(\mathcal{ED}_{\mathsf{fin}}) \geq \mathsf{cov}^*(\mathcal{I}) \geq \mathfrak{b}(\mathsf{pLoc})$.

Proof. We only prove (ii). The proof of (i) is similar but easier and uses 3.9 instead of 3.10. See also the similar proof of Corollary 3.11. The first inequality is [HH, Proposition 3.2].

For the second inequality, first note that $\mathfrak{b}(\mathsf{pLoc}) = \min\{\mathfrak{b}, \mathfrak{b}(\mathsf{pbdLoc})\}$. Assume $\mathcal{F} \subseteq \mathcal{I}$ is of size $< \mathfrak{b}(\mathsf{pLoc})$. We need to show $\mathcal{F}$ is not a witness for $\mathsf{cov}^*(\mathcal{I})$. Use Proposition 3.10 and its notation. Since $|\mathcal{F}| < \mathfrak{b}$, there is $f \in \omega^\omega$ with $f \geq^* f_B$ for all $B \in \mathcal{F}$. By $|\mathcal{F}| < \mathfrak{b}(\mathsf{pbdLoc})$ there is $h_f \in \mathcal{H}_{f,\bar{\epsilon}}$ such that for all $B \in \mathcal{F}$ and almost all $n \in \mathrm{dom}(h_f)$, $g_{B,f}(n) \in h_f(n)$. Therefore $|B \cap A_{h_f}| < \aleph_0$ for all $B \in \mathcal{F}$, as required. ∎

The following is motivated by Conjecture 3.35.

Proposition 4.3. *$\mathfrak{s} > \mathsf{cov}^*(\mathcal{I}_{1/n})$ is consistent and holds in the Mathias model.*

Proof. Denote the ω_2-stage countable support iteration of Mathias forcing by $\mathbb{M}_{\omega_2}$. It is well-known that $\mathfrak{s} = \mathfrak{c} = \aleph_2$ in the resulting Mathias model

([17]) $\mathfrak{b}(\mathsf{pLoc})$ was originally introduced by Blass [Bl1] as linear evasion number $\mathfrak{e}_\ell$. For the equality $\mathfrak{b}(\mathsf{pLoc}) = \mathfrak{e}_\ell$ see [BrS]. $\mathfrak{b}(\mathsf{bdLoc})$ was first considered by Pawlikowski [Pa] (see also [BJ, Theorem 2.7.10]), who showed that this cardinal is equal to the transitive additivity of measure.

(see Subsection 1.3). Let $(I_n : n \in \omega)$ be a partition of ω into intervals of length 2^n in the ground model which satisfies CH.

Let $\dot{X}$ be an $\mathbb{M}_{\omega_2}$-name for an infinite subset of ω. By replacing $\dot{X}$ by an infinite subset if necessary, we may assume that the trivial condition forces that $|\dot{X} \cap I_n| \leq 1$ for all n. Also let $p \in \mathbb{M}_{\omega_2}$. By the Laver property of $\mathbb{M}_{\omega_2}$ (see Subsection 1.3 again), there are $q \leq p$ and $Y \in [\omega]^\omega$ with $|Y \cap I_n| \leq n+1$ for all n such that q forces $\dot{X} \subseteq Y$. Clearly $Y \in \mathcal{I}_{1/n}$. Hence the ground model witnesses $\mathsf{cov}^*(\mathcal{I}_{1/n}) = \aleph_1$ in the generic extension. ∎

Duality now suggests that there should be an ω^ω-bounding, P-point preserving forcing which increases $\mathsf{non}^*(\mathcal{I}_{1/n})$. This would confirm Conjecture 3.35.

4.3. Some proofs. Here we collect some proofs of known results for which we could not find a reference.

PROPOSITION 4.4. $\mathfrak{ge}(\mathsf{conv}) = \mathfrak{d}$.

Proof. Since $\mathsf{conv} \leq_{\mathrm{K}} \mathsf{Fin} \times \mathsf{Fin}$ (see the discussion after Figure 1 in Subsection 2.1), $\mathfrak{ge}(\mathsf{conv}) \leq \mathfrak{ge}(\mathsf{Fin} \times \mathsf{Fin})$ holds by Observation 3.6. On the other hand, Hong and Zhang [HZ1, Theorem 3.6] proved that $\mathfrak{ge}(\mathsf{Fin} \times \mathsf{Fin}) = \mathfrak{d}$. Hence it suffices to prove $\mathfrak{d} \leq \mathfrak{ge}(\mathsf{conv})$.

Let $\mathcal{F}$ be a filter base on $2^{<\omega}$ with $|\mathcal{F}| < \mathfrak{d}$. We need to show that there is $I \in \mathsf{conv}$ such that $F \cap I$ is infinite for all $F \in \mathcal{F}$. Since $\mathcal{F}$ is a filter base, for all $n \in \omega$ there is $s \in 2^n$ such that $\{t \supseteq s : t \in F\}$ is infinite for all $F \in \mathcal{F}$. By König's Lemma, the tree of such s's has an infinite branch, that is, there is $z \in 2^\omega$ such that for all $n \in \omega$ and all $F \in \mathcal{F}$, the set $\{t \supseteq z{\restriction}n : t \in F\}$ is infinite. For $F \in \mathcal{F}$ define $f_F \in \omega^\omega$ by

$$f_F(n) := \min\{|t| : t \in F \text{ and } t \supseteq z{\restriction}n\}.$$

Since $|\mathcal{F}| < \mathfrak{d}$, there is $g \in \omega^\omega$ with $g \not\leq^* f_F$ for all $F \in \mathcal{F}$. Letting $I_n = \{t \in 2^{<\omega} : |t| \leq g(n) \text{ and } t \supseteq z{\restriction}n\}$ for $n \in \omega$, we see that if we rewrite $I = \bigcup_n I_n$ as a sequence, it converges to z. So $I \in \mathsf{conv}$. On the other hand, $F \cap I$ is infinite, as required. ∎

PROPOSITION 4.5. $\mathfrak{ge}(\mathcal{R}) = \mathsf{cov}(\mathcal{M})$.

Proof. As in the previous proof, use $\mathfrak{ge}(\mathcal{R}) \leq \mathfrak{ge}(\mathcal{ED})$ (which follows from $\mathcal{R} \leq_{\mathrm{K}} \mathcal{ED}$ (Figure 1 in Subsection 2.1) and Observation 3.6) and $\mathfrak{ge}(\mathcal{ED}) = \mathsf{cov}(\mathcal{M})$ [HZ1, Theorem 3.7] to see that is suffices to show $\mathsf{cov}(\mathcal{M}) \leq \mathfrak{ge}(\mathcal{R})$.

Let $\mathcal{F}$ be a filter base on ω with $|\mathcal{F}| < \mathsf{cov}(\mathcal{M})$. Again we prove that, for some $I \in \mathcal{R}$, $F \cap I$ is infinite for all $F \in \mathcal{F}$. Let $f : [\omega]^2 \to 2$ be such that $f(\{n, m\}) = 1$ iff $\{n, m\}$ is an edge of the random graph. Now build a tree $T \subseteq \omega^{<\omega}$ such that for all $\sigma \in T$, the set of successors $A_\sigma = \{n \in \omega : \sigma\hat{\ }n \in T\}$ is infinite and

- $A_\tau \subset A_\sigma$ for $\tau \supseteq \sigma$,
- for all $i \in \mathrm{dom}(\sigma)$, either $f(\{\sigma(i), n\}) = 1$ for all $n \in A_\sigma$ or $f(\{\sigma(i), n\}) = 0$ for all $n \in A_\sigma$,
- $A_\sigma \cap F$ is infinite for all $F \in \mathcal{F}$.

Let $A_{\langle\rangle} = \omega$. Suppose $\sigma \in T$ with $|\sigma| \geq 1$. Since $\mathcal{F}$ is a filter base, for either $j = 0$ or $j = 1$, $\{n \in A_{\sigma\restriction|\sigma|-1} : f(\{\sigma(|\sigma|-1), n\}) = j\}$ has infinite intersection with all $F \in \mathcal{F}$. Let A_σ be the set which does. This completes the construction of T.

Notice that $[T]$, the set of branches of T, is homeomorphic to the Baire space ω^ω. For $F \in \mathcal{F}$, let $B_F = \{x \in [T] : |\mathrm{ran}(x) \cap F| < \aleph_0\}$. By construction of T, B_F is a meager subset of $[T]$ for all $F \in \mathcal{F}$. Since $|\mathcal{F}| < \mathsf{cov}(\mathcal{M})$, we may find $c \in [T]$ with $c \notin B_F$ for all $F \in \mathcal{F}$. Letting $C = \mathrm{ran}(c)$, we see that

- $|C \cap F| = \aleph_0$ for all $F \in \mathcal{F}$,
- for all $i \in \omega$, $f\restriction\{\{c(i), c(j)\} : i < j\}$ is a constant function with, say, value $g(i)$ (by construction of T).

Using again the fact that $\mathcal{F}$ is a filter base, we see that for either $j = 0$ or $j = 1$, $I_j := \{i \in \omega : g(i) = j\} \subseteq C$ has infinite intersection with all $F \in \mathcal{F}$. Thus the homogeneous set $I = I_j$ is as required. ∎

The following proposition implies the results $\mathsf{cof}(\mathsf{thin}) = \mathsf{cof}(\mathcal{SC}) = \mathfrak{c}$ mentioned earlier. (For the definition of $\mathsf{cof}(\mathcal{I}, \mathcal{J})$ and its relation to $\mathsf{cof}(\mathcal{I})$ and $\mathsf{cof}(\mathcal{J})$, see before Observation 3.4.)

Proposition 4.6. $\mathsf{cof}(\mathsf{thin}, \mathcal{SC}) = \mathfrak{c}$.

Proof. Let a_n, $n \in \omega$, be a sequence of natural numbers such that $(a_n + 2^n)/a_{n+1}$ converges to 0. Partition ω into intervals J_m, $m \in \omega$, of length $(2^m)!$. Let $(f_n : n \in J_m)$ list all bijections between the set 2^m of binary sequences of length m and the set $\{i < 2^m\}$ of numbers below 2^m. For $x \in 2^\omega$ define $A_x = \{a_n + f_n(x\restriction m) : m \in \omega \text{ and } n \in J_m\}$. By choice of a_n, all A_x are thin. Also, they are pairwise almost disjoint. Furthermore, given distinct x_i, $i < k$, for all m such that all initial segments $x_i\restriction m$ are distinct there is an $n \in J_m$ such that the numbers $f_n(x_i\restriction m)$, $i < k$, are k consecutive numbers. This implies that if $B \in \mathcal{SC}$, say $B = \bigcup_{i<k} B_i$ where each B_i is an SC-set, then B can contain at most k sets of the form A_x. Thus, if $\mathcal{F} \subseteq \mathcal{SC}$ is a family of less than $\mathfrak{c}$ many sets, there is $x \in 2^\omega$ such that A_x is not contained in any member of $\mathcal{F}$, and $\mathsf{cof}(\mathsf{thin}, \mathcal{SC}) = \mathfrak{c}$ follows. ∎

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