Remarks about van der Waerden ideal

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Arithmetic progressions and van der Waerden theorem

An arithmetic progression of length l is the finite sequence $\{a + id : i = 0, 1, \ldots, l - 1\}$ where $a, d \in \mathbb{N}$.

Van der Waerden Theorem (finite version).

For any given natural numbers k and l, there is some natural number $W(k, l)$ such that if the integers $\{1, 2, ..., W(k, l)\}$ are colored, each with one of k different colors, then there exists an arithmetic progression of length at least l, all of which elements are of the same color.

Van der Waerden theorem and AP-sets

Definition.

A set $A \subseteq \mathbb{N}$ is called an AP-set if it contains arbitrary long arithmetic progressions.

Van der Waerden Theorem (infinite version).

If an AP-set is partitioned into finitely many pieces then at least one of them is again an AP-set.

Sets which are not AP-sets form a proper ideal on $\mathbb{N}$
— van der Waerden ideal denoted by $\mathcal{W}$

Van der Waerden ideal and other ideals

Szemerédi Theorem.

$$\mathcal{W} \subseteq \mathcal{Z} \quad \text{where } \mathcal{Z} = \{A \subseteq \mathbb{N} : \limsup_{n\to\infty} \frac{|A \cap n|}{n} = 0\}$$

Erdős Conjecture.

$$\mathcal{W} \subseteq \mathcal{I}_{1/n} \quad \text{where } \mathcal{I}_{1/n} = \{A \subseteq \mathbb{N} : \sum_{a \in A} \frac{1}{a} < \infty\}$$

What sets belong to $\mathcal{W}$?

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Example B. $\{n^2 : n \in \omega\}$ contains infinitely many arithmetic progressions of length 3 (known by Pythagoras), but no arithmetic progression of length 4 (proved by Euler).

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Example B. $\{n^2 : n \in \omega\}$ contains infinitely many arithmetic progressions of length 3 (known by Pythagoras), but no arithmetic progression of length 4 (proved by Euler).

Example C. The set of the prime numbers does not belong to the van der Waerden ideal (Green-Tao).

Van der Waerden ideal $\mathcal{W}$

The van der Waerden ideal $\mathcal{W}$ is

- a tall ideal — because every infinite $A \subseteq \mathbb{N}$ contains an infinite subset with no arithmetic progressions of length 3
- not a P-ideal — consider for example the sets

$$A_k = \{2^n + k : n \in \omega\} \text{ for } k \in \omega$$

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- F_σ-ideal — because $\mathcal{W} = \bigcup_{n\in\mathbb{N}} \mathcal{W}_n$ where

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The family $\mathcal{W}_n$

- is not an ideal for every $n \in \mathbb{N}$
- generates a proper ideal $\langle \mathcal{W}_n \rangle$

Strictly increasing sequence of ideals

The ideal $\langle \mathcal{W}_n \rangle$ is a tall F_σ-ideal for every $n \geq 3$.

Fact.

$$\mathcal{W} = \bigcup_{n \geq 3} \langle \mathcal{W}_n \rangle$$

and $\langle \mathcal{W}_n \rangle \subseteq \langle \mathcal{W}_{n+1} \rangle$ for every $n \in \mathbb{N}$.

Strictly increasing sequence of ideals

Proposition 1.

For every $n \geq 3$ there exists $A \subset \mathbb{N}$ such that

$$A \in \mathcal{W}_{n+1} \setminus \langle \mathcal{W}_n \rangle$$

Strictly increasing sequence of ideals

Proposition 1.

For every $n \geq 3$ there exists $A \subset \mathbb{N}$ such that

$$A \in \mathcal{W}_{n+1} \setminus \langle \mathcal{W}_n \rangle$$

Proof. Consider

$$A = \left\{ \sum_{i=0}^{k} c_i \cdot n^{2i} : k \in \omega, c_i = 0, \ldots, n-1, c_k \neq 0 \right\}$$

Strictly increasing sequence of ideals

Claim 1. Show $A \in \mathcal{W}_{n+1}$ (straightforward calculation)

Claim 2. Show $A \notin \langle \mathcal{W}_n \rangle$ (use Hales-Jewett theorem)

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Let $L(n)$ ... be the set of finite words in the alphabet $\{0, 1, \dots, n-1\}$.

A variable word $w(x)$ is a finite word in the alphabet $\{0, 1, \dots, n-1, x\}$ in which the variable x occurs at least once.

Hales-Jewett theorem

Hales-Jewett theorem.

For every $n, r \in \mathbb{N}$ there exists a number $HJ(n, r)$ such that if words in $L(n)$ of length $HJ(n, r)$ are colored by r colors then there exists a variable word $w(x)$ such that $w(0), w(1), \ldots,$ $w(n-1)$ have the same color.

The symbol $w(i)$ denotes the word in $L(n)$ which is produced from $w(x)$ by replacing all the occurences of the variable x by the letter of the alphabet in brackets.

Some questions

Conjecture. $A \in \langle \mathcal{W}_n \rangle$ if and only if there exists $k \in \mathbb{N}$ such that A does not contain a copy of n^k.

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Question 1. Is it true that whenever a set A does not contain a copy of 3^2 then $A \in \langle \mathcal{W}_3 \rangle$?

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Conjecture. $A \in \langle \mathcal{W}_n \rangle$ if and only if there exists $k \in \mathbb{N}$ such that A does not contain a copy of n^k.

Question 1. Is it true that whenever a set A does not contain a copy of 3^2 then $A \in \langle \mathcal{W}_3 \rangle$?

Question 2. Does the set $\{n^2 : n \in \omega\}$ belong to the ideal $\langle \mathcal{W}_3 \rangle$?

The square of squares...

$a^2 - d_1 - d_2$	$a^2 - d_2$	$a^2 + d_1 - d_2$
$a^2 - d_1$	a^2	$a^2 + d_1$
$a^2 - d_1 + d_2$	$a^2 + d_2$	$a^2 + d_1 + d_2$

...transforms...

$$a^2 + d_1 - d_2$$

$$a^2 - d_2 \qquad\qquad a^2 + d_1$$

$$a^2 - d_1 - d_2 \qquad\qquad a^2 \qquad\qquad a^2 + d_1 + d_2$$

$$a^2 - d_1 \qquad\qquad a^2 + d_2$$

$$a^2 - d_1 + d_2$$

...transforms...

$a^2 - d_2$	$a^2 + d_1 - d_2$	$a^2 + d_1$
$a^2 - d_1 - d_2$	a^2	$a^2 + d_1 + d_2$
$a^2 - d_1$	$a^2 - d_1 + d_2$	$a^2 + d_2$

...into a magic square of squares

$a^2 + d_1$	$a^2 + d_1 - d_2$	$a^2 - d_2$
$a^2 - d_1 - d_2$	a^2	$a^2 + d_1 + d_2$
$a^2 + d_2$	$a^2 - d_1 + d_2$	$a^2 - d_1$

1000€ problem

Problem 1. Can a 3x3 magic square be constructed with nine distinct square numbers?

Asked by Martin LaBar (1984), republished by Martin Gardner (1996) who offered $100 to the first person to construct such a square (or to prove its impossibility).

Problem 2. Provide a new example of 3x3 magic square with seven distinct square entries different from rotations, symmetries and k^2 multiples of the known example or provide any example with eight square entries.

Christian Boyer offers 1000€ prize + a bottle of champagne since 2010 (was only 100€ from 2005 to 2009).