# $\mathscr{I}$-ULTRAFILTERS FOR IDEAL $\mathscr{I}$ ON $\omega$

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ABSTRACT. This paper covers the talk held by the author at 35th Winter School on Abstract Analysis - Section Topology in Hejnice, 2007. It summarizes some presented general results concerning $\mathscr{I}$-ultrafilters where $\mathscr{I}$ is an ideal on $\omega$ and provides proofs which where omitted in the talk.

## 1. INTRODUCTION

**Definition 1.1** (Baumgartner)**.** Let $\mathscr{I}$ be a family of subsets of a set $X$ such that $\mathscr{I}$ contains all singletons and is closed under subsets. Given an ultrafilter $\mathscr{U}$ on $\omega$, we say that $\mathscr{U}$ is an $\mathscr{I}$-*ultrafilter* if for any $F : \omega \to X$ there is $A \in \mathscr{U}$ such that $F[A] \in \mathscr{I}$.

It is easy to see that $\mathscr{I}$-ultrafilters are downwards closed in the $\leq_{RK}$-ordering and if we have $\mathscr{I} \subseteq \mathscr{J}$ then every $\mathscr{I}$-ultrafilter is a $\mathscr{J}$-ultrafilter.

Particular examples of $\mathscr{I}$-ultrafilters are nowhere dense ultrafilters, measure zero ultrafilters or countably closed ultrafilters defined by taking $X = 2^\omega$ and $\mathscr{I}$ to contain all the nowhere dense sets, the sets with closure of measure zero, or the sets with countable closure, respectively. The class of $\alpha$-ultrafilters was defined for an indecomposable countable ordinal $\alpha$ by taking $X = \omega_1$ and $\mathscr{I}$ to consist of the subsets of $\omega_1$ with order type less than $\alpha$. Consistency results about existence of these ultrafilters and some inclusions among the appropriate classes of ultrafilters were obtained by Baumgartner [2], Brendle [4], Barney [1]. It was proved by Shelah [10] that consistently there are no nowhere dense ultrafilters, consequently all mentioned ultrafilters (except the $\alpha$-ultrafilters for which the question is still open) may not exist.

*Key words and phrases.* $\mathscr{I}$-ultrafilter, density ideal, summable ideal, van der Waerden ideal, analytic P-ideal.

We investigate here $\mathscr{I}$-ultrafilters in the setting $X = \omega$ (or $\mathbb{N}$). We assume that the family $\mathscr{I}$ contains all finite subsets of $\omega$ and is usually an ideal. In fact, it is not important whether we deal with ideals or not because if we replace the family $\mathscr{I}$ in the definition of $\mathscr{I}$-ultrafilter by $\langle \mathscr{I} \rangle$ (the ideal generated by $\mathscr{I}$), we get the same concept (noticed in [1], "proved" in my thesis).

An ideal $\mathscr{I} \subseteq \mathscr{P}(\omega)$ is called *tall* if every $A \notin \mathscr{I}$ contains an infinite subset that belongs to the ideal $\mathscr{I}$. (Some authors call such ultrafilters dense.) For every $A, B \subseteq \omega$ we say that $A$ is *almost contained* in $B$ and we write $A \subseteq^* B$ if $A \setminus B$ is finite. Let us also recall that an ideal $\mathscr{I}$ is called a *$P$-ideal* if whenever $A_n \in \mathscr{I}$, $n \in \omega$, then there is $A \in \mathscr{I}$ such that $A_n \subseteq^* A$ for every $n$.

**Example 1.2.** Density ideal $\mathscr{Z}_0 = \{A \subseteq \omega : \limsup_{n\to\infty} \frac{|A\cap n|}{n} = 0\}$, summable ideal $\mathcal{I}_{1/n} = \{A \subseteq \mathbb{N} : \sum_{n\in A} \frac{1}{n} < \infty\}$ and van der Waerden ideal $\mathscr{W} = \{A \subseteq \omega : A$ does not contain arithmetic progressions of arbitrary length$\}$ are tall ideals, but only the first two are $P$-ideals.

## 2. Some general results

Are there any $\mathscr{I}$-ultrafilters at all? It turns out that for some ideals the answer is positive. To prove this we shall recall that $\chi(\mathscr{I})$, the *character of $\mathscr{I}$*, is the minimal cardinality of a base for $\mathscr{I}$, i.e. $\chi(\mathscr{I}) = \min\{|\mathcal{B}| : \mathcal{B} \subseteq \mathscr{I} \wedge (\forall I \in \mathscr{I})(\exists B \in \mathcal{B})\ I \subseteq^* B\}$.

**Theorem 2.1.** *If $\mathscr{I}$ is a maximal ideal on $\omega$ such that $\chi(\mathscr{I}) = \mathfrak{c}$ then $\mathscr{I}$-ultrafilters exist.*

*Proof.* Enumerate all functions from $\omega$ to $\omega$ as $\{f_\alpha : \alpha < \mathfrak{c}\}$. By transfinite induction on $\alpha < \mathfrak{c}$ we will construct filter bases $\mathscr{F}_\alpha$ satisfying

(i) $\mathscr{F}_0$ is the Fréchet filter
(ii) $\mathscr{F}_\alpha \subseteq \mathscr{F}_\beta$ whenever $\alpha \leq \beta$
(iii) $\mathscr{F}_\gamma = \bigcup_{\alpha<\gamma} \mathscr{F}_\alpha$ for $\gamma$ limit
(iv) $(\forall \alpha)\ |\mathscr{F}_\alpha| \leq |\alpha| \cdot \omega$
(v) $(\forall \alpha)\ (\exists F \in \mathscr{F}_{\alpha+1})\ f_\alpha[F] \in \mathscr{I}$

Suppose we know already $\mathscr{F}_\alpha$. If there is a set $F \in \mathscr{F}_\alpha$ such that $f_\alpha[F] \in \mathscr{I}$ then put $\mathscr{F}_{\alpha+1} = \mathscr{F}_\alpha$. Hence we may assume that $f_\alpha[F] \notin \mathscr{I}$. Then $\omega \setminus f_\alpha[F] \in \mathscr{I}$ for every $F \in \mathscr{F}_\alpha$ and since $\chi(\mathscr{I}) = \mathfrak{c} > |\mathscr{F}_\alpha|$ we can find $M \in \mathscr{I}$ such that $M \cap f_\alpha[F]$ is infinite for every $F \in \mathscr{F}_\alpha$. To complete the induction step let $\mathscr{F}_{\alpha+1}$ be the filter base generated by $\mathscr{F}_\alpha$ and $f_\alpha^{-1}[M]$.

It is obvious that any ultrafilter that extends the filter base $\mathscr{F} = \bigcup_{\alpha<\mathfrak{c}} \mathscr{F}_\alpha$ is an $\mathscr{I}$-ultrafilter. □

Unfortunately, there are many interesting ideals which are not maximal, e.g. all the three in Example 1.2, and to which we cannot apply Theorem 2.1. What about ideals which are not maximal? At first, we give a necessary condition on $\mathscr{I}$.

**Proposition 2.2.** *If the ideal $\mathscr{I}$ is not tall then there are no $\mathscr{I}$-ultrafilters.*

*Proof.* Suppose that for $A \in [\omega]^\omega \setminus \mathscr{I}$ we have $\mathscr{I} \cap \mathscr{P}(A) = [A]^{<\omega}$ and let $e_A : \omega \to A$ be an increasing enumeration of the set $A$.

Now assume for the contrary that there exists an $\mathscr{I}$-ultrafilter $\mathscr{U} \in \omega^*$. According to the definition of an $\mathscr{I}$-ultrafilter there exists $U \in \mathscr{U}$ such that $e_A[U] \in \mathscr{I}$. Since $e_A[U] \subseteq A$ the set $e_A[U]$ is finite. It follows that $U$ is finite because $e_A$ is one-to-one — a contradiction to the assumption that no set in $\mathscr{U}$ is finite. □

Under some additional set-theoretic assumptions the necessary condition from Proposition 2.2 is also sufficient. We recall the definition of the *pseudointersection number*:
$\mathfrak{p} = \min\{|\mathscr{F}| : \mathscr{F} \subseteq [\omega]^\omega \text{ centered}, \neg((\exists A \in [\omega]^\omega)(\forall F \in \mathscr{F}) A \subseteq^* F)\}$

**Proposition 2.3.** *($\mathfrak{p}= \mathfrak{c}$) If $\mathscr{I}$ is a tall ideal then $\mathscr{I}$-ultrafilters exist.*

*Proof.* Enumerate all functions from $\omega$ to $\omega$ as $\{f_\alpha : \alpha < \mathfrak{c}\}$. By transfinite induction on $\alpha < \mathfrak{c}$ we will construct filter bases $\mathscr{F}_\alpha$ satisfying

(i) $\mathscr{F}_0$ is the Fréchet filter
(ii) $\mathscr{F}_\alpha \subseteq \mathscr{F}_\beta$ whenever $\alpha \leq \beta$
(iii) $\mathscr{F}_\gamma = \bigcup_{\alpha<\gamma} \mathscr{F}_\alpha$ for $\gamma$ limit
(iv) $(\forall\alpha)$ $|\mathscr{F}_\alpha| \leq |\alpha| \cdot \omega$
(v) $(\forall\alpha)$ $(\exists F \in \mathscr{F}_{\alpha+1})$ $f_\alpha[F] \in \mathscr{I}$

Suppose we know already $\mathscr{F}_\alpha$. If there is a set $F \in \mathscr{F}_\alpha$ such that $f_\alpha[F] \in \mathscr{I}$ then put $\mathscr{F}_{\alpha+1} = \mathscr{F}_\alpha$. Hence we may assume that $f_\alpha[F] \notin \mathscr{I}$, in particular $f_\alpha[F]$ is infinite, for every $F \in \mathscr{F}_\alpha$.

Since $|\mathscr{F}_\alpha| < \mathfrak{c} = \mathfrak{p}$ there exists $M \in [\omega]^\omega$ such that $M \subseteq^* f_\alpha[F]$ for every $F \in \mathscr{F}_\alpha$. The ideal $\mathscr{I}$ is tall, so there is $A \in \mathscr{I}$ which is an infinite subset of $M$ and we have $A \subseteq^* f_\alpha[F]$ and $f_\alpha^{-1}[A] \cap F$ is infinite for every $F \in \mathscr{F}_\alpha$. It follows that $f_\alpha^{-1}[A]$ is compatible with $\mathscr{F}_\alpha$. To complete the induction step let $\mathscr{F}_{\alpha+1}$ be the filter base generated by $\mathscr{F}_\alpha$ and $f_\alpha^{-1}[A]$.

It is easy to see that every ultrafilter that extends $\mathscr{F} = \bigcup_{\alpha<\mathfrak{c}} \mathscr{F}_\alpha$ is an $\mathscr{I}$-ultrafilter. □

Is it possible to weaken somehow the assumption $\mathfrak{p} = \mathfrak{c}$? Yes, but it seems we have to restrict the class of considered ideals. We will show that if $\mathscr{I}$ is a tall analytic $P$-ideal or $F_\sigma$-ideal then it is enough to assume that a selective ultrafilter exists.

A free ultrafilter $\mathscr{U}$ is called a *selective ultrafilter* (or a Ramsey ultrafilter) if for all partitions of $\mathbb{N}$, $\{R_i : i \in \omega\}$, either for some $i$, $R_i \in \mathscr{U}$, or $(\exists U \in \mathscr{U})$ $(\forall i \in \omega)$ $|U \cap R_i| \leq 1$. It is proved in [5] that selective ultrafilters are minimal in Rudin-Keisler order on ultrafilters.

A free ultrafilter $\mathscr{U}$ is called a *$P$-point* if for all partitions of $\mathbb{N}$, $\{R_i : i \in \omega\}$, either for some $i$, $R_i \in \mathscr{U}$, or $(\exists U \in \mathscr{U})$ $(\forall i \in \omega)$ $|U \cap R_i| < \omega$. Equivalently: $\mathscr{U} \in \mathbb{N}^*$ is a $P$-point if and only if whenever $U_n \in \mathscr{U}$, $n \in \omega$, there is $U \in \mathscr{U}$ such that $U \subseteq^* U_n$ for each $n$. The class of $P$-points is downwards closed under Rudin-Keisler order (see e.g. [5]).

A free ultrafilter $\mathscr{U}$ is called a *$Q$-point* if for every partition $\{Q_n : n \in \omega\}$ of $\omega$ into finite sets there exists $U \in \mathscr{U}$ such that $|U \cap Q_n| \leq 1$ for every $n \in \omega$. $Q$-points in general are not downwards closed in the $\leq_{RK}$-ordering.

We say that a free ultrafilter $\mathscr{U}$ is a *hereditary $Q$-point* if it is a $Q$-point such that for every $\mathscr{V} \leq_{RK} \mathscr{U}$ the ultrafilter $\mathscr{V}$ is again a $Q$-point.

Notice that a free ultrafilter $\mathscr{U} \in \omega^*$ is selective if and only if $\mathscr{U}$ is $P$-point and $Q$-point.

**Lemma 2.4.** *If $\mathscr{I}$ is a tall $F_\sigma$-ideal or a tall analytic $P$-ideal and $\mathscr{U}$ is a $Q$-point then $\mathscr{U} \cap \mathscr{I} \neq \emptyset$.*

*Proof.* Let us first prove the lemma for an analytic $P$-ideal $\mathscr{I}$. Solecki in [11] proved that there exists a lower semicontinuous submeasure $\varphi : \mathscr{P}(\omega) \to [0, \infty]$ such that $\mathscr{I} = \mathrm{Exh}(\varphi) = \{A \subseteq \omega :$

$\lim_{n\to\infty}\varphi(A\setminus n)=0\}$. The ideal $\mathrm{Exh}(\varphi)$ is tall if and only if $\lim_{n\to\infty}\varphi(\{n\})=0$. So we can define by induction an increasing sequence $\langle n_k\rangle_{k\in\omega}$ such that $\varphi(\{m\})<\frac{1}{2^k}$ whenever $m\geq n_k$. Now, since $\mathscr{U}$ is a $Q$-point there exists $U\in\mathscr{U}$ such that $|U\cap[n_k,n_{k+1})|\leq 1$ for every $k\in\omega$. It is not difficult to check that $U\in\mathrm{Exh}(\varphi)=\mathscr{I}$.

If $\mathscr{I}$ is a tall $F_\sigma$-ideal then according to a characterisation of Mazur [9] there is again a lower semicontinuous submeasure $\varphi:\mathscr{P}(\omega)\to[0,\infty]$ such that $\mathscr{I}=\mathrm{Fin}(\varphi)=\{A\subseteq\omega:\varphi(A)<\infty\}$. Since $\mathrm{Exh}(\varphi)\subseteq\mathrm{Fin}(\varphi)$ and every $Q$-point contains a set from $\mathrm{Exh}(\varphi)$ it also contains a set from $\mathrm{Fin}(\varphi)=\mathscr{I}$. □

**Proposition 2.5.** *If $\mathscr{I}$ is $F_\sigma$ ideal or analytic $P$-ideal then every selective ultrafilter is an $\mathscr{I}$-ultrafilter.*

*Proof.* If $\mathscr{U}$ is a selective ultrafilter then for every $f:\omega\to\omega$ the ultrafilter $f(\mathscr{U})$ is also selective and in particular a $Q$-point. Hence $f(\mathscr{U})\cap\mathscr{I}\neq\emptyset$ and it follows that there is $U\in\mathscr{U}$ such that $f[U]\in\mathscr{I}$. □

Notice that actually every hereditary $Q$-point is an $\mathscr{I}$-ultrafilter.

*Remark* 2.6. If we assume $\mathrm{MA}_{\mathrm{ctble}}$ then for every (tall) ideal $\mathscr{I}\subseteq\mathscr{P}(\omega)$ there is a $Q$-point which is not an $\mathscr{I}$-ultrafilter (= Proposition 2.4.7. in [8]).

## 3. Description of $P$-points via $\mathscr{I}$-ultrafilters

If $X=2^\omega$ then $P$-points are precisely the $\mathscr{I}$-ultrafilters for $\mathscr{I}$ consisting of all finite and converging sequences, if $X=\omega_1$ then $P$-points are precisely the $\mathscr{I}$-ultrafilters for $\mathscr{I}=\{A\subseteq\omega_1: A$ has order type $\leq\omega\}$ (see [2]). Is there a family $\mathscr{I}$ of subsets of natural numbers such that $P$-points are precisely the corresponding $\mathscr{I}$-ultrafilters? The following theorem shows that such a family cannot be an $F_\sigma$ ideal.

**Theorem 3.1.** *($MA_{\mathrm{ctble}}$) For every $F_\sigma$-ideal $\mathscr{I}\subseteq\mathscr{P}(\omega)$ there is a $P$-point that is not an $\mathscr{I}$-ultrafilter.*

*Proof.* Since $\mathscr{I}$ is $F_\sigma$ there is a lower semicontinuous submeasure $\varphi:\mathscr{P}(\omega)\to[0,\infty]$ such that $\mathscr{I}=\mathrm{Fin}(\varphi)=\{A\subseteq\omega:\varphi(A)<\infty\}$ (see [9]). Enumerate all partitions of $\omega$ (into infinite sets) as $\{\mathscr{R}_\alpha:\alpha<\mathfrak{c}\}$. By transfinite induction on $\alpha<\mathfrak{c}$ we will construct filter bases $\mathscr{F}_\alpha$, $\alpha<\mathfrak{c}$, so that the following conditions are satisfied:

(i) $\mathscr{F}_0$ is the Fréchet filter
(ii) $\mathscr{F}_\alpha \subseteq \mathscr{F}_\beta$ whenever $\alpha \leq \beta$
(iii) $\mathscr{F}_\gamma = \bigcup_{\alpha<\gamma} \mathscr{F}_\alpha$ for $\gamma$ limit
(iv) $(\forall\alpha)\ |\mathscr{F}_\alpha| \leq |\alpha| \cdot \omega$
(v) $(\forall\alpha)\ (\forall F \in \mathscr{F}_\alpha)\ \varphi(F) = \infty$
(vi) $(\forall\alpha)\ (\exists F \in \mathscr{F}_{\alpha+1})$ either $(\exists R_n^\alpha \in \mathscr{R}_\alpha)\ F \subseteq R_n^\alpha$ or $(\forall R_n^\alpha \in \mathscr{R}_\alpha)\ |F \cap R_n^\alpha| < \omega$

Induction step: Suppose we already know $\mathscr{F}_\alpha$ and we construct $\mathscr{F}_{\alpha+1}$.

Case A. $(\exists K \in [\omega]^{<\omega})\ (\forall F \in \mathscr{F}_\alpha)\ \varphi(F \cap \bigcup_{n\in K} R_n^\alpha) = \infty$

If for every $n \in K$ there is a set $F_n \in \mathscr{F}_\alpha$ such that $\varphi(F_n \cap R_n^\alpha) < \infty$ then $\varphi(\bigcap_{n\in K} F_n \cap \bigcup_{n\in K} R_n^\alpha) < \infty$ — a contradiction to the assumption of *Case A*. Thus for some $n_0 \in K$ we have $\varphi(F \cap R_{n_0}^\alpha) = \infty$ for every $F \in \mathscr{F}_\alpha$ and to complete the induction step let $\mathscr{F}_{\alpha+1}$ be the filter base generated by $\mathscr{F}_\alpha$ and the set $R_{n_0}^\alpha$.

Case B. $(\forall K \in [\omega]^{<\omega})\ (\exists F_K \in \mathscr{F}_\alpha)\ \varphi(F_K \cap \bigcup_{n\in K} R_n^\alpha) < \infty$

Consider $P = \{\langle K, n\rangle \in [\omega]^{<\omega} \times \omega : K \subseteq \bigcup_{i\leq n} R_i^\alpha,\ K \cap R_n^\alpha \neq \emptyset\}$ and define $\langle K, n\rangle \leq_P \langle L, m\rangle$ if $\langle K, n\rangle = \langle L, m\rangle$ or $K \supset L$, $\min(K \setminus L) > \max L$, $n > m$ and $(K \setminus L) \cap \bigcup_{i\leq m} R_i^\alpha = \emptyset$. This is a countable poset which we can apply Martin's Axiom on. We only have to find a suitable collection of its dense subsets – for $F \in \mathscr{F}_\alpha$ and $k, j \in \omega$ let $D_{F,k} = \{\langle K, n\rangle \in P : \varphi(K \cap F) \geq k\}$ and $D_j = \{\langle K, n\rangle \in P : n \geq j\}$.

*Claim: $D_{F,k}$ is dense in $(P, \leq_P)$ for every $F \in \mathscr{F}_\alpha$ and $k \in \omega$; $D_j$ is dense in $(P, \leq_P)$ for every $j \in \omega$.*

Consider $\langle L, m\rangle \in P$ arbitrary. Since $L$ is finite there exists $p \geq m$ such that $[0, \max L] \subseteq \bigcup_{i\leq p} R_i^\alpha$. According to the assumption there is $F_p \in \mathscr{F}_\alpha$ such that $\varphi(F_p \cap \bigcup_{i\leq p} R_i^\alpha) < \infty$. It follows that $\varphi((F_p \cap F) \setminus \bigcup_{i\leq p} R_i^\alpha) = \infty$. We can choose a finite set $L' \subseteq (F_p \cap F) \setminus \bigcup_{i\leq p} R_i^\alpha$ such that $\varphi(L') \geq k$ because $\varphi$ is lower semicontinuous. Let $n = \max\{i : L' \cap R_i^\alpha \neq \emptyset\}$ and $K = L \cup L'$. Note that from the choice of $p$ we have $\min L' > \max L$ and it follows that $\langle K, n\rangle \leq_P \langle L, m\rangle$ and $\langle K, n\rangle \in D_{F,k}$. So $D_{F,k}$ is dense. For $j \leq m$ we have $\langle L, m\rangle \in D_j$ and for any $j > m$ we can choose arbitrary $r \in R_j^\alpha$ such that $r > \max L$. Let $K' = L \cup \{r\}$. Of course, $\langle K', j\rangle \leq_P \langle L, m\rangle$ and $\langle K', j\rangle \in D_j$. So $D_j$ is dense. And the claim is proved.

The family $\mathscr{D} = \{D_{F,k} : F \in \mathscr{F}_\alpha, k \in \omega\} \cup \{D_j : j \in \omega\}$ consists of dense subsets in $P$ and $|\mathscr{D}| < \mathfrak{c}$. Therefore there is a $\mathscr{D}$-generic filter $\mathscr{G}$. Let $U = \bigcup\{K : \langle K, n\rangle \in \mathscr{G}\}$. It remains to check that:

- $(\forall F \in \mathscr{F}_\alpha)\ \varphi(U \cap F) = \infty$

Take $k \in \omega$ arbitrary. For every $K \in \mathscr{G} \cap D_{F,k}$ we have $U \supseteq K$ and $k \leq \varphi(K \cap F) \leq \varphi(U \cap F)$. Hence $\varphi(U \cap F) = \infty$.

- $(\forall R^\alpha_n \in \mathscr{R}_\alpha)\ |U \cap R^\alpha_n| < \omega$

Take $\langle K_n, j_n\rangle \in \mathscr{G} \cap D_n$ where $j_n = \min\{j : (\exists K \in [\omega]^{<\omega})\langle K, j\rangle \in \mathscr{G} \cap D_n\}$. Now observe that for $\langle K, m\rangle \in \mathscr{G}$ we have $K \cap R^\alpha_n = \emptyset$ if $m < n$ and that $K \cap R^\alpha_n = K_n \cap R^\alpha_n$ if $m \geq n$. To see the latter consider $\langle L, m'\rangle \in \mathscr{G}$ such that $\langle L, m'\rangle \leq_P \langle K, m\rangle$ and $\langle L, m'\rangle \leq_P \langle K_n, j_n\rangle$ (such a condition exists because $\mathscr{G}$ is a filter) for which we get $L \cap R^\alpha_n = K \cap R^\alpha_n$ and $L \cap R^\alpha_n = K_n \cap R^\alpha_n$. It follows that $U \cap R^\alpha_n = K_n \cap R^\alpha_n$ is finite.

To complete the induction step let $\mathscr{F}_{\alpha+1}$ be the filter base generated by $\mathscr{F}_\alpha$ and the set $U$.

It follows from condition (vi) that every ultrafilter which extends filter base $\mathscr{F} = \bigcup_{\alpha<\mathfrak{c}} \mathscr{F}_\alpha$ is a $P$-point. Because of condition (v) there exists an ultrafilter extending $\mathscr{F}$ which extends the dual filter of $\mathrm{Fin}(\varphi) = \mathscr{I}$, i.e. it is not an $\mathscr{I}$-ultrafilter. □

For analytic $P$-ideals the answer is not so easy. On one hand we know from the previous proposition that there is a $P$-point not $\mathcal{I}_{1/n}$-ultrafilter, on the other hand every $P$-point is a $\mathcal{Z}_0$-ultrafilter. So, is there any hope to characterize $P$-points as $\mathscr{I}$-ultrafilters for some analytic $P$-ideal $\mathscr{I}$?

**Theorem 3.2** ([6])**.** *(CH) Let $\mathscr{I}$ be an analytic $P$-ideal. There exists a $P$-point which is not an $\mathscr{I}$-ultrafilter if and only if $\mathscr{P}(\omega)/\mathscr{I}$ does not have a countable splitting family.*

Thus $P$-points are contained in the class of $\mathscr{I}$-ultrafilters for some analytic $P$-ideals. However, the following proposition shows that $P$-ideals in general are not good candidates for characterisation of $P$-points via $\mathscr{I}$-ultrafilters.

**Proposition 3.3.** *($\mathfrak{p}=\mathfrak{c}$) If $\mathscr{I}$ is a tall $P$-ideal then there is an $\mathscr{I}$-ultrafilter which is not a $P$-point.*

*Proof.* We proved in Proposition 2.3 that assuming $\mathfrak{p} = \mathfrak{c}$ there exist $\mathscr{I}$-ultrafilters for every tall ideal $\mathscr{I}$. We will show that if $\mathscr{I}$ is a tall

$P$-ideal then the square of an $\mathscr{I}$-ultrafilter is again an $\mathscr{I}$-ultrafilter and it is not a $P$-point. So let us recall the definition of the product of ultrafilters (see e. g. [5]): If $\mathscr{U}$ and $\mathscr{V}$ are ultrafilters on $\omega$ then $\mathscr{U} \cdot \mathscr{V} = \{A \subseteq \omega \times \omega : \{n : \{m : \langle n, m\rangle \in A\} \in \mathscr{V}\} \in \mathscr{U}\}$ is an ultrafilter on $\omega \times \omega$. By the square of ultrafilter $\mathscr{U}$ we mean the ultrafilter $\mathscr{U} \cdot \mathscr{U}$.

Notice that the partition $\{\{n\} \times \omega : n \in \omega\}$ of $\omega \times \omega$ witnesses the fact that no product of free ultrafilters on $\omega$ is a $P$-point. Hence to complete the proof of Proposition 3.3 it is sufficient to check that if $\mathscr{U}$ is an $\mathscr{I}$-ultrafilter then $\mathscr{U} \cdot \mathscr{U}$ is again an $\mathscr{I}$-ultrafilter, i. e. for every $f : \omega \times \omega \to \omega$ there is $U \in \mathscr{U} \cdot \mathscr{U}$ such that $f[U] \in \mathscr{I}$.

So assume $\mathscr{U}$ is an $\mathscr{I}$-ultrafilter and $f : \omega \times \omega \to \omega$ an arbitrary function. For every $n \in \omega$ define $f_n : \omega \to \omega$ by $f_n(m) = f(\langle n, m\rangle)$.

Since $\mathscr{U}$ is an $\mathscr{I}$-ultrafilter there exists $V_n \in \mathscr{U}$ such that $f_n[V_n] \in \mathscr{I}$ for every $n$. Now we can find a set $A \in \mathscr{I}$ such that $f_n[V_n] \subseteq^* A$ for every $n$ because $\mathscr{I}$ is a $P$-ideal.

Obviously, for every $n \in \omega$ we have $f_n^{-1}[f_n[V_n]] \in \mathscr{U}$. It follows that either $f_n^{-1}[f_n[V_n] \cap A]$ or $f_n^{-1}[f_n[V_n] \setminus A]$ belongs to $\mathscr{U}$. Let $I_0 = \{n \in \omega : f_n^{-1}[f_n[V_n] \cap A] \in \mathscr{U}\}$ and $I_1 = \{n \in \omega : f_n^{-1}[f_n[V_n] \setminus A] \in \mathscr{U}\}$. Since $\mathscr{U}$ is an ultrafilter it contains one of the sets $I_0$, $I_1$.

<u>*Case A.*</u> $I_0 \in \mathscr{U}$

Put $U = \{\{n\} \times f_n^{-1}[f_n[V_n] \cap A] : n \in I_0\}$. It is easy to see that $U \in \mathscr{U} \cdot \mathscr{U}$ and $f[U] = \bigcup_{n \in I_0} f_n[V_n] \cap A \subseteq A \in \mathscr{I}$.

<u>*Case B.*</u> $I_1 \in \mathscr{U}$

Since $f_n[V_n] \setminus A$ is finite and $\mathscr{U}$ is an ultrafilter, there exists $k_n \in f_n[V_n] \setminus A$ such that $f_n^{-1}\{k_n\} \in \mathscr{U}$. Define $g : \omega \to \omega$ by $g(n) = k_n$. Since $\mathscr{U}$ is an $\mathscr{I}$-ultrafilter there exists $V \in \mathscr{U}$ such that $g[V] \in \mathscr{I}$. It remains to put $U = \{\{n\} \times f_n^{-1}\{k_n\} : n \in I_1 \cap V\}$. It is easy to check that $U \in \mathscr{U} \cdot \mathscr{U}$ and $f[U] \subseteq g[V] \in \mathscr{I}$. □

Is it possible to characterise $P$-points as $\mathscr{I}$-ultrafilters for some non-$P$-ideal $\mathscr{I}$?

## 4. Three other families of subsets of $\omega$

It is difficult to formulate any general statements about ideals which are neither (analytic) $P$-ideals nor $F_\sigma$ ideals because there is no 'nice' description of such ideals. So we study in this section just three particular families of subsets of $\omega$.

**Definition 4.1.** A set $A \in [\omega]^\omega$ with an (increasing) enumeration $A = \{a_n : n \in \mathbb{N}\}$ is called

- *thin* (see [3]) if $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 0$.
- *almost thin* if $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} < 1$.
- *(SC)-set* if $\lim_{n\to\infty} a_{n+1} - a_n = \infty$.

It is easy to check that every thin set is almost thin and every almost thin set is an $(SC)$-set. Hence the corresponding classes of ultrafilters are also in inclusion. In fact, thin ultrafilters and almost thin ultrafilters coincide although the ideals generated by thin and almost thin sets are two distinct $F_{\sigma\delta}$ ideals. The ideal generated by $(SC)$-sets is an $F_{\sigma\delta}$ ideal too and the corresponding $(SC)$-ultrafilters need not be thin.

**Proposition 4.2.** *For $\mathscr{U} \in \omega^*$ the following are equivalent:*

(1) *$\mathscr{U}$ is a thin ultrafilter.*
(2) *$\mathscr{U}$ is an almost thin ultrafilter.*
(3) *$\mathscr{U}$ is a hereditary Q-point.*

Proposition 4.2 was proved in [7].

**Theorem 4.3** (= Proposition 2.3.3. in [8])**.** *Every P-point is an $(SC)$-ultrafilter.*

*Proof.* Let $\mathscr{U}$ be a $P$-point. Consider an arbitrary function $f : \omega \to \omega$. Our aim is to find $U \in \mathscr{U}$ such that $f[U] \in (SC)$.

Take arbitrary $U_0 \in \mathscr{U}$. If $f[U_0] \in (SC)$ then set $U = U_0$. Otherwise, we will proceed by induction. Suppose we know already $U_i \in \mathscr{U}$, $i = 0, 1, \ldots, k-1$, such that $U_i \subseteq U_{i-1}$ for $i > 0$ and the difference of two successive elements of $f[U_i]$ is greater or equal to $2^i$ for every $i < k$. Enumerate $f[U_{k-1}] = \{u_n : n \in \omega\}$. Since $\mathscr{U}$ is an ultrafilter either $f^{-1}[\{u_{2n} : n \in \omega\}] \cap U_{k-1}$ or $f^{-1}[\{u_{2n+1} : n \in \omega\}] \cap U_{k-1}$ belongs to $\mathscr{U}$. Denote this set by $U_k$. If $f[U_k] \in (SC)$ then let $U = U_k$. If $f[U_k] \notin (SC)$ then we may continue the induction because the difference of two successive elements of $f[U_k]$ is greater or equal to $2 \cdot 2^{k-1} = 2^k$.

If we obtain an infinite sequence of sets $U_n \in \mathscr{U}$ such that $U_n \supseteq U_{n+1}$ and the difference of two successive elements of $f[U_n]$ is greater or equal to $2^n$ for every $n$ then since $\mathscr{U}$ is a $P$-point there is $U \in \mathscr{U}$ such that $U \subseteq^* U_n$ for every $n \in \omega$. For this $U$ we have $f[U] \subseteq^* f[U_n]$ for every $n \in \omega$. Thus for every $k \in \omega$ all but finitely many pairs of successive elements in $f[U]$ have difference greater or equal to $2^k$ and it follows that $f[U] \in (SC)$. $\square$

**Theorem 4.4.** *($MA_{\mathrm{ctble}}$)*

(1) *There is a $P$-point that is not a thin ultrafilter.*
(2) *There exists a thin ultrafilter which is not a $P$-point.*

*Proof.* Since the ideal generated by thin sets is contained in $F_\sigma$ ideal $\mathcal{I}_{1/n}$ part (1) is a consequence of Proposition 3.1. A thin ultrafilter which is not a $P$-point was constructed in [7]. $\square$

## References


[1] Barney, C., Ultrafilters on the natural numbers, *J. Symbolic Logic* **68**(2003), no. 3, 764–784.

[2] Baumgartner, J., Ultrafilters on $\omega$, *J. Symbolic Logic* **60**, no. 2, 624–639, 1995.

[3] Blass, A., Frankiewicz, R., Plebanek, G., Ryll-Nardzewski, C., A note on extensions of asymptotic density, *Proc. Amer. Math. Soc.* **129**, 97–103, 2001.

[4] Brendle, J., Between $P$-points and nowhere dense ultrafilters, *Israel J. Math.* **113**, 205–230, 1999.

[5] Comfort, W. W., Negrepontis, S., *The Theory of Ultrafilters*, Springer, Berlin, New York, 1974.

[6] Filipów, R., Mrożek, N., Recław, I., Szuca, P., Ideal convergence of bounded sequences, *preprint*.

[7] Flašková, J., Thin Ultrafilters, *Acta Univ. Carolinae – Math. et Phys.* **46**, no. 2, 13 – 19, 2005.

[8] Flašková, J., Ultrafilters and small sets, Ph.D. Thesis, Charles University, Prague.

[9] Mazur, K., $F_\sigma$-ideals and $\omega_1\omega_1^*$-gaps in the Boolean algebra $\mathscr{P}\omega/I$, *Fund. Math.* **138**, 103 – 111, 1991.

[10] Shelah, S., There may be no nowhere dense ultrafilters, in: *Proceedings of the logic colloquium Haifa '95*, Lecture notes Logic, 11, 305–324, Springer, Berlin, 1998.

[11] Solecki, S., Analytic ideals and their applications, *Ann. Pure and Applied Logic* **99**, 51 – 72, 1999.



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