Van der Waerden spaces and some other subclasses of sequentially compact spaces

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Sequentially compact spaces

All topological spaces are Hausdorff.

Definition.

A topological space X is called sequentially compact if for every sequence $\langle x_n \rangle_{n\in\omega}$ in X there exists a converging subsequence $\langle x_{n_k} \rangle_{k\in\omega}$.

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Is it possible to choose the subsequence so that the set of indices is "large"?

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Definition A. (Kojman)

A topological space X is called van der Waerden if for every sequence $\langle x_n \rangle_{n \in \omega}$ in X there exists a converging subsequence $\langle x_{n_k} \rangle_{k \in \omega}$ so that $\{n_k : k \in \omega\}$ is an AP-set.

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For example, compact metric spaces or every succesor ordinal with the order topology satisfy $(*)$.

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Every van der Waerden space is sequentially compact.

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Theorem (Kojman)

There exists a compact, sequentially compact, separable space which is first-countable at all points but one, which is not van der Waerden.

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!!! only finite T_2 spaces fullfill the condition!!!

Two F_σ-ideals

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- (Folkman-Rado-Sanders)
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- Ideal $\mathcal{I}_{ipr}$ is an F_σ-ideal

$\mathcal{I}_{1/n}$-spaces and $\mathcal{I}_{\mathrm{ipr}}$

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Definition C.

A topological space X is called $\mathcal{I}_{1/n}$-space if for every sequence $\langle x_n \rangle_{n\in\omega}$ in X there exists a converging subsequence $\langle x_{n_k} \rangle_{k\in\omega}$ so that $\{n_k : k \in \omega\}$ does not belong to $\mathcal{I}_{1/n}$.

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$\mathcal{I}$-spaces

Theorem 1.

If a Hausdorff space X satisfies the following condition

(∗) The closure of every countable set in X is compact and first-countable.

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- Theorem 1. is true for an arbitrary F_σ-ideal on ω.

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Note: $\Psi(\mathcal{A})$ is regular, first countable and separable.

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Proof. Consider the one-point compactification of $\Psi(\mathcal{A})$ for a suitable MAD family $\mathcal{A}$.

$\mathcal{I}_{1/n}$ & van der Waerden spaces

Erdős-Turán Conjecture.

Every set $A \notin \mathcal{I}_{1/n}$ is an AP-set.

If Erdős-Turán Conjecture is true then every $\mathcal{I}_{1/n}$-space is van der Waerden.

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Theorem 3.

($\mathrm{MA}_{\sigma-\mathrm{cent.}}$) There exists a van der Waerden space which is not an $\mathcal{I}_{1/n}$-space.

- Theorem 3. is true for an arbitrary F_σ P-ideal on ω.

Outline of the proof

Lemma

Assume $A \subseteq \mathbb{N}$ is an AP-set and $f : \mathbb{N} \to \mathbb{N}$.
There is an AP-set $C \subseteq A$ such that

(1) either f is constant on C

(2) or f is finite-to-one on C and $f[C] \in \mathcal{I}_{1/n}$.

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Proposition

($\mathrm{MA}_{\sigma-\mathrm{cent.}}$) There exists a MAD family $\mathcal{A} \subseteq \mathcal{I}_{1/n}$ so that for every AP-set $B \subseteq \mathbb{N}$ and every finite-to-one function $f : B \to \mathbb{N}$ there exists an AP-set $C \subseteq B$ and $A \in \mathcal{A}$ so that $f[C] \subseteq A$.

Some questions

Is it consistent that there is a van der Waerden space which is not an $\mathcal{I}_{ipr}$-space?

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Theorem 4.

($\mathrm{MA}_{\sigma-\mathrm{cent.}}$) There exists an $\mathcal{I}_{ipr}$-space which is not an $\mathcal{I}_{1/n}$-space.

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Theorem 4.

($\mathrm{MA}_{\sigma-\mathrm{cent.}}$) There exists an $\mathcal{I}_{ipr}$-space which is not an $\mathcal{I}_{1/n}$-space.

Is it consistent that there is an $\mathcal{I}_{ipr}$-space which is not van der Waerden?

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Proposition (Kojman)

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Is the product of two $\mathcal{I}_{ipr}$-spaces an $\mathcal{I}_{ipr}$-space?

References

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