Some ultrafilters on natural numbers

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Selective ultrafilters and P-points

Definition A.

A free ultrafilter $\mathcal{U}$ is called a selective ultrafilter (or a Ramsey ultrafilter) if for all partitions of ω, $\{R_i : i \in \omega\}$, either for some i, $R_i \in \mathcal{U}$, or $(\exists U \in \mathcal{U})\ (\forall i \in \omega)\ |U \cap R_i| \leq 1$.

A free ultrafilter $\mathcal{U}$ is called a P-point if for all partitions of ω, $\{R_i : i \in \omega\}$, either for some i, $R_i \in \mathcal{U}$, or $(\exists U \in \mathcal{U})\ (\forall i \in \omega)$ $|U \cap R_i| < \omega$.

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Theorem (Shelah)

It is consistent that there are no P-points.

Ultrafilter sums and products

Definition B.

Let $\mathcal{U}$ and $\mathcal{V}_n$, $n \in \omega$, be ultrafilters on ω.
$\mathcal{U}$-sum of ultrafilters $\mathcal{V}_n$, $\sum_{\mathcal{U}} \langle \mathcal{V}_n : n \in \omega \rangle$, is an ultrafilter on $\omega \times \omega$ defined by $M \in \sum_{\mathcal{U}} \langle \mathcal{V}_n : n \in \omega \rangle$ if and only if $\{n : \{m : \langle n, m \rangle \in A\} \in \mathcal{V}_n\} \in \mathcal{U}$.

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The product of ultrafilters $\mathcal{U}$ and $\mathcal{V}$, denoted by $\mathcal{U} \cdot \mathcal{V}$, is a $\mathcal{U}$-sum of ultrafilters $\mathcal{V}_n$, where $\mathcal{V}_n = \mathcal{V}$ for every $n \in \omega$.

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- $\mathcal{U} \cdot \mathcal{U}$ is usually abbreviated as $\mathcal{U}^2$
- $\mathcal{U}^{n+1}$ for $n > 1$ is defined recursively as $\mathcal{U} \cdot \mathcal{U}^n$
- $\mathcal{U}^\omega = \sum_{\mathcal{U}} \langle \mathcal{U}^n : n \in \omega \rangle$

$\mathcal{I}$-ultrafilters

Definition C. (Baumgartner)

Let $\mathcal{I}$ be a family of subsets of a set X such that $\mathcal{I}$ contains all singletons and is closed under subsets.
An ultrafilter $\mathcal{U}$ on ω is called an $\mathcal{I}$-ultrafilter if for every $F : \omega \to X$ there exists $A \in \mathcal{U}$ such that $F[A] \in \mathcal{I}$.

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- if $\mathcal{I} \subseteq \mathcal{J}$ then every $\mathcal{I}$-ultrafilter is a $\mathcal{J}$-ultrafilter
- $\mathcal{I}$-ultrafilters and $\langle\mathcal{I}\rangle$-ultrafilters coincide

where $\langle\mathcal{I}\rangle$ is the ideal generated by $\mathcal{I}$

$\mathcal{I}$-ultrafilters for $X = 2^\omega$

family $\mathcal{I}$	corresponding $\mathcal{I}$-ultrafilters
converging sequences and finite sets	P-points
discrete sets	discrete ultrafilters
scattered sets	scattered ultrafilters
$\{A : \mu(\bar{A}) = 0\}$	measure zero ultrafilters
nowhere dense sets	nowhere dense ultrafilters

$\mathcal{I}$-ultrafilters for $X = 2^\omega$

Theorem (Baumgartner)

($MA_{\sigma-centered}$)

1. There is a nwd ultrafilter which is not measure zero.
2. There is a measure zero ultrafilter which is not scattered.

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Theorem (Baumgartner)

1. For every free ultrafilter $\mathcal{U}$ on ω, $\mathcal{U}^\omega$ is not discrete.
2. If $\mathcal{U}$ is a scattered ultrafilter then $\mathcal{U}^\omega$ is scattered.
3. If $\mathcal{U}$ is a P-point then $\mathcal{U}^n$ is discrete for all $n \in \omega$.

$\mathcal{I}$-ultrafilters for $X = 2^\omega$

If $\mathcal{I}$ is the family of subsets of 2^ω with countable closure then the corresponding $\mathcal{I}$-ultrafilters are called countable closed ultrafilters (Brendle).

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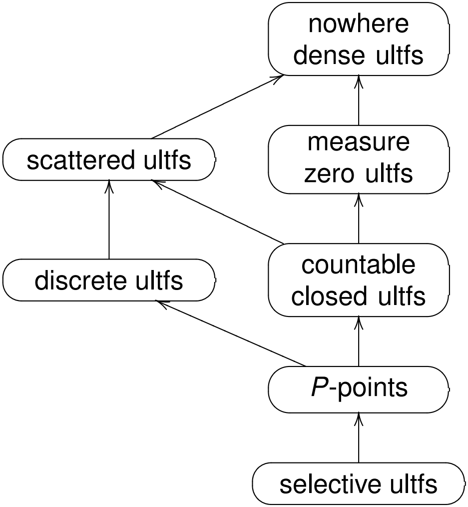
- Every P-point is countable closed.
- Every countable closed ultrafilter is both measure zero and scattered.

Between P-points and nowhere dense ultrafilters



$\mathcal{I}$-ultrafilters for $X = 2^\omega$

Theorem (Brendle)

($\mathrm{MA}_{\sigma-centered}$) There is a discrete ultrafilter which is not measure zero.

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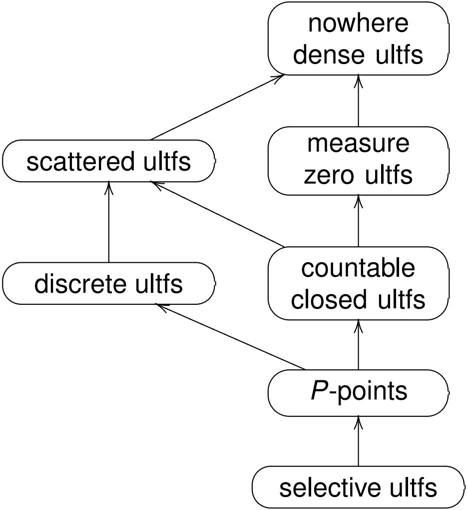
Theorem (Brendle)

If $\mathcal{U}$ is a P-point then $\mathcal{U}^\omega$ is countable closed.

Between P-points and nowhere dense ultrafilters



assuming ($\mathrm{MA}_{\sigma-centered}$)
no arrow can be added

Some forcing results

Theorem (Shelah)

1. It is consistent that there are no nowhere dense ultrafilters.
2. It is consistent that there are no P-points, but there is a nowhere dense ultrafilters.

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Theorem (Brendle)

1. It is consistent that there are no measure zero ultrafilters, but there is a nowhere dense ultrafilter.
2. It is consistent that there are no countable closed ultrafilters, but there is a measure zero ultrafilter.

Small subsets of ω

Let A be a subset of ω with an increasing enumeration $A = \{a_n : n \in \omega\}$. We say that A is

thin if $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 0$

(SC)-set if $\lim_{n\to\infty} a_{n+1} - a_n = \infty$

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$\mathcal{I}_{1/n} = \{A \subseteq \mathbb{N} : \sum_{a\in A} \frac{1}{a} < \infty\}$

$\mathcal{Z}_0 = \{A \subseteq \mathbb{N} : \limsup_{n\to\infty} \frac{|A\cap n|}{n} = 0\}$

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The corresponding $\mathcal{I}$-ultrafilters are called thin ultrafilters, (SC)-ultrafilters, $\mathcal{I}_{1/n}$-ultrafilters, $\mathcal{Z}_0$-ultrafilters.

$\mathcal{I}$-ultrafilters for $X = \omega$

It follows from the inclusion between appropriate families of subsets of ω that:

- every thin ultrafilter is both (SC)-ultrafilter and $\mathcal{I}_{1/n}$-ultrafilter
- every (SC)-ultrafilter and $\mathcal{I}_{1/n}$-ultrafilter as well is $\mathcal{Z}_0$-ultrafilter

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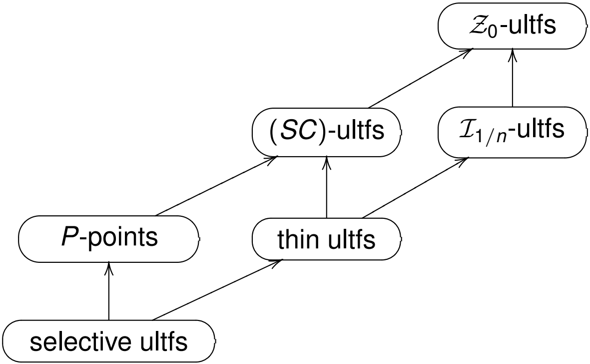
Theorem (Flašková)

1. Every P-point is (SC)-ultrafilter.
2. Every selective ultrafilter is thin.

$\mathcal{I}$-ultrafilters for $X = \omega$



$\mathcal{I}$-ultrafilters for $X = \omega$

Theorem (Flašková)

(MA_{ctble})

1. There exists a P-point which is not $\mathcal{I}_{1/n}$-ultrafilter.
2. There exists a thin ultrafilter which is not a P-point.

$\mathcal{I}$-ultrafilters for $X = \omega$

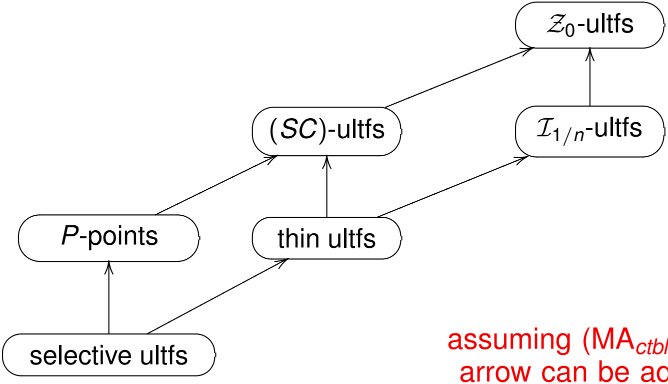
Theorem (Flašková)

(MA_{ctble})
1. There exists a P-point which is not $\mathcal{I}_{1/n}$-ultrafilter.
2. There exists a thin ultrafilter which is not a P-point.

Theorem (Flašková)

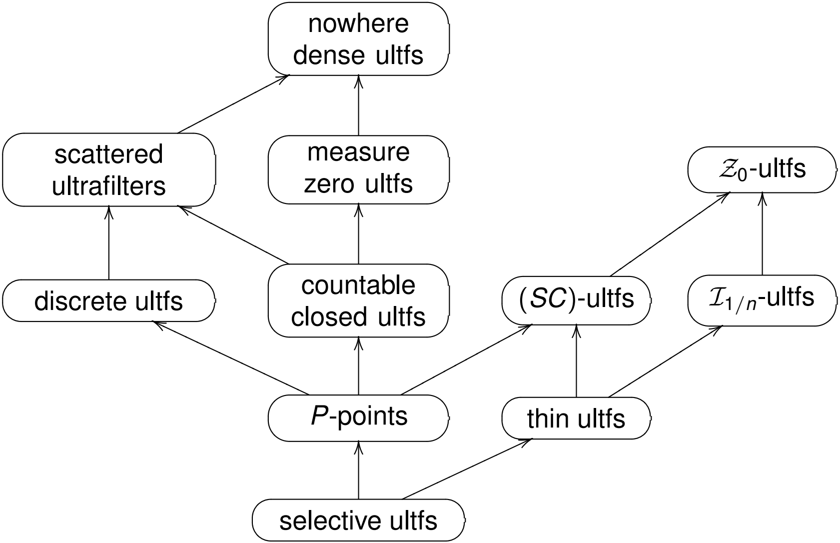
1. For every $\mathcal{U} \in \omega^*$, $\mathcal{U}^2$ is neither thin nor (SC)-ultrafilter.
2. Assume $\mathcal{I}$ is a P-ideal on ω.
If $\mathcal{U}$ and $\mathcal{V}_n$, $n \in \omega$, are $\mathcal{I}$-ultrafilters then $\mathcal{U}$-sum of ultrafilters $\mathcal{V}_n$ is $\mathcal{I}$-ultrafilter.

$\mathcal{I}$-ultrafilters for $X = \omega$



assuming (MA_{ctble}) no arrow can be added

Two diagrams in one



Some recent results

Theorem 1.

(MA_{ctble}) There exists a thin ultrafilter which is not a nowhere dense ultrafilter.

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Theorem 2.

Every measure zero ultrafilter is a $\mathcal{Z}_0$-ultrafilter.

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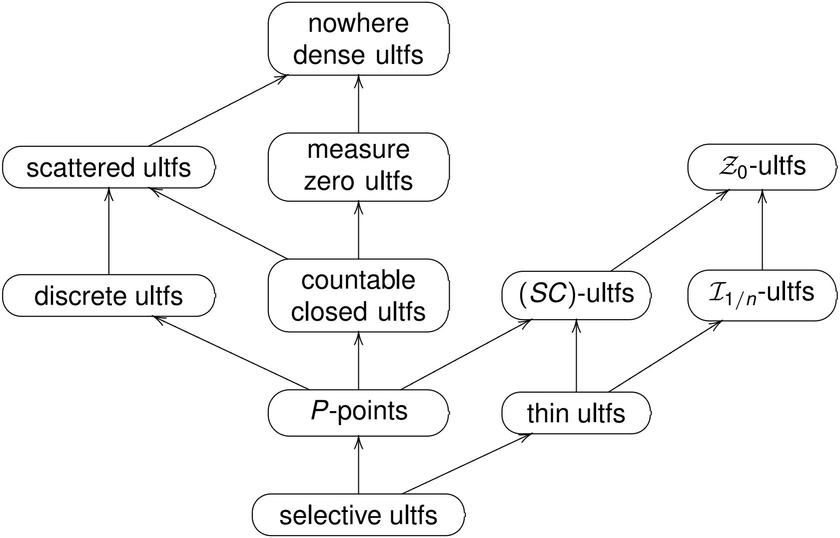
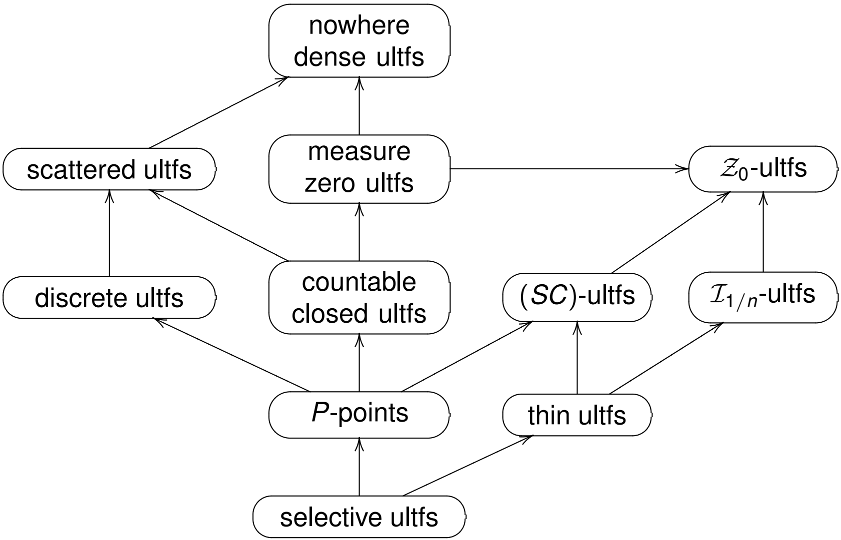


Complete(?) picture



Open Problem

Conjecture.

($MA_{\sigma-centered}$) There exists a discrete ultrafilter which is not a $\mathcal{Z}_0$-ultrafilter.

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Theorem (Brendle)

($MA_{\sigma-centered}$) There exists a discrete ultrafilter which is not a measure zero ultrafilter.

References

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