# Some remarks concerning van der Waerden ideal

Jana Flašková[1]

[1]Department of Mathematics
University of West Bohemia in Pilsen

Winter School on Abstract Analysis – section Set Theory
January 2013, Hejnice

# Arithmetic progressions and van der Waerden theorem

An arithmetic progression of length $l$ is the finite sequence $\{a + id : i = 0, 1, \ldots, l - 1\}$ where $a, d \in \mathbb{N}$.

Van der Waerden Theorem (finite version).

For any given natural numbers $k$ and $l$, there is some natural number $W(k, l)$ such that if the integers $\{1, 2, ..., W(k, l)\}$ are colored, each with one of $k$ different colors, then there exists an arithmetic progression of length at least $l$, all of which elements are of the same color.

# Van der Waerden theorem and AP-sets

**Definition.**

A set $A \subseteq \mathbb{N}$ is called an AP-set if it contains arbitrary long arithmetic progressions.

**Van der Waerden Theorem (infinite version).**

If an AP-set is partitioned into finitely many pieces then at least one of them is again an AP-set.

Sets which are not AP-sets form a proper ideal on $\mathbb{N}$
— van der Waerden ideal denoted by $\mathcal{W}$

# Van der Waerden ideal and other ideals

**Szemerédi Theorem.**

$$\mathcal{W} \subseteq \mathcal{Z} \quad \text{where} \quad \mathcal{Z} = \{A \subseteq \mathbb{N} : \limsup_{n\to\infty} \frac{|A \cap n|}{n} = 0\}$$

**Erdős Conjecture.**

$$\mathcal{W} \subseteq \mathcal{I}_{1/n} \quad \text{where} \quad \mathcal{I}_{1/n} = \{A \subseteq \mathbb{N} : \sum_{a \in A} \frac{1}{a} < \infty\}$$

# What sets belong to $\mathcal{W}$?

**Example A.** $\{n! : n \in \omega\}$ or $\{2^n : n \in \omega\}$ do not contain arithmetic progressions of length 3.

# What sets belong to $\mathcal{W}$?

**Example A.** $\{n! : n \in \omega\}$ or $\{2^n : n \in \omega\}$ do not contain arithmetic progressions of length 3.

**Example B.** $\{n^2 : n \in \omega\}$ contains infinitely many arithmetic progressions of length 3 (known by Pythagoras), but no arithmetic progression of length 4 (proved by Euler).

# What sets belong to $\mathcal{W}$?

Example A. $\{n! : n \in \omega\}$ or $\{2^n : n \in \omega\}$ do not contain arithmetic progressions of length 3.

Example B. $\{n^2 : n \in \omega\}$ contains infinitely many arithmetic progressions of length 3 (known by Pythagoras), but no arithmetic progression of length 4 (proved by Euler).

Example C. The set of the prime numbers does not belong to the van der Waerden ideal (Green-Tao).

# Van der Waerden ideal $\mathcal{W}$

The van der Waerden ideal $\mathcal{W}$ is

- a tall ideal — because every infinite $A \subseteq \mathbb{N}$ contains an infinite subset with no arithmetic progressions of length 3
- not a $P$-ideal — consider for example the sets

$$A_k = \{2^n + k : n \in \omega\} \text{ for } k \in \omega$$

# Van der Waerden ideal $\mathcal{W}$

The van der Waerden ideal $\mathcal{W}$ is

- $F_\sigma$-ideal — because $\mathcal{W} = \bigcup_{n\in\mathbb{N}} \mathcal{W}_n$ where

$$\mathcal{W}_n = \{A \subseteq \mathbb{N} : A \text{ contains no a. p. of length } n\}$$

# Van der Waerden ideal $\mathcal{W}$

The van der Waerden ideal $\mathcal{W}$ is

- $F_\sigma$-ideal — because $\mathcal{W} = \bigcup_{n\in\mathbb{N}} \mathcal{W}_n$ where

$$\mathcal{W}_n = \{A \subseteq \mathbb{N} : A \text{ contains no a. p. of length } n\}$$

The family $\mathcal{W}_n$

- is not an ideal for every $n \in \mathbb{N}$
- generates a proper ideal $\langle \mathcal{W}_n \rangle$

# Strictly increasing sequence of ideals

The ideal $\langle \mathcal{W}_n \rangle$ is a tall $F_\sigma$-ideal for every $n \geq 3$.

**Fact.**

$$\mathcal{W} = \bigcup_{n \geq 3} \langle \mathcal{W}_n \rangle$$

and $\langle \mathcal{W}_n \rangle \subseteq \langle \mathcal{W}_{n+1} \rangle$ for every $n \in \mathbb{N}$.

# Strictly increasing sequence of ideals

**Proposition 1.**

For every $n \geq 3$ there exists $A \subset \mathbb{N}$ such that

$$A \in \mathcal{W}_{n+1} \setminus \langle \mathcal{W}_n \rangle$$

# Strictly increasing sequence of ideals

**Proposition 1.**

For every $n \geq 3$ there exists $A \subset \mathbb{N}$ such that

$$A \in \mathcal{W}_{n+1} \setminus \langle \mathcal{W}_n \rangle$$

**Proof.** Consider

$$A = \left\{ \sum_{i=0}^{k} c_i \cdot n^{2i} : k \in \omega, c_i = 0, \ldots, n-1, c_k \neq 0 \right\}$$

# Strictly increasing sequence of ideals

**Claim 1.** Show $A \in \mathcal{W}_{n+1}$ (straightforward calculation)

**Claim 2.** Show $A \notin \langle \mathcal{W}_n \rangle$ (use Hales-Jewett theorem)

# Strictly increasing sequence of ideals

Claim 1. Show $A \in \mathcal{W}_{n+1}$ (straightforward calculation)

Claim 2. Show $A \notin \langle \mathcal{W}_n \rangle$ (use Hales-Jewett theorem)

Let $L(n)$ ... be the set of finite words in the alphabet $\{0, 1, \ldots, n-1\}$.

A variable word $w(x)$ is a finite word in the alphabet $\{0, 1, \ldots, n-1, x\}$ in which the variable $x$ occurs at least once.

# Hales-Jewett theorem

**Hales-Jewett theorem.**

For every $n, r \in \mathbb{N}$ there exists a number $HJ(n, r)$ such that if words in $L(n)$ of length $HJ(n, r)$ are colored by $r$ colors then there exists a variable word $w(x)$ such that $w(0), w(1), \dots,$ $w(n-1)$ have the same color.

The symbol $w(i)$ denotes the word in $L(n)$ which is produced from $w(x)$ by replacing all the occurences of the variable $x$ by the letter of the alphabet in brackets.

# Some questions

**Conjecture.** $A \in \langle \mathcal{W}_n \rangle$ if and only if there exists $k \in \mathbb{N}$ such that $A$ does not contain a copy of $n^k$.

# Some questions

**Conjecture.** $A \in \langle \mathcal{W}_n \rangle$ if and only if there exists $k \in \mathbb{N}$ such that $A$ does not contain a copy of $n^k$.

**Question 1.** Is it true that whenever a set $A$ does not contain a copy of $3^2$ then $A \in \langle \mathcal{W}_3 \rangle$?

# Some questions

Conjecture. $A \in \langle \mathcal{W}_n \rangle$ if and only if there exists $k \in \mathbb{N}$ such that $A$ does not contain a copy of $n^k$.

Question 1. Is it true that whenever a set $A$ does not contain a copy of $3^2$ then $A \in \langle \mathcal{W}_3 \rangle$?

Question 2. Does the set $\{n^2 : n \in \omega\}$ belong to the ideal $\langle \mathcal{W}_3 \rangle$?

# Cofinality number of $\mathcal{W}$

$$\mathrm{cof}^{(*)}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge (\forall I \in \mathcal{I})(\exists A \in \mathcal{A})(I \subseteq^* A)\}$$

**Proposition 2.** $\mathrm{cof}^*(\mathcal{W}) = 2^{\aleph_0}$

# Cofinality number of $\mathcal{W}$

$$\mathrm{cof}^{(*)}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge (\forall I \in \mathcal{I})(\exists A \in \mathcal{A})(I \subseteq^* A)\}$$

**Proposition 2.** $\mathrm{cof}^*(\mathcal{W}) = 2^{\aleph_0}$

**Sketch of the proof:**

1. Show that there exists a perfect set $P \subseteq {}^\omega\omega$ such that every $f \in P$ satisfies $f(n+1) > 2f(n)$ for every $n \in \omega$ and whenever $f_0, f_1, \ldots f_k \in P$ are distinct, there exist infinitely many $n \in \omega$ such that $\{f_0(n), f_1(n), \ldots f_k(n)\}$ is a set of $k+1$ successive integers.

# Cofinality number of $\mathcal{W}$

$$\mathrm{cof}^{(*)}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \land (\forall I \in \mathcal{I})(\exists A \in \mathcal{A})(I \subseteq^* A)\}$$

**Proposition 2.** $\mathrm{cof}^*(\mathcal{W}) = 2^{\aleph_0}$

**Sketch of the proof:**

1. Show that there exists a perfect set $P \subseteq {}^{\omega}\omega$ such that every $f \in P$ satisfies $f(n+1) > 2f(n)$ for every $n \in \omega$ and whenever $f_0, f_1, \ldots f_k \in P$ are distinct, there exist infinitely many $n \in \omega$ such that $\{f_0(n), f_1(n), \ldots f_k(n)\}$ is a set of $k+1$ successive integers.
2. $A_f = \{f(n) : n \in \omega\} \in \mathcal{W}$ for every $f \in P$

# Cofinality number of $\mathcal{W}$

$$\mathrm{cof}^{(*)}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge (\forall I \in \mathcal{I})(\exists A \in \mathcal{A})(I \subseteq^* A)\}$$

**Proposition 2.** $\mathrm{cof}^*(\mathcal{W}) = 2^{\aleph_0}$

**Sketch of the proof:**

1. Show that there exists a perfect set $P \subseteq {}^\omega\omega$ such that every $f \in P$ satisfies $f(n+1) > 2f(n)$ for every $n \in \omega$ and whenever $f_0, f_1, \ldots f_k \in P$ are distinct, there exist infinitely many $n \in \omega$ such that $\{f_0(n), f_1(n), \ldots f_k(n)\}$ is a set of $k+1$ successive integers.
2. $A_f = \{f(n) : n \in \omega\} \in \mathcal{W}$ for every $f \in P$
3. $\{f \in P : A_f \subseteq^* B\}$ is finite for every $B \in \mathcal{W}$