Ultrafilters and divergent series

Jana Blobner[1] Peter Vojtáš[2]

[1]Department of Mathematics
University of West Bohemia in Pilsen

[2]Department of Software Engineering
Faculty of Mathematics and Physics
Charles University

Winter School, 31 January 2017, Hejnice

Ultrafilters on ω

Definition.

$\mathcal{U} \subseteq \mathcal{P}(\omega)$ is an ultrafilter if

- $\mathcal{U} \neq \emptyset$ and $\emptyset \notin \mathcal{U}$
- if $U_1, U_2 \in \mathcal{U}$ then $U_1 \cap U_2 \in \mathcal{U}$
- if $U \in \mathcal{U}$ and $U \subseteq V \subseteq \omega$ then $V \in \mathcal{U}$.
- for every $M \subseteq \omega$ either M or $\omega \setminus M$ belongs to $\mathcal{U}$

Example. fixed (or principal) ultrafilter $\{A \subseteq \omega : n \in A\}$

Free ultrafilters may be viewed as points in the remainder ω^* of the Čech-Stone compactification $\beta\omega$.

Divergent series

The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$

Divergent series

The harmonic series $$\sum_{n=1}^{\infty} \frac{1}{n}$$

Some divergent series grow to infinity slowlier:

$$\sum_{n\in\mathbb{N}} \frac{1}{2n}, \quad \sum_{n\in\mathbb{N}} \frac{1}{n \cdot \ln n}, \quad \sum_{n\in\mathbb{N}} \frac{1}{n \cdot \ln n \cdot \ln \ln n}$$

Some divergent series grow to infinity faster:

$$\sum_{n\in\mathbb{N}} \frac{\ln n}{n}, \quad \sum_{n\in\mathbb{N}} \frac{1}{\sqrt{n}}, \quad \sum_{n\in\mathbb{N}} \frac{1}{\ln n}$$

Summable ideals

Definition.

For a given a divergent series $\sum_{n\in\mathbb{N}} g(n) = +\infty$ such that $\lim_{n\to\infty} g(n) = 0$ the family

$$\mathcal{I}_g = \{A \subseteq \mathbb{N} : \sum_{n\in A} g(n) < +\infty\}$$

is a tall proper ideal which we call summable ideal determined by the series $\sum_{n\in\mathbb{N}} g(n)$ (by the function g).

Summable ideals

Lemma.

Assume g_1 and g_2 are two sequences of positive real numbers such that $\sum_{n\in\mathbb{N}} g_i(n) = +\infty$ and $\lim_{n\to\infty} g_i(n) = 0$ for $i = 1, 2$. Then

1. If $g_1 \leq^* g_2$ then $\mathcal{I}_{g_1} \supset \mathcal{I}_{g_2}$.
2. If $\lim_{n\to\infty} \frac{g_1(n)}{g_2(n)} \in \mathbb{R}$, then $\mathcal{I}_{g_1} \supset \mathcal{I}_{g_2}$.

Rapid ultrafilters

Definition.

A free ultrafilter $\mathcal{U}$ on ω is called rapid if the enumeration functions of its sets form a dominating family in $(\omega^\omega, \leq^*)$.

Rapid ultrafilters

Definition.

A free ultrafilter $\mathcal{U}$ on ω is called rapid if the enumeration functions of its sets form a dominating family in $(\omega^\omega, \leq^*)$.

Theorem (Booth?).

(CH) Rapid ultrafilters exist.

Theorem (Miller).

In Laver's model there are no rapid ultrafilters.

Characterization of rapid ultrafilters

Theorem (Vojtáš).

The following are equivalent for an ultrafilter $\mathcal{U} \in \omega^*$:

- $\mathcal{U}$ is rapid
- $\mathcal{U} \cap \mathcal{I}_g \neq \emptyset$ for every tall summable ideal $\mathcal{I}_g$

Characterization of rapid ultrafilters

Theorem (Vojtáš).

The following are equivalent for an ultrafilter $\mathcal{U} \in \omega^*$:

- $\mathcal{U}$ is rapid
- $\mathcal{U} \cap \mathcal{I}_g \neq \emptyset$ for every tall summable ideal $\mathcal{I}_g$

One can add two more equivalent conditions:

- $(\forall f : \omega \to \mathbb{N}$ one-to-one$)$ $(\exists U \in \mathcal{U})$ such that $f[U] \in \mathcal{I}_g$ for every tall summable ideal $\mathcal{I}_g$

Characterization of rapid ultrafilters

Theorem (Vojtáš).

The following are equivalent for an ultrafilter $\mathcal{U} \in \omega^*$:

- $\mathcal{U}$ is rapid
- $\mathcal{U} \cap \mathcal{I}_g \neq \emptyset$ for every tall summable ideal $\mathcal{I}_g$

One can add two more equivalent conditions:

- $(\forall f : \omega \to \mathbb{N}$ one-to-one$)$ $(\exists U \in \mathcal{U})$ such that $f[U] \in \mathcal{I}_g$ for every tall summable ideal $\mathcal{I}_g$
- $(\forall f : \omega \to \mathbb{N}$ finite-to-one$)$ $(\exists U \in \mathcal{U})$ such that $f[U] \in \mathcal{I}_g$ for every tall summable ideal $\mathcal{I}_g$
  (= $\mathcal{U}$ is a weak $\mathcal{I}_g$-ultrafilter for every $\mathcal{I}_g$)

How many summable ideals decide?

Proposition.

There is a family $\mathcal{D}$ of tall summable ideals such that $|\mathcal{D}| = \mathfrak{d}$ and an ultrafilter $\mathcal{U} \in \omega^*$ is rapid if and only if it has a nonempty intersection with every tall summable ideal in $\mathcal{D}$.

How many summable ideals decide?

Proposition.

There is a family $\mathcal{D}$ of tall summable ideals such that $|\mathcal{D}| = \mathfrak{d}$ and an ultrafilter $\mathcal{U} \in \omega^*$ is rapid if and only if it has a nonempty intersection with every tall summable ideal in $\mathcal{D}$.

Is it possible that an ultrafilter has a nonempty intersection with “many" tall summable ideals simultaneously and it is not a rapid ultrafilter?

How many summable ideals decide?

Proposition (Cancino Manríquez).

For any family $\mathcal{D}$ of tall summable ideals such that $|\mathcal{D}| < \mathfrak{d}$ there is an ultrafilter $\mathcal{U} \in \omega^*$ which meets all ideals $\mathcal{I} \in \mathcal{D}$, but $\mathcal{U}$ is not a rapid ultrafilter.

How many summable ideals decide?

Proposition (Cancino Manríquez).

For any family $\mathcal{D}$ of tall summable ideals such that $|\mathcal{D}| < \mathfrak{d}$ there is an ultrafilter $\mathcal{U} \in \omega^*$ which meets all ideals $\mathcal{I} \in \mathcal{D}$, but $\mathcal{U}$ is not a rapid ultrafilter.

Can we find a family of tall summable ideals of cardinality at least $\mathfrak{d}$ such that an ultrafilter has a nonempty intersection with every ideal in the family, but is not a rapid ultrafilter?

The ℓ^p-hierarchy

For $1 \leq p < \infty$ the ℓ^p-space is defined as follows

$$\ell^p = \{(x_n)_{n\in\mathbb{N}} \in {}^{\mathbb{N}}\mathbb{R} : \sum_{n\in\mathbb{N}} |x_n|^p < \infty\}$$

The ℓ^p-hierarchy

For $1 \leq p < \infty$ the ℓ^p-space is defined as follows

$$\ell^p = \{(x_n)_{n\in\mathbb{N}} \in {}^{\mathbb{N}}\mathbb{R} : \sum_{n\in\mathbb{N}} |x_n|^p < \infty\}$$

The space ℓ^1 is the space of all absolutely convergent series.

The ℓ^p-hierarchy

For $1 \leq p < \infty$ the ℓ^p-space is defined as follows

$$\ell^p = \{(x_n)_{n\in\mathbb{N}} \in {}^{\mathbb{N}}\mathbb{R} : \sum_{n\in\mathbb{N}} |x_n|^p < \infty\}$$

The space ℓ^1 is the space of all absolutely convergent series.

$\left(\frac{1}{n}\right)_n \in \bigcap_{k\in\mathbb{N}} \ell^{1+\frac{1}{k}} \setminus \ell^1$ and also $\left(\frac{\ln n}{n}\right)_n \in \bigcap_{k\in\mathbb{N}} \ell^{1+\frac{1}{k}} \setminus \ell^1$

$\left(\frac{1}{\sqrt{n}}\right)_n \in \bigcap_{k\in\mathbb{N}} \ell^{2+\frac{1}{k}} \setminus \ell^2$, but $\left(\frac{1}{\ln n}\right)_n \notin \ell^p$ for any $1 < p < +\infty$

Rapid ultrafilters once more

Proposition 1.

The following are equivalent for an ultrafilter $\mathcal{U} \in \omega^*$:

- $\mathcal{U}$ is rapid
- $\mathcal{U} \cap \mathcal{I}_g \neq \emptyset$ for every tall summable ideal $\mathcal{I}_g$
- $\mathcal{U} \cap \mathcal{I}_g \neq \emptyset$ for every tall summable ideal $\mathcal{I}_g$ with $g \notin \ell^2$
- $\mathcal{U} \cap \mathcal{I}_g \neq \emptyset$ for every tall summable ideal $\mathcal{I}_g$ with $g \notin \ell^k$ for a given $k > 2$

The bottom part of the hierarchy

Conjecture 2.

For any $k \in \mathbb{N}$ there is an ultrafilter $\mathcal{U}$ in ZFC such that $\mathcal{U} \cap \mathcal{I}_g \neq \emptyset$ for every tall summable ideal $\mathcal{I}_g$ with $g \in \ell^k$.

The bottom part of the hierarchy

Conjecture 2.

For any $k \in \mathbb{N}$ there is an ultrafilter $\mathcal{U}$ in ZFC such that $\mathcal{U} \cap \mathcal{I}_g \neq \emptyset$ for every tall summable ideal $\mathcal{I}_g$ with $g \in \ell^k$.

- if $g \in \ell^k$ and $h \notin \ell^k$ then $h \not\leq^* g$

g-summable ultrafilters

Definition.

An ultrafilter $\mathcal{U} \in \omega^*$ is called g-summable ultrafilter if for every one-to-one $f : \omega \to \mathbb{N}$ there exists $U \in \mathcal{U}$ such that $f[U] \in \mathcal{I}_g$.

g-summable ultrafilters

Definition.

An ultrafilter $\mathcal{U} \in \omega^*$ is called g-summable ultrafilter if for every one-to-one $f : \omega \to \mathbb{N}$ there exists $U \in \mathcal{U}$ such that $f[U] \in \mathcal{I}_g$.

Rapid ultrafilter is g-summable for every $\mathcal{I}_g$.

g-summable ultrafilters

Definition.

An ultrafilter $\mathcal{U} \in \omega^*$ is called g-summable ultrafilter if for every one-to-one $f : \omega \to \mathbb{N}$ there exists $U \in \mathcal{U}$ such that $f[U] \in \mathcal{I}_g$.

Rapid ultrafilter is g-summable for every $\mathcal{I}_g$.

Theorem (J.B.)

There exists $\mathcal{U} \in \omega^*$ such that $\mathcal{U}$ is an $\frac{1}{n}$-summable ultrafilter.

g-summable ultrafilters

Theorem 3.

If Conjecture 2. holds then
g-summable ultrafilters exist in ZFC for every $k \in \mathbb{N}$ and $g \in \ell^k$.

g-summable ultrafilters

Theorem 3.

If Conjecture 2. holds then
g-summable ultrafilters exist in ZFC for every $k \in \mathbb{N}$ and $g \in \ell^k$.

Open questions:

Do $\frac{1}{\ln n}$-summable ultrafilters exist in ZFC?

g-summable ultrafilters

Theorem 3.

If Conjecture 2. holds then
g-summable ultrafilters exist in ZFC for every $k \in \mathbb{N}$ and $g \in \ell^k$.

Open questions:

Do $\frac{1}{\ln n}$-summable ultrafilters exist in ZFC?

Do weak $\mathcal{I}_g$-ultrafilters exist in ZFC? (open also for $g(n) = \frac{1}{n}$)

References

P. Vojtáš, On ω^* and absolutely divergent series, *Topology Proceedings* **19**, 335–348, 1994.

J. Flašková, Rapid ultrafilters and summable ideals, *arXiv:1208.3690*

J. Flašková, More than a 0-point, *Comment. Math. Univ. Carolin.* **47**, no. 4, 617–621, 2006.