# Rapid ultrafilters and summable ideals

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**Abstract**

This note answers a question raised in [3]: Is it consistent that for an arbitrary tall summable ideal $\mathscr{I}_g$ there exists an $\mathscr{I}_g$-ultrafilter which is not rapid? Assuming CH we construct such ultrafilters for every tall summable ideal $\mathscr{I}_g$.

## 1 Introduction

This note follows up the author's paper "$\mathscr{I}$-ultrafilters and summable ideals" [3] in which the connections between rapid ultrafilters and $\mathscr{I}_g$-ultrafilters have been studied. We will use the same notation and recall the most important definitions and facts in this introduction.

An ultrafilter $\mathscr{U}$ is called a *rapid ultrafilter* if the enumeration functions of sets in $\mathscr{U}$ form a dominating family in $({}^\omega\omega, \leq^*)$, where the enumeration function of a set $A$ is the unique strictly increasing function $e_A$ from $\omega$ onto $A$. An ultrafilter $\mathscr{U}$ is called a *Q-point* if for every partition $\{Q_n : n \in \omega\}$ of $\omega$ into finite sets there is $A \in \mathscr{U}$ such that $|A \cap Q_n| \leq 1$ for every $n \in \omega$. Clearly, every $Q$-point is a rapid ultrafilter, but the converse is not true (see e.g. [5]).

For a function $g : \omega \to (0, \infty)$ such that $\sum_{n \in \omega} g(n) = +\infty$ the family

$$\mathscr{I}_g = \{A \subseteq \omega : \sum_{a \in A} g(a) < +\infty\}$$

is an ideal on $\omega$, which we call the *summable ideal determined by function g*. A summable ideal is tall if and only if $\lim_{n \to \infty} g(n) = 0$.

The following description of rapid ultrafilters can be found in [6]:

**Theorem 1.1.** *For an ultrafilter $\mathscr{U} \in \omega^*$ the following are equivalent:*

*1. $\mathscr{U}$ is rapid*

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2. $\mathscr{U} \cap \mathscr{I}_g \neq \emptyset$ for every tall summable ideal $\mathscr{I}_g$

The definition of an $\mathscr{I}$-ultrafilter was given by Baumgartner in [1]: Let $\mathscr{I}$ be a family of subsets of a set $X$ such that $\mathscr{I}$ contains all singletons and is closed under subsets. Given an ultrafilter $\mathscr{U}$ on $\omega$, we say that $\mathscr{U}$ is an *$\mathscr{I}$-ultrafilter* if for every function $F : \omega \to X$ there exists $A \in \mathscr{U}$ such that $F[A] \in \mathscr{I}$.

We say that an ultrafilter $\mathscr{U}$ is a *hereditarily rapid ultrafilter* if it is a rapid ultrafilter such that for every $\mathscr{V} \leq_{RK} \mathscr{U}$ the ultrafilter $\mathscr{V}$ is again a rapid ultrafilter. Since every hereditarily rapid ultrafilter is obviously a rapid ultrafilter, the existence of hereditarily rapid ultrafilters is not provable in ZFC because Miller proved in [5] that there are no rapid ultrafilters in Laver model. On the other hand, every selective ultrafilter is hereditarily rapid, thus the existence of hereditarily rapid ultrafilters is consistent with ZFC.

The following characterization of hereditarily rapid ultrafilters follows from the definition, Theorem 1.1 and from the fact that the class of $\mathscr{I}$-ultrafilters is downwards closed with respect to the Rudin-Keisler order on ultrafilters.

**Theorem 1.2.** *For an ultrafilter $\mathscr{U} \in \omega^*$ the following are equivalent:*

1. *$\mathscr{U}$ is hereditarily rapid*

2. *$\mathscr{U}$ is an $\mathscr{I}$-ultrafilter for every tall summable ideal $\mathscr{I}$*

It was proved in [3] that $Q$-points (and consequently rapid ultrafilters) need not be $\mathscr{I}_g$-ultrafilters in a strong sense.

**Theorem 1.3.** *($MA_{\text{ctble}}$) There is a $Q$-point which is not an $\mathscr{I}_g$-ultrafilter for any summable ideal $\mathscr{I}_g$.*

**Corollary 1.4.** *($MA_{\text{ctble}}$) For an arbitrary summable ideal $\mathscr{I}_g$ there exists a rapid ultrafilter which is not an $\mathscr{I}_g$-ultrafilter.*

We also showed in [3] that $\mathscr{I}_g$-ultrafilters need not be $Q$-points by the following counterpart of Theorem 1.3.

**Theorem 1.5.** *($MA_{\text{ctble}}$) There exists $\mathscr{U} \in \omega^*$ such that $\mathscr{U}$ is an $\mathscr{I}_g$-ultrafilter for every tall summable ideal $\mathscr{I}_g$ and $\mathscr{U}$ is not a $Q$-point.*

However, we did not prove a counterpart for Corollary 1.4. Asssuming Martin's axiom for countable posets an $\mathscr{I}_{1/n}$-ultrafilter which is not rapid was constructed in [2], but the question remained open for an arbitrary tall summable ideal $\mathscr{I}_g$.

The aim of this note is to provide a construction of an $\mathscr{I}_g$-ultrafilter which is not rapid for an arbitrary tall summable ideal $\mathscr{I}_g$.

## 2 Katětov order and summable ideals

Let us first recall the definition of Katětov order $\leq_K$ for ideals on $\omega$: For $\mathscr{I}$ and $\mathscr{J}$ ideals on $\omega$ we write $\mathscr{I} \leq_K \mathscr{J}$ if there is a function $f : \omega \to \omega$ such that $f^{-1}[A] \in \mathscr{J}$ for all $A \in \mathscr{I}$.

The structure of the summable ideals ordered by Katětov order was investigated by Meza [4]. We are particularly interested how the comparability of two ideals in Katětov order reflects to the corresponding classes of $\mathscr{I}$-ultrafilters.

Obviously, if $\mathscr{I} \leq_K \mathscr{J}$ then every $\mathscr{I}$-ultrafilter is a $\mathscr{J}$-ultrafilter. This implication cannot be reversed in general. However, in Theorem 3.2 we prove that assuming CH the converse is also true whenever $\mathscr{I}$ and $\mathscr{J}$ are tall summable ideals.

**Lemma 2.1.** *Assume $\mathscr{I}_g$ is a tall summable ideal determined by a monotone function $g$ and $h(n) = g(2n)$ for every $n \in \omega$. Then $\mathscr{I}_h = \mathscr{I}_g$.*

*Proof.* Since the function $g$ is monotone, $g(2k) + g(2k+1) \leq 2g(2k) = 2h(k)$ and $g(2k-1) + g(2k) \geq 2g(2k) = 2h(k)$ for every $k$. From the inequalities

$$\sum_{n \in \omega} 2h(n) \leq g(0) + \sum_{n \in \omega} g(n) \leq g(0) + \sum_{n \in \omega} 2h(n)$$

then follows that $\mathscr{I}_h = \mathscr{I}_g$. $\qquad\square$

**Lemma 2.2.** *Assume $f \in \omega^\omega$, $\mathscr{I}_g$, $\mathscr{I}_h$ are tall summable ideals and $\mathscr{I}_g \not\leq_K \mathscr{I}_h$. If $H$ is an infinite subset of $\omega$ such that $H \notin \mathscr{I}_h$ and $f[H] \notin \mathscr{I}_g$ then there exists $A \subseteq f[H]$ such that $A \in \mathscr{I}_g$ and $f^{-1}[A] \cap H \notin \mathscr{I}_h$.*

*Proof.* We may assume that both $g$ and $h$ are monotone functions because every tall summable ideal is isomorphic to one determined by monotone function. Enumerate $f[H] = \{h_n : n \in \omega\}$ and define $\tilde{f} : \omega \to \omega$ by

$$\tilde{f}(n) = \begin{cases} 2h_n & \text{for } n \in H \\ 2h_n + 1 & \text{for } n \notin H \end{cases}$$

Notice that $\tilde{f}[H] \cap \tilde{f}[\omega \setminus H] = \emptyset$ and use Lemma 2.1 to show that $\tilde{f}[H] \notin \mathscr{I}_g$.

Since $\mathscr{I}_g \not\leq_K \mathscr{I}_h$ we have $\mathscr{I}_g \upharpoonright \tilde{f}[H] \not\leq_K \mathscr{I}_h$, so there is $\tilde{A} \in \mathscr{I}_g \upharpoonright \tilde{f}[H]$ such that $\tilde{f}^{-1}[\tilde{A}] \notin \mathscr{I}_h$. Now, define $A = \{h_n : 2h_n \in \tilde{A}\}$. As in Lemma 2.1, one can verify that $A \in \mathscr{I}_g$ because $\tilde{A} \in \mathscr{I}_g$.

Finally. observe that $f^{-1}[A] \cap H \supseteq \tilde{f}^{-1}[\tilde{A}] \notin \mathscr{I}_h$. $\qquad\square$

# 3   Main result

We will use the fact that rapid ultrafilters are precisely those ultrafilters which have nonempty interesection with every tall summable ideal. Thus in order to construct an $\mathscr{I}_g$-ultrafilter which is not rapid, we want to construct an $\mathscr{I}_g$-ultrafilter which has an empty intersection with some other summable ideal $\mathscr{I}_h$.

**Lemma 3.1.** *Assume $\mathscr{I}_g$ and $\mathscr{I}_h$ are two tall summable ideals such that $\mathscr{I}_g \not\leq_K \mathscr{I}_h$. Assume $\mathscr{F}$ is a countable filter base such that $\mathscr{F} \cap \mathscr{I}_h = \emptyset$ and a function $f \in \omega^\omega$ is given. Then there exists $G \subseteq \omega$ such that $f[G] \in \mathscr{I}_g$ and $G \cap F \notin \mathscr{I}_h$ for every $F \in \mathscr{F}$.*

*Proof.* Since $\mathscr{F}$ is countable we can enumerate $\mathscr{F} = \{F_n : n \in \omega\}$. We may also assume that $F_n \supseteq F_{n+1}$ for every $n$. The family $\{F_n : n \in \omega\}$ is a countable collection of $\mathscr{I}_h$-positive sets, therefore there is an $\mathscr{I}_h$-positive set $H$ such that $|H \setminus F_n| < \omega$ for every $n \in \omega$.

If there $f[H] \in \mathscr{I}_g$ then put $G = H$ and we are done.

If $f[H] \notin \mathscr{I}_g$ we may apply Lemma 2.2. So there exists $A \subseteq f[H]$ such that $A \in \mathscr{I}_g$ and $f^{-1}[A] \cap H \notin \mathscr{I}_h$. Let $G = f^{-1}[A]$.

- $f[G] = A \in \mathscr{I}_g$
- Since $G \cap H \notin \mathscr{I}_h$ and $(G \cap H) \setminus F$ is finite for every $F \in \mathscr{F}$ it follows that $G \cap F \notin \mathscr{I}_h$ for every $F \in \mathscr{F}$. $\qquad\qquad\square$

**Theorem 3.2.** *(CH) For arbitrary tall summable ideals $\mathscr{I}_g$ and $\mathscr{I}_h$ such that $\mathscr{I}_g \not\leq_K \mathscr{I}_h$ there is an $\mathscr{I}_g$-ultrafilter which is not an $\mathscr{I}_h$-ultrafilter.*

*Proof.* Enumerate all functions in $^\omega\omega$ as $\{f_\alpha : \alpha < \omega_1\}$. By transfinite induction on $\alpha < \omega_1$ we will construct filter bases $\mathscr{F}_\alpha$ such that the following conditions are satisfied:

   (i) $\mathscr{F}_0$ is the Fréchet filter
   (ii) $\mathscr{F}_\alpha \supseteq \mathscr{F}_\beta$ whenever $\alpha \geq \beta$
   (iii) $\mathscr{F}_\gamma = \bigcup_{\alpha < \gamma} \mathscr{F}_\alpha$ for $\gamma$ limit
   (iv) $(\forall \alpha)\ |\mathscr{F}_\alpha| \leq \omega$
   (v) $(\forall \alpha)\ \mathscr{F}_\alpha \cap \mathscr{I}_h = \emptyset$
   (vi) $(\forall \alpha)\ (\exists F \in \mathscr{F}_{\alpha+1})\ f_\alpha[F] \in \mathscr{I}_g$

   Conditions (i)–(iii) allow us to start the induction and keep it going. Moreover (iii) ensures that (iv)–(vi) are satisfied at limit stages of the construction, so it is necessary to verify conditions (iv)–(vi) only at non-limit steps.

   Induction step: Suppose we already know $\mathscr{F}_\alpha$.

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Due to (iv) and (v) we may apply Lemma 3.1 to $f_\alpha$ and $\mathscr{F}_\alpha$. Let $\mathscr{F}_{\alpha+1}$ be the filter base generated by $\mathscr{F}_\alpha$ and $G$. The filter base $\mathscr{F}_{\alpha+1}$ satisfies (iv)–(vi).

Finally, let $\mathscr{F} = \bigcup_{\alpha < \omega_1} \mathscr{F}_\alpha$. Every ultrafilter which extends $\mathscr{F}$ is an $\mathscr{I}_g$-ultrafilter because of condition (vi). Because of condition (v) $\mathscr{F}$ has empty intersection with $\mathscr{I}_h$ and thus can be extended to an ultrafilter $\mathscr{U}$ with $\mathscr{U} \cap \mathscr{I}_h = \emptyset$. It follows that $\mathscr{U}$ is not an $\mathscr{I}_h$-ultrafilter. $\qquad\square$

**Proposition 3.3.** *For every tall summable ideal $\mathscr{I}_g$ there is a tall summable ideal $\mathscr{I}_h$ such that $\mathscr{I}_g \not\leq_K \mathscr{I}_h$.*

*Proof.* Since $\mathscr{I}_g$ is a tall summable ideal we may fix a partition of $\omega$ into finite consecutive intervals $I_n$, $n \in \omega$ such that

(i) $I_0 \neq \emptyset$
(ii) $|I_{n+1}| \geq n|\bigcup_{j \leq n} I_j|$ for every $n \in \omega$
(iii) for every $n > 0$ if $m \in I_n$ then $g(m) < \frac{1}{2^n}$

Now define $h : \omega \to (0, \infty)$ by

$$h(m) = \begin{cases} 1 & \text{for } m \in I_0 \\ \frac{1}{n} & \text{for } m \in I_n \text{ with } n \geq 1 \end{cases}$$

It remains to verify that $\mathscr{I}_g \not\leq_K \mathscr{I}_h$. We will show that for every $f : \omega \to \omega$ there exists $A \in \mathscr{I}_g$ such that $f^{-1}[A] \notin \mathscr{I}_h$.

Consider $f : \omega \to \omega$ arbitrary. For every $n \in \omega$ define

$$B_n = \{m \in I_n : f(m) < \min I_n\} \qquad C_n = \{m \in I_n : f(m) \geq \min I_n\}$$

*Case I. $A_0 = \{n \in \omega : |B_n| \geq |C_n|\}$ is infinite*

Since $B_n \cup C_n = I_n$ we have $|B_n| \geq \frac{1}{2}|I_n| \geq \frac{n}{2}|\bigcup_{j<n} I_j| = \frac{n}{2}(\min I_n - 1)$. Thus for every $n \in A_0$ there exists $m_n \in f[B_n]$ such that $|f^{-1}(m_n) \cap B_n| \geq \frac{n}{2}$.

If $A = \{m_n : n \in A_0\}$ is finite then, of course $A \in \mathscr{I}_g$. Otherwise there exists an infinite set $A \subseteq \{m_n : n \in A_0\}$ such that $A \in \mathscr{I}_g$ because $\mathscr{I}_g$ is a tall ideal. In both cases $\tilde{A}_0 = \{n \in A_0 : m_n \in A\}$ is infinite and $f^{-1}[A] \notin \mathscr{I}_h$ because

$$\sum_{a \in f^{-1}[A]} h(a) \geq \sum_{n \in \tilde{A}_0} \sum_{a \in f^{-1}[A] \cap I_n} h(a) \geq \sum_{n \in \tilde{A}_0} |f^{-1}(m_n) \cap I_n| \cdot \frac{1}{n} \geq \sum_{n \in \tilde{A}_0} \frac{1}{2} = \infty$$

*Case II. $A_0 = \{n \in \omega : |B_n| \geq |C_n|\}$ is finite*

According to the assumption there is $n_0 \in \omega$ such that $|B_n| < |C_n|$ for every $n \geq n_0$. Pick $m_n \in C_n$ for every $n \geq n_0$. Put $M = \{m_n : n \geq n_0\}$

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and $A = f[M]$. Since $m_n \in C_n$ one has $f(m_n) \geq \min I_n$ and therefore $g(f(m_n)) \leq \frac{1}{2^n}$. It is easy to see that $A \in \mathscr{I}_g$ because

$$\sum_{a \in A} g(a) \leq \sum_{n \geq n_0} g(f(m_n)) \leq \sum_{n \geq n_0} \frac{1}{2^n} = \frac{1}{2^{n_0 - 1}}.$$

It remains to verify that $f^{-1}[A] \notin \mathscr{I}_h$. To see this notice that

$$\sum_{a \in f^{-1}[A]} h(a) \geq \sum_{a \in M} h(a) = \sum_{n \geq n_0} h(m_n) = \sum_{n \geq n_0} \frac{1}{n} = \infty.$$

$\square$

**Theorem 3.4.** *(CH) For an arbitrary tall summable ideal $\mathscr{I}_g$ there is an $\mathscr{I}_g$-ultrafilter which is not rapid.*

*Proof.* This is an immediate consequence of Theorem 3.2, Proposition 3.3 and the characterization of rapid ultrafilters in Theorem 1.1. $\square$

# 4   One possible generalization and its limits

Once we have proved Theorem 3.4, which so to speak reverses Corollary 1.4, we may ask whether it is possible that an ultrafilter is an $\mathscr{I}_g$-ultrafilter for "many" tall summable ideals simultaneously and still not a rapid ultrafilter. Certainly, "many" cannot mean all tall summable ideals, because of Theorem 1.2. We will show that in fact $\mathfrak{d}$ many may be too much, but less than $\mathfrak{b}$ is not.

**Proposition 4.1.** *There exists a family $\mathcal{D}$ of tall summable ideals such that $|\mathcal{D}| = \mathfrak{d}$ and an ultrafilter $\mathscr{U} \in \omega^*$ is rapid if and only if it has a nonempty intersection with every tall summable ideal in $\mathcal{D}$.*

*Proof.* Let us first construct the family $\mathcal{D}$: Assume $\mathcal{F} \subseteq {}^\omega\omega$ is a dominating family and $|\mathcal{F}| = \mathfrak{d}$. Without loss of generality we may assume that all functions in $\mathcal{F}$ are strictly increasing and $f(j+1) \geq f(j) + j + 1$ for every $j \in \omega$. For every $f \in \mathcal{F}$ define $g_f : \omega \to \mathbb{R}^+$ by

$$g_f(m) = \begin{cases} 1 & \text{if } m < f(0) \\ \frac{1}{j+1} & \text{if } m \in [f(j), f(j+1)) \end{cases}$$

Let $\mathcal{D} = \{\mathscr{I}_{g_f} : f \in \mathcal{F}\}$.

Now, one implication is clear since every rapid ultrafilter has a nonempty intersection with all tall summable ideals, in particular it has a nonempty intersection with every ideal from $\mathcal{D}$.

It remains to verify that if an ultrafilter has nonempty intersection with every ideal in $\mathcal{D}$, then it has nonempty intersection with all tall summable ideals and therefore is rapid. To this end, assume $\mathscr{I}_g$ is an arbitrary tall summable ideal. One can define a strictly increasing function $f_g$ such that for every $j \in \omega$:

- $f_g(j+1) \geq f_g(j) + j + 1$

- if $m \geq f_g(j)$ then $g(m) \leq \frac{1}{2^j}$

Remember that family $\mathcal{F}$ was dominating. Hence there exists $f \in \mathcal{F}$ and $k_0 \in \omega$ such that $f(k) \geq f_g(k)$ for every $k \geq k_0$. For a every $n \geq f(k_0)$ there exists a unique $j \geq k_0$ such that $n \in [f(j), f(j+1))$. Since $n \geq f(j) \geq f_g(j)$ we get $g(n) \leq \frac{1}{2^j} \leq \frac{1}{j+1} = g_f(n)$. From $g \leq^* g_f$ follows that $\mathscr{I}_{g_f} \subseteq \mathscr{I}_g$. Thus every ultrafilter $\mathscr{U} \in \omega^*$ which has a nonempty intersection with all ideals from $\mathcal{D}$ has a nonempty intersection with $\mathscr{I}_g$ and since $\mathscr{I}_g$ was arbitrary, $\mathscr{U}$ is a rapid ultrafilter, $\qquad\square$

**Proposition 4.2.** *If $\mathcal{D}$ is a family of tall summable ideals and $|\mathcal{D}| < \mathfrak{b}$ then there exists a tall summable ideal $\mathscr{I}_g$ such that $\mathscr{I}_g \subseteq \mathscr{I}_h$ for every $\mathscr{I}_h \in \mathcal{D}$.*

*Proof.* For every $\mathscr{I}_h \in \mathcal{D}$ define a strictly increasing function $f_h \in {}^\omega\omega$ such that whenever $m \geq f_h(j)$ then $h(m) \leq \frac{1}{2^j}$.

According to the assumptions, the family of functions $\mathcal{F} = \{f_h : \mathscr{I}_h \in \mathcal{D}\}$ is bounded, so there exists $f \in {}^\omega\omega$ such that $f_h \leq^* f$ for every $f_h \in \mathcal{F}$. We may assume that $f$ is strictly increasing. Define $g : \omega \to \mathbb{R}^+$ by

$$g(m) = \begin{cases} 1 & \text{if } m < f(0) \\ \frac{1}{j+1} & \text{if } m \in [f(j), f(j+1)) \end{cases}$$

For a given function $f_h \in \mathcal{F}$ there exists $k_h \in \omega$ such that $f_h(k) \leq f(k)$ for every $k \geq k_h$. For every $n \geq f(k_h)$ there is exactly one $j \geq k_h$ such that $n \in [f(j), f(j+1))$. Since $n \geq f(j) \geq f_h(j)$ we get $h(n) \leq \frac{1}{2^j} \leq \frac{1}{j+1} = g(n)$. From $h \leq^* g$ follows that $\mathscr{I}_g \subseteq \mathscr{I}_h$. $\qquad\square$

**Corollary 4.3.** *(CH) If $\mathcal{D}$ is a family of tall summable ideals and $|\mathcal{D}| < \omega_1$ then there exists an ultrafilter $\mathscr{U} \in \omega^*$ such that $\mathscr{U}$ is an $\mathscr{I}$-ultrafilter for every $\mathscr{I} \in \mathcal{D}$, but $\mathscr{U}$ is not a rapid ultrafilter.*

*Proof.* Combine Theorem 3.4 and Proposition 4.2. $\qquad\square$

# 5  Open questions

Let $\mathcal{D}$ be a family of tall summable ideals.

**Question 5.1.** *What is the minimal size of the family $\mathcal{D}$ if rapid ultrafilters can be characterized as those ultrafilters on the natural numbers which have a nonempty intersection with all ideals in the family $\mathcal{D}$?*

Due to Proposition 4.1 the size of such a family is at most $\mathfrak{d}$. But is $\mathfrak{d}$ really the minimum?

**Question 5.2.** *Is it true that whenever the cardinality of $\mathcal{D}$ is less than $\mathfrak{d}$ then there exist an ultrafilter on the natural numbers which is an $\mathscr{I}_g$-ultrafilter for every $\mathscr{I}_g \in \mathcal{D}$, but not a rapid ultrafilter?*

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