SUMS AND PRODUCTS OF CERTAIN $\mathscr{I}$-ULTRAFILTERS

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Abstract. We investigate ultrafilter sums of $\mathscr{I}$-ultrafilters and weak $\mathscr{I}$-ultrafilters. We prove that $\mathscr{I}$-ultrafilters are closed under $\mathscr{I}$-sums if $\mathscr{I}$ is a P-ideal on ω. We give two examples of ideals such that corresponding $\mathscr{I}$-ultrafilters are not closed even under products.

Introduction

The definition of $\mathscr{I}$-ultrafilter was given by Baumgartner in [1].

Definition A. Let $\mathscr{I}$ be a family of subsets of a set X such that $\mathscr{I}$ contains all singletons and is closed under subsets. Given an ultrafilter $\mathscr{U}$ on ω, we say that $\mathscr{U}$ is an *$\mathscr{I}$-ultrafilter* if for any $F : \omega \to X$ there is $A \in \mathscr{U}$ such that $F[A] \in \mathscr{I}$.

Baumgartner defined in his article discrete ultrafilters, scattered ultrafilters, measure zero ultrafilters and nowhere dense ultrafilters which he obtained by taking $X = 2^\omega$, the Cantor set, and $\mathscr{I}$ the collection of discrete sets, scattered sets, sets with closure of measure zero, nowhere dense sets respectively. Another example of $\mathscr{I}$-ultrafilters are ordinal ultrafilters which he defined also in [1] by taking $X = \omega_1$ and $\mathscr{I} = \{A \subseteq \omega_1 : A \text{ has order type } \leq \alpha\}$ for an indecomposable ordinal α.

We study in this paper $\mathscr{I}$-ultrafilters in the setting $X = \mathbb{N}$ and $\mathscr{I}$ is an ideal on $\mathbb{N}$ or another family of "small" subsets of natural numbers that contains all finite sets and is closed under subsets. We consider also two modifications of the notion: weak $\mathscr{I}$-ultrafilters

2000 *Mathematics Subject Classification.* 54D40.
Key words and phrases. $\mathscr{I}$-ultrafilter, ultrafilter sum, ultrafilter product, 0-point, summable ultrafilter, van der Waerden ideal.

and $\mathscr{I}$-close ultrafilters, which we obtain by restricting the family of functions in definition.

Definition B. Let $\mathscr{I}$ be a family of subsets of a set X such that $\mathscr{I}$ contains all singletons and is closed under subsets. Given an ultrafilter $\mathscr{U}$ on ω, we say that $\mathscr{U}$ is a *weak $\mathscr{I}$-ultrafilter* (*$\mathscr{I}$-close ultrafilter*) if for any finite-to-one (one-to-one) function $F : \omega \to X$ there is $A \in \mathscr{U}$ such that $F[A] \in \mathscr{I}$.

Baumgartner in [1] studied closure of $\mathscr{I}$-ultrafilters under ultrafilter sums for the case $X = \mathbb{R}$ and $\mathscr{I}$ a family of subsets of $\mathbb{R}$. The main purpose of this article is to investigate the closure of (weak) $\mathscr{I}$-ultrafilters and $\mathscr{I}$-close ultrafilters under ultrafilter sums and products for the case $X = \omega$ (or $\mathbb{N}$) and $\mathscr{I}$ is an ideal on ω (or $\mathbb{N}$).

So let us first recall the definition of ultrafilter sums and products (for details see [2]).

Definition C. If $\mathscr{U}$ and $\mathscr{V}_n$, $n \in \mathbb{N}$, are ultrafilters on $\mathbb{N}$ then $\sum_{\mathscr{U}} \langle \mathscr{V}_n : n \in \mathbb{N} \rangle$ is the ultrafilter on $\mathbb{N} \times \mathbb{N}$ defined by $M \in \sum_{\mathscr{U}} \langle \mathscr{V}_n : n \in \mathbb{N} \rangle$ if and only if $\{n : \{m : \langle n, m \rangle \in M\} \in \mathscr{V}_n\} \in \mathscr{U}$. We often identify isomorphic ultrafilters so we regard $\sum_{\mathscr{U}} \langle \mathscr{V}_n : n \in \mathbb{N} \rangle$ as an ultrafilter on $\mathbb{N}$. The ultrafilter $\sum_{\mathscr{U}} \langle \mathscr{V}_n : n \in \mathbb{N} \rangle$ is called the *$\mathscr{U}$-sum of ultrafilters $\mathscr{V}_n$, $n \in \mathbb{N}$.* If $\mathscr{V}_n = \mathscr{V}$ for every $n \in \mathbb{N}$ then we write $\sum_{\mathscr{U}} \langle \mathscr{V}_n : n \in \mathbb{N} \rangle = \mathscr{U} \cdot \mathscr{V}$ and ultrafilter $\mathscr{U} \cdot \mathscr{V}$ is called the *product of ultrafilters $\mathscr{U}$ and $\mathscr{V}$*.

Notation. Let us introduce the following notation often used in the sequel: For every $M \subseteq \mathbb{N} \times \mathbb{N}$ let $M(n) = \{m : \langle n, m \rangle \in M\}$ and $\widetilde{M} = \{n : M(n) \in \mathscr{V}_n\}$. With this notation we have $M \in \sum_{\mathscr{U}} \{\mathscr{V}_n : n \in \mathbb{N}\}$ if and only if $\widetilde{M} \in \mathscr{U}$.

Definition D. Let $\mathcal{C}$ and $\mathcal{D}$ be classes of ultrafilters. We say that $\mathcal{C}$ is *closed under $\mathcal{D}$-sums* provided that whenever $\{\mathscr{V}_n : n \in \omega\} \subseteq \mathcal{C}$ and $\mathscr{U} \in \mathcal{D}$ then $\sum_{\mathscr{U}} \langle \mathscr{V}_n : n \in \omega \rangle \in \mathcal{C}$. If $\mathcal{D}$ is a class of $\mathscr{I}$-ultrafilters then we simply say that $\mathcal{C}$ is closed under $\mathscr{I}$-sums.

1. Small subsets of natural numbers

We denote by $\mathbb{N}$ the set of all natural numbers (without zero). Let us recall that a set $A \subseteq \mathbb{N}$ has asymptotic density zero if $d^*(A) = \limsup_{n \to \infty} \frac{|A \cap n|}{n} = 0$. All sets with asymptotic density zero form an ideal on $\mathbb{N}$ that we call the density ideal and denote as $\mathcal{Z}_0$. The summable ideal $\mathcal{I}_{1/n}$ is the family $\{A \subseteq \mathbb{N} : \sum_{a \in A} \frac{1}{a} < +\infty\}$. It is not difficult to prove that every set in the summable ideal has asymptotic density zero. However, it suffices to consider the set of all prime numbers to see that the converse is not true.

It is important for our future considerations that both the density ideal and the summable ideal are P-ideals. Remember that an ideal $\mathscr{I} \subseteq \mathscr{P}(\mathbb{N})$ is a *P-ideal* if whenever $A_n \in \mathscr{I}$, $n \in \omega$, then there exists $A \in \mathscr{I}$ which contains all but finitely many elements of each A_n (we use the notation $A_n \subseteq^* A$ for this).

Actually, $\mathcal{Z}_0$-close ultrafilters were introduced as *0-points* by Gryzlov who constructed such ultrafilters in ZFC (see [5], [6]). Gryzlov's result was strengthened in [4] where a *summable ultrafilter* (an $\mathcal{I}_{1/n}$-close ultrafilter in our present terminology) was constructed in ZFC.

Definition 1.1. Let A be a subset of $\mathbb{N}$ with an increasing enumeration $A = \{a_n : n \in \mathbb{N}\}$. We say that A is

(1) *thin* if $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 0$
(2) *almost thin* if $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} < 1$
(3) *(SC)-set* if $\lim_{n\to\infty} a_{n+1} - a_n = \infty$

It follows from the definition that every thin set is almost thin. The converse is not true, see for example the set $\{2^n : n \in \omega\}$. It is also easy to see that every almost thin set is an (SC)-set and every (SC)-set belongs to the summable ideal $\mathcal{I}_{1/n}$. The set of all squares of natural numbers $\{n^2 : n \in \omega\}$ has property (SC) and it is not almost thin. The set $B = \{n!, n!+1 : n \in \mathbb{N}\} \in \mathcal{I}_{1/n}$ does not have the (SC)-property. In fact, the set B witnesses the fact that neither thin sets nor almost thin sets nor (SC)-sets form an ideal on $\mathbb{N}$.

We say that $A \subseteq \mathbb{N}$ contains an arithmetic progression of length n if there exist $a \in \mathbb{N}$ and $d > 0$ such that all the members of arithmetic progression $a + j \cdot d$ for $j = 0, \ldots, n-1$ belong to the set A.

Definition 1.2. *Van der Waerden ideal* is the family $\mathscr{W} = \{A \subseteq \mathbb{N} : A \text{ does not contain arithmetic progressions of arbitrary length}\}$.

2. $\mathscr{I}$-ultrafilters for P-ideals

Proposition 2.1. *If $\sum_{\mathscr{U}}\langle \mathscr{V}_n : n \in \mathbb{N}\rangle$ is an $\mathscr{I}$-ultrafilter then $\mathscr{U}$ is an $\mathscr{I}$-ultrafilter and $\{n : \mathscr{V}_n$ is an $\mathscr{I}$-ultrafilter$\} \in \mathscr{U}$.*

Proof. For arbitrary function $f : \mathbb{N} \to \mathbb{N}$ define $\tilde{f} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by $\tilde{f}(n, m) = f(n)$. According to the assumption there exists set $M \in \sum_{\mathscr{U}}\langle \mathscr{V}_n : n \in \mathbb{N}\rangle$ such that $\tilde{f}[M] \in \mathscr{I}$. Consider $M_2 = \{n : M(n) \neq \emptyset\}$. It is obvious that $\widetilde{M} \subseteq M_2$ and since $\widetilde{M} \in \mathscr{U}$ we have $M_2 \in \mathscr{U}$. Now observe that $f[M_2] = \tilde{f}[M] \in \mathscr{I}$ and thus $\mathscr{U}$ is an $\mathscr{I}$-ultrafilter.

For the other part assume that $\{n : \mathscr{V}_n$ is an $\mathscr{I}$-ultrafilter$\} \notin \mathscr{U}$. Then $I_0 = \{n : \mathscr{V}_n$ is not an $\mathscr{I}$-ultrafilter$\} \in \mathscr{U}$. For every $n \in I_0$ fix a function $f_n : \mathbb{N} \to \mathbb{N}$ such that $f_n[V] \notin \mathscr{I}$ for every $V \in \mathscr{V}_n$ and define $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by $g(n, m) = f_n(m)$ if $n \in I_0$ and arbitrarily if $n \notin I_0$. It remains to check that for every $M \in \sum_{\mathscr{U}}\langle \mathscr{V}_n : n \in \mathbb{N}\rangle$ we have $g[M] \notin \mathscr{I}$ to prove that $\sum_{\mathscr{U}}\langle \mathscr{V}_n : n \in \mathbb{N}\rangle$ is not an $\mathscr{I}$-ultrafilter. According to the assumption $f_n[M(n)] \notin \mathscr{I}$ if $n \in \widetilde{M} \cap I_0$. Now $g[M] \supseteq g[\bigcup\{\{n\} \times M(n) : n \in \widetilde{M} \cap I_0\}] = \bigcup_{n \in \widetilde{M} \cap I_0} f_n[M(n)] \notin \mathscr{I}$. □

For P-ideals the necessary condition from Proposition 2.1 is also a sufficient condition for $\mathscr{I}$-ultrafilters to be closed under $\mathscr{I}$-sums.

Proposition 2.2. *Let $\mathscr{I}$ be a P-ideal on $\mathbb{N}$. If $\mathscr{U}$ is an $\mathscr{I}$-ultrafilter and $\{n : \mathscr{V}_n$ is $\mathscr{I}$-ultrafilter$\} \in \mathscr{U}$ then $\sum_{\mathscr{U}}\langle \mathscr{V}_n : n \in \mathbb{N}\rangle$ is an $\mathscr{I}$-ultrafilter.*

Proof. Suppose $\mathscr{U}$ is an $\mathscr{I}$-ultrafilter and that for some $U_0 \in \mathscr{U}$ the ultrafilters $\mathscr{V}_n$, $n \in U_0$, are also $\mathscr{I}$-ultrafilters. Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be an arbitrary function. We want to find $M \in \sum_{\mathscr{U}}\langle \mathscr{V}_n : n \in \mathbb{N}\rangle$ such that $f[M] \in \mathscr{I}$.

For every $n \in U_0$ define $f_n : \mathbb{N} \to \mathbb{N}$ by $f_n(m) = f(\langle n, m\rangle)$ for every $m \in \mathbb{N}$. According to the assumption for every $n \in U_0$ there exists $V_n \in \mathscr{V}_n$ such that $f_n[V_n] \in \mathscr{I}$. Since $\mathscr{I}$ is a P-ideal there is a set $A \in \mathscr{I}$ such that $f_n[V_n] \subseteq^* A$ for every $n \in U_0$.

The set $f_n^{-1}[f_n[V_n]]$ belongs obviously to $\mathscr{V}_n$. Therefore either $f_n^{-1}[f_n[U_n] \cap A]$ or $f_n^{-1}[f_n[U_n] \setminus A]$ belongs to $\mathscr{V}_n$ for every $n \in U_0$. Define $I_0 = \{n \in U_0 : f_n^{-1}[f_n[V_n] \cap A] \in \mathscr{V}_n\}$ and $I_1 = \{n \in U_0 : f_n^{-1}[f_n[V_n] \setminus A] \in \mathscr{V}_n\}$. Since $\mathscr{U}$ is an ultrafilter one of the sets I_0, I_1 belongs to the ultrafilter $\mathscr{U}$.

Case A. $I_0 \in \mathscr{U}$

Put $M = \{\{n\} \times f_n^{-1}[f_n[V_n] \cap A] : n \in I_0\}$. It is easy to see that $M \in \sum_{\mathscr{U}} \langle \mathscr{V}_n : n \in \mathbb{N}\rangle$ and $f[M] = \bigcup_{n\in I_0} f_n[V_n] \cap A \subseteq A \in \mathscr{I}$.

Case B. $I_1 \in \mathscr{U}$

Since $f_n[U_n] \setminus A$ is finite and $\mathscr{V}_n$ is an ultrafilter, there exists $k_n \in f_n[U_n] \setminus A$ such that $f_n^{-1}\{k_n\} \in \mathscr{V}_n$ for every $n \in I_1$. Define $g : \mathbb{N} \to \mathbb{N}$ by $g(n) = k_n$ if $n \in I_1$ and arbitrarily otherwise. Since $\mathscr{U}$ is an $\mathscr{I}$-ultrafilter there exists $U \in \mathscr{U}$ such that $g[U] \in \mathscr{I}$. It remains to put $M = \{\{n\} \times f_n^{-1}\{k_n\} : n \in I_1 \cap U\}$. It is easy to check that $M \in \sum_{\mathscr{U}} \langle \mathscr{V}_n : n \in \mathbb{N}\rangle$ and $f[M] \subseteq g[U] \in \mathscr{I}$. □

Proposition 2.3. *Let $\mathscr{I}$ be a P-ideal on $\mathbb{N}$. If $\{n : \mathscr{V}_n$ is weak $\mathscr{I}$-ultrafilter$\} \in \mathscr{U}$ then $\sum_{\mathscr{U}} \langle \mathscr{V}_n : n \in \mathbb{N}\rangle$ is a weak $\mathscr{I}$-ultrafilter.*

Proof. Assume that $U_0 = \{n : \mathscr{V}_n$ is weak $\mathscr{I}$-ultrafilter$\} \in \mathscr{U}$. For a given finite-to-one function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ let f_n be the restriction of f onto $\{n\} \times \mathbb{N}$. For every $n \in \mathbb{N}$ the function f_n is again finite-to-one, so for every $n \in U_0$ there exists $V_n \in \mathscr{V}_n$ such that $f_n[V_n] \in \mathscr{I}$. Take $A \in \mathscr{I}$ such that $f_n[V_n] \subseteq^* A$ for every $n \in U_0$ (such a set exists because $\mathscr{I}$ is a P-ideal). The function f is finite-to-one hence we get $V_n \subseteq^* f_n^{-1}[A]$ for every $n \in \mathbb{N}$. It follows that $B_n = f_n^{-1}[A] \in \mathscr{V}_n$. Then the set $M = \{\langle n, k\rangle : n \in U_0, k \in B_n\}$ belongs to $\sum_{\mathscr{U}} \langle \mathscr{V}_n : n \in \mathbb{N}\rangle$ and $f[M] = \bigcup_{n\in U_0} f_n[B_n] = A \in \mathscr{I}$. Hence $\sum_{\mathscr{U}} \langle \mathscr{V}_n : n \in \mathbb{N}\rangle$ is a weak $\mathscr{I}$-ultrafilter. □

Proposition 2.4. *Let $\mathscr{I}$ be a P-ideal on $\mathbb{N}$. If $\{n : \mathscr{V}_n$ is $\mathscr{I}$-close ultrafilter$\} \in \mathscr{U}$ then $\sum_{\mathscr{U}} \langle \mathscr{V}_n : n \in \mathbb{N}\rangle$ is an $\mathscr{I}$-close ultrafilter.*

Proof. It suffices to replace words 'finite-to-one' by 'one-to-one' in the previous proof. □

The following proposition is in some sense a counterpart for Proposition 2.3.

Proposition 2.5. *Let $\mathscr{I}$ be an ideal on $\mathbb{N}$ and assume there exists function $g : \mathbb{N} \to \mathbb{N}$ such that $g(n) > n$ for every $n \in \mathbb{N}$ and $A \notin \mathscr{I}$ implies $g[A] \notin \mathscr{I}$ for every $A \subseteq \mathbb{N}$. If $\sum_{\mathscr{U}} \langle \mathscr{V}_n : n \in \mathbb{N}\rangle$ is a weak $\mathscr{I}$-ultrafilter then $\{n : \mathscr{V}_n$ is a weak $\mathscr{I}$-ultrafilter$\} \in \mathscr{U}$.*

Proof. We want to prove that if $\{n : \mathscr{V}_n$ is weak $\mathscr{I}$-ultrafilter$\} \notin \mathscr{U}$ then $\sum_{\mathscr{U}} \langle \mathscr{V}_n : n \in \mathbb{N}\rangle$ is not a weak $\mathscr{I}$-ultrafilter.

If $U_1 = \{n : \mathscr{V}_n$ is not weak $\mathscr{I}$-ultrafilter$\} \in \mathscr{U}$ then for every $n \in U_1$ there exists a finite-to-one function $f_n : \mathbb{N} \to \mathbb{N}$ such that

$f_n[V]$ does not belong to the ideal $\mathscr{I}$ for every $V \in \mathscr{V}_n$. Define $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ so that $f(n,m) = g^n[f_n(m)]$ whenever $n \in U_1$ and f is one-to-one on $(\mathbb{N} \setminus U_1) \times \mathbb{N}$. It is not difficult to check that the function f is finite-to-one. For every $M \in \sum_{\mathscr{U}} \langle \mathscr{V}_n : n \in \mathbb{N} \rangle$ there exists $n \in U_1$ and $V_n \in \mathscr{V}_n$ such that $M \supseteq \{n\} \times V_n$ and $f[M] \supseteq g^n[f_n[V_n]]$. But the latter set is not in the ideal $\mathscr{I}$ according to the assumption. Hence $f[M] \notin \mathscr{I}$ for every $M \in \sum_{\mathscr{U}} \langle \mathscr{V}_n : n \in \mathbb{N} \rangle$ and $\sum_{\mathscr{U}} \langle \mathscr{V}_n : n \in \mathbb{N} \rangle$ is not a weak $\mathscr{I}$-ultrafilter. □

Proposition 2.6. *Let $\mathscr{I}$ be an ideal on $\mathbb{N}$ and assume $A \notin \mathscr{I}$ implies $A + 1 \notin \mathscr{I}$ and $2A \notin \mathscr{I}$ for each $A \subseteq \mathbb{N}$. If the ultrafilter $\sum_{\mathscr{U}} \langle \mathscr{V}_n : n \in \mathbb{N} \rangle$ is $\mathscr{I}$-close then $\{n : \mathscr{V}_n$ is $\mathscr{I}$-close$\} \in \mathscr{U}$.*

Proof. Assume for the contrary that $\{n : \mathscr{V}_n$ is $\mathscr{I}$-close$\} \notin \mathscr{U}$. Then $U_1 = \{n : \mathscr{V}_n$ is not $\mathscr{I}$-close$\}$ belongs to $\mathscr{U}$ and for every $n \in U_1$ there exists a one-to-one function $f_n : \mathbb{N} \to \mathbb{N}$ such that $f_n[V]$ does not belong to the ideal $\mathscr{I}$ for every $V \in \mathscr{V}_n$. Define $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ in the following way: $f(n,m) = 2^n(2f_n(m) + 1)$. The function f is one-to-one. For every $M \in \sum_{\mathscr{U}} \langle \mathscr{V}_n : n \in \mathbb{N} \rangle$ there exists $n \in U_1$ and $V_n \in \mathscr{V}_n$ such that $M \supseteq \{n\} \times V_n$ and $f[M] \supseteq 2^n(2f_n[V_n] + 1)$. But the latter set is not in the ideal $\mathscr{I}$ according to the assumption. So $f[M] \notin \mathscr{I}$ for every $M \in \sum_{\mathscr{U}} \langle \mathscr{V}_n : n \in \mathbb{N} \rangle$ and $\sum_{\mathscr{U}} \langle \mathscr{V}_n : n \in \mathbb{N} \rangle$ is not an $\mathscr{I}$-close ultrafilter. □

Corollary 2.7. *Let $\mathscr{I}$ be a P-ideal on $\mathbb{N}$ as in Proposition 2.6. Assume $\mathscr{V}_n$, $n \in \mathbb{N}$, and $\mathscr{U}$ are free ultrafilters on $\mathbb{N}$. The ultrafilter $\sum_{\mathscr{U}} \langle \mathscr{V}_n : n \in \mathbb{N} \rangle$ is an $\mathscr{I}$-close ultrafilter if and only if $\{n : \mathscr{V}_n$ is $\mathscr{I}$-close$\} \in \mathscr{U}$.*

Proof. Combine Proposition 2.4 and Proposition 2.6. □

Corollary 2.8. *Assume $\mathscr{I}$ is a P-ideal on $\mathbb{N}$ as in Proposition 2.6. For an ultrafilter $\mathscr{V} \in \mathbb{N}^*$ the following conditions are equivalent:*

(1) *$\mathscr{V}$ is an $\mathscr{I}$-close ultrafilter.*
(2) *$(\forall \mathscr{U} \in \mathbb{N}^*)$ $\mathscr{U} \cdot \mathscr{V}$ is an $\mathscr{I}$-close ultrafilter.*
(3) *$(\exists \mathscr{U} \in \mathbb{N}^*)$ $\mathscr{U} \cdot \mathscr{V}$ is an $\mathscr{I}$-close ultrafilter.*

Proof. If $\mathscr{V}$ is $\mathscr{I}$-close then $\mathscr{U} \cdot \mathscr{V} = \sum_{\mathscr{U}} \langle \mathscr{V}_n : n \in \mathbb{N} \rangle$ where $\mathscr{V}_n = \mathscr{V}$ for every n. Hence $\{n : \mathscr{V}_n$ is an $\mathscr{I}$-close ultrafilter$\} = \mathbb{N} \in \mathscr{U}$ for every $\mathscr{U} \in \mathbb{N}^*$ and $\mathscr{U} \cdot \mathscr{V}$ is an $\mathscr{I}$-close ultrafilter according to Proposition 2.4.

The implication (2) $\Rightarrow$ (3) is trivial.

If $\mathscr{U} \cdot \mathscr{V}$ is an $\mathscr{I}$-close ultrafilter for some $\mathscr{U} \in \mathbb{N}^*$ then it follows from Proposition 2.6 that $\mathscr{V}$ is an $\mathscr{I}$-close ultrafilter. □

3. Some remarks on other ideals

In this chapter we will show that it is not possible to remove from Corollary 2.8 the assumption on ideal $\mathscr{I}$ to be a P-ideal. Ideals generated by thin sets and (SC)-sets are not P-ideals and it turns out that thin ultrafilters and (SC)-ultrafilters are not closed even under products.

Proposition 3.1. *The ultrafilter $\mathscr{U} \cdot \mathscr{U}$ is not an (SC)-close ultrafilter for every $\mathscr{U} \in \mathbb{N}^*$.*

Proof. Assume $\mathscr{U}$ is a free ultrafilter on $\mathbb{N}$. Consider $h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $h(\langle n, m\rangle) = 2^M + M + n - m$ where $M = \max\{n, m\}$. Notice that h is a one-to-one function. We will show in the following that $h[U]$ is not an (SC)-set for every $U \in \mathscr{U} \cdot \mathscr{U}$.

Since $U \in \mathscr{U} \cdot \mathscr{U}$ we have $\widetilde{U} \in \mathscr{U}$. Choose two distinct elements $n_1, n_2 \in \widetilde{U}$ and denote $V = U(n_1) \cap U(n_2)$. The set V is infinite because it belongs to the free ultrafilter $\mathscr{U}$. For every $v \in V$ we have $\langle n_1, v\rangle$, $\langle n_2, v\rangle \in U$. If $v \geq \max\{n_1, n_2\}$ then $h(\langle n_1, v\rangle) = 2^v + n_1$, $h(\langle n_2, v\rangle) = 2^v + n_2$ and $h(\langle n_2, v\rangle) - h(\langle n_1, v\rangle) = n_2 - n_1$. So we have infinitely pairs of elements in U such that $h(u_2) - h(u_1) = n_2 - n_1$, in particular $h[U]$ is not an (SC)-set. □

Corollary 3.2. *For every $\mathscr{U} \in \omega^*$ the ultrafilter $\mathscr{U} \cdot \mathscr{U}$ is not:*

1. *thin ultrafilter.*
2. *weak thin ultrafilter.*
3. *thin-close ultrafilter.*
4. *weak (SC)-ultrafilter.*
5. *(SC)-ultrafilter.*

Van der Waerden ideal is not a P-ideal either. At the moment we do not know if (weak) $\mathscr{W}$-ultrafilters ($\mathscr{W}$-close ultrafilters) are closed under corresponding products and sums, but some necessary conditions are clear.

Proposition 3.3. *If $\mathscr{U} \cdot \mathscr{V}$ is a weak $\mathscr{W}$-ultrafilter then $\mathscr{U}$ and $\mathscr{V}$ are weak $\mathscr{W}$-ultrafilters.*

Proof. If $\mathscr{U}$ is not a weak $\mathscr{W}$-ultrafilter then there exists a finite-to-one function $g : \mathbb{N} \to \mathbb{N}$ such that $g[U] \notin \mathscr{W}$ for every $U \in \mathscr{U}$. Define $G : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by $G(n,m) = 2^m(2g(n)+1)$. It is easy to check that function G is finite-to-one.

For every $M \in \mathscr{U} \cdot \mathscr{V}$ we have $\widetilde{M} \in \mathscr{U}$ and hence $g[\widetilde{M}] \notin \mathscr{W}$. Thus we can find for every $k \in \mathbb{N}$ a finite set $K \subseteq \widetilde{M}$ such that $g[K]$ is an arithmetic progression of length k. Now, let $V_0 = \bigcap_{n \in K} M(n)$. From the definition of product, $\widetilde{M}$ and $M(n)$ we know that $V_0 \in \mathscr{V}$. We have $G[M] \supseteq G[K \times V_0] = \bigcup_{v \in V_0} 2^v(2g[K]+1)$. Obviously, the latter set contains infinitely many arithmetic progressions of length k since any shift of arithmetic progression is again an arithmetic progression. It follows that $G[M] \notin \mathscr{W}$ and $\mathscr{U} \cdot \mathscr{V}$ is not a weak $\mathscr{W}$-ultrafilter.

The other part follows from Proposition 2.5 for example if we consider $g(n) = n+1$ or $g(n) = 2n$. □

Proposition 3.4. *If $\mathscr{U} \cdot \mathscr{V}$ is a $\mathscr{W}$-close ultrafilter then $\mathscr{U}$ and $\mathscr{V}$ are $\mathscr{W}$-close ultrafilters.*

Proof. It suffices to modify slightly the proof of Proposition 3.3. Observe that if function g in the proof is one-to-one then G is also one-to-one. The other part follows from Proposition 2.6. □

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