# The relation of rapid ultrafilters and $Q$-points to van der Waerden ideal

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# $Q$-points and rapid ultrafilters

Definition.

A free ultrafilter $\mathcal{U}$ is called a $Q$-point if for every $\{Q_i : i \in \omega\}$, a partition of $\omega$ into finite sets, there exists $U \in \mathcal{U}$ such that $(\forall i \in \omega)\ |U \cap Q_i| \leq 1$.

A free ultrafilter $\mathcal{U}$ is called rapid if for every $\{Q_i : i \in \omega\}$, a partition of $\omega$ into finite sets, there exists $U \in \mathcal{U}$ such that $(\forall i \in \omega)\ |U \cap Q_i| \leq i$.

Alternative definition of rapid ultrafilters:

A free ultrafilter $\mathcal{U}$ is called rapid if the enumeration functions of its sets form a dominating family in $(\omega^\omega, \leq^*)$.

# Existence of $Q$-points and rapid ultrafilters

Every $Q$-point is rapid, but the converse is not true.

Theorem (Booth?).

(CH) $Q$-points exist.

Theorem (Miller).

In Laver's model there are no rapid ultrafilters.

In every model where $Q$-points are known not to exist,
rapid ultrafilters do not exist either.

# Generic existence

Definition (Canjar).

We say that $Q$-points (respectively rapid ultrafilters) exist generically if every filter of character $< \mathfrak{d}$ is included in a $Q$-point (respectively rapid ultrafilter).

Theorem (Canjar).

The following are equivalent:

- $\mathrm{cov}(\mathcal{M}) = \mathfrak{d}$,
- $Q$-points exist generically,
- Rapid ultrafilters exist generically.

# Product of ultrafilters

Definition.

Let $\mathcal{U}$ and $\mathcal{V}$, $n \in \omega$, be ultrafilters on $\omega$.
The product of ultrafilters $\mathcal{U}$ and $\mathcal{V}$, denoted by $\mathcal{U} \times \mathcal{V}$, is an ultrafilter on $\omega \times \omega$ defined by $A \in \mathcal{U} \times \mathcal{V}$ if and only if $\{n : \{m : \langle n, m\rangle \in A\} \in \mathcal{V}\} \in \mathcal{U}$.

It is known that $\mathcal{U} \times \mathcal{V}$ is never a $Q$-point.

Theorem (Miller).

$\mathcal{U} \times \mathcal{V}$ is a rapid ultrafilter if and only if $\mathcal{V}$ is rapid.

# AP-sets and van der Waerden ideal

Definition.

A set $A \subseteq \omega$ is called an AP-set if it contains arbitrary long arithmetic progressions.

Sets which are not AP-sets form a proper ideal on $\omega$.

It is van der Waerden ideal $\mathcal{W}$.

The van der Waerden ideal $\mathcal{W}$ is $F_\sigma$-ideal, not a $P$-ideal.

# Difference between $Q$-points and rapid ultrafilters

**Lemma 1.**

Every $Q$-point has a nonempty intersection with the ideal $\mathcal{W}$.

**Proof of Lemma 1.**

1. Let $\omega = \bigcup_{n\in\omega} I_n$ where $I_n = [2^n, 2^{n+1})$.
2. $\exists U_0$ in the ultrafilter such that $|U_0 \cap I_n| \leq 1$ for every $n$.
3. Either $U_1 = \bigcup_{n\ odd} I_n$ or $U_2 = \bigcup_{n\ even} I_n$ is in the ultrafilter.
4. The set $U = U_0 \cap U_i$ is in $\mathcal{W}$.

**Theorem 2.**

($\mathrm{MA}_{\mathrm{ctble}}$) There is a rapid ultrafilter $\mathcal{U}$ such that $\mathcal{U} \cap \mathcal{W} = \emptyset$.

# Proof of Theorem 2

## An alternative characterization of rapid ultrafilters

**Definition.**

For a function $g : \omega \to [0, \infty)$ with $\sum_{n \in \omega} g(n) = \infty$ the family

$$\mathcal{I}_g = \{A \subseteq \omega : \sum_{a \in A} g(a) < +\infty\}$$

is a summable ideal determined by function $g$.

A summable ideal $\mathcal{I}_g$ is tall if and only if $\lim_{n \to \infty} g(n) = 0$.

**Theorem (Vojtáš).**

An ultrafilter $\mathcal{U} \in \omega^*$ is rapid if and only if $\mathcal{U} \cap \mathcal{I}_g \neq \emptyset$ for every tall summable ideal $\mathcal{I}_g$.

# Proof of Theorem 2

## Outline of the construction

1. List all tall summable ideals as $\{\mathcal{I}_{g_\alpha} : \alpha < \mathfrak{c}\}$.

2. For $\alpha < \mathfrak{c}$ construct filter bases $\mathcal{F}_\alpha$ such that for every $\alpha < \mathfrak{c}$ the following hold:
   - (i) $\mathcal{F}_0$ is the Fréchet filter
   - (ii) $\mathcal{F}_\alpha \supseteq \mathcal{F}_\beta$ whenever $\alpha \geq \beta$
   - (iii) $\mathcal{F}_\gamma = \bigcup_{\alpha<\gamma} \mathcal{F}_\alpha$ for $\gamma$ limit
   - (iv) $(\forall \alpha)\ |\mathcal{F}_\alpha| \leq |\alpha + 1| \cdot \omega$
   - (v) $(\forall \alpha)\ (\forall F \in \mathcal{F}_\alpha)\ F$ is an AP-set
   - (vi) $(\forall \alpha)\ (\exists F \in \mathcal{F}_{\alpha+1})\ F \in \mathcal{I}_{g_\alpha}$

3. At successor stage use the following lemma:

# Proof of Theorem 2

## Succesor stage

**Lemma 2a.**

($\mathrm{MA}_{\mathrm{ctble}}$) Assume $\mathcal{I}_g$ is a tall summable ideal, $\mathcal{F}$ is a filter base on $\omega$ with $|\mathcal{F}| < \mathfrak{c}$ and $\mathcal{F} \cap \mathcal{W} = \emptyset$.
Then there exists $G \in [\omega]^\omega$ such that $G \in \mathcal{I}_g$ and $G \cap F$ is an AP-set for every $F \in \mathcal{F}$.

**Proof of Lemma 2a:**

If $\mathcal{F} \cap \mathcal{I}_g = \emptyset$ then consider $P = \{K \in [\omega]^{<\omega} : \sum_{a \in K} g(a) < 1\}$ with a partial order $\leq_P$ defined by: $K \leq_P L$ if and only if $K = L$ or $K \supset L$ and $\min(K \setminus L) > \max L$.

$D_{F,k} = \{K \in P : K \cap F \text{ contains an a. p. of length } k\}$ are dense

# $\mathcal{W}$-ultrafilters

Definition.

An ultrafilter $\mathcal{U} \in \omega^*$ is called

- a weak $\mathcal{W}$-ultrafilter if for every finite-to-one $f : \omega \to \omega$ there exists $U \in \mathcal{U}$ such that $f[U] \in \mathcal{W}$.
- an $\mathcal{W}$-ultrafilter if for every $f : \omega \to \omega$ there exists $U \in \mathcal{U}$ such that $f[U] \in \mathcal{W}$.

Every $\mathcal{W}$-ultrafilter is a weak $\mathcal{W}$-ultrafilter.

Every weak $\mathcal{W}$-ultrafilter has a nonempty intersection with the van der Waerden ideal.

# $\mathcal{W}$-ultrafilters and $Q$-points

**Lemma 3.**

Every $Q$-point is a weak $\mathcal{W}$-ultrafilter.

**Proposition 4.**

($\mathrm{MA}_{\mathrm{ctble}}$) There is a $Q$-point which is not a $\mathcal{W}$-ultrafilter.

**Theorem 5.**

($\mathrm{MA}_{\mathrm{ctble}}$) There is a $\mathcal{W}$-ultrafilter which is not a $Q$-point.

# Proof of Theorem 5.

## Property (♠)

**Definition.**

A filter base $\mathcal{F}$ has property (♠) if

$$(\forall F \in \mathcal{F})\,(\forall k \in \omega)\,(\exists n \in \omega)\,|F \cap [2^n, 2^{n+1})| > k.$$

**Lemma 5a.**

Every filter base $\mathcal{F}$ which has property (♠) can be extended into an ultrafilter which is not a $Q$-point.

# Proof of Theorem 5.

## Outline of the construction

1. List all functions ${}^{\omega}\omega = \{f_\alpha : \alpha < \mathfrak{c}\}$.

2. For $\alpha < \mathfrak{c}$ construct filter bases $\mathcal{F}_\alpha$ such that for every $\alpha < \mathfrak{c}$ the following hold:
   (i) $\mathcal{F}_0$ is the Fréchet filter
   (ii) $\mathcal{F}_\alpha \supseteq \mathcal{F}_\beta$ whenever $\alpha \geq \beta$
   (iii) $\mathcal{F}_\gamma = \bigcup_{\alpha<\gamma} \mathcal{F}_\alpha$ for $\gamma$ limit
   (iv) $(\forall\alpha)\ |\mathcal{F}_\alpha| \leq |\alpha + 1| \cdot \omega$
   (v) $(\forall\alpha)\ \mathcal{F}_\alpha$ has property ($\spadesuit$)
   (vi) $(\forall\alpha)\ (\exists F \in \mathcal{F}_{\alpha+1})\ f_\alpha[F] \in \mathcal{W}$

3. At successor stage use the following lemma:

# Proof of Theorem 5.

## Successor stage

**Lemma 5b.**

($MA_{\text{ctble}}$) Assume $\mathcal{F}$ is a filter base with $|\mathcal{F}| < \mathfrak{c}$ with the property (♠). Assume $f \in {}^{\omega}\omega$.
Then there is $G \in [\omega]^{\omega}$ such that $f[G] \in \mathcal{W}$ and the filter base generated by $\mathcal{F}$ and $G$ has property (♠).

**Proof of Lemma 5b:**

If neither a set from $\mathcal{F}$ nor $f^{-1}[K]$ for some finite set $K$ has the required property then consider
$P = \{K \in [\omega]^{<\omega} : f[K]$ contains no a. p. of length 3$\}$
with a partial order $\leq_P$ defined by: $K \leq_P L$ if and only if $K = L$ or $K \supset L$ and $\min(K \setminus L) > \max L$.
$D_{F,k} = \{K \in P : (\exists n \in \omega)|K \cap F \cap [2^n, 2^{n+1})| \geq k\}$ are dense

# Questions

**Theorem 2.**

($MA_{\text{ctble}}$) There is a rapid ultrafilter $\mathcal{U}$ such that $\mathcal{U} \cap \mathcal{W} = \emptyset$.

**Question A.**

Does there consistently exist an idempotent ultrafilter which is a rapid ultrafilter?

**Theorem 5.**

($MA_{\text{ctble}}$) There is a $\mathcal{W}$-ultrafilter which is not a $Q$-point.

**Question B.**

Does there (consistently) exist a $\mathcal{W}$-ultrafilter which is not a rapid ultrafilter?

# References

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