# Ultrafilters on $\mathbb{N}$ and van der Waerden ideal

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# $P$-points

All the ultrafilters in this talk are free.

The free ultrafilters on $\mathbb{N}$ correspond to the points in the remainder of the Čech-Stone compactification of $\mathbb{N}$.

Definition.

An ultrafilter $p \in \mathbb{N}^*$ is called a $P$-point if for every family $V_i$, $i \in \omega$ of open neighborhoods of $p$ there exists an open neighborhood $V$ such that $V \subseteq V_i$ for every $i \in \omega$.

An ultrafilter $\mathcal{U}$ is a $P$-point if for every $\{R_i : i \in \omega\}$, a partition of $\omega$ with $R_i \notin \mathcal{U}$, there exists $U \in \mathcal{U}$ such that $(\forall i \in \omega)\ |U \cap R_i| < \omega$.

# $Q$-points and rapid ultrafilters

**Definition.**

An ultrafilter $\mathcal{U}$ is called a $Q$-point if for every $\{Q_i : i \in \omega\}$, a partition of $\omega$ into finite sets, there exists $U \in \mathcal{U}$ such that $(\forall i \in \omega)\ |U \cap Q_i| \leq 1$.

An ultrafilter $\mathcal{U}$ is called rapid if for every $\{Q_i : i \in \omega\}$, a partition of $\omega$ into finite sets, there exists $U \in \mathcal{U}$ such that $(\forall i \in \omega)\ |U \cap Q_i| \leq i$.

**Alternative definition of rapid ultrafilters:**

An ultrafilter $\mathcal{U}$ is rapid if the enumeration functions of its sets form a dominating family in $(\omega^\omega, \leq^*)$.

# Some facts about ultrafilters

Assuming CH or Martin's axiom for countable posets all the above mentioned ultrafilters exist.

**Theorem (Shelah).**

It is consistent with ZFC that there are no *P*-points.

**Theorem (Miller).**

In Laver's model there are no rapid ultrafilters.

# Some questions about ultrafilters

**Problem I.**

No model is known in which neither $P$-points nor $Q$-points exist.

Every $Q$-point is rapid, but the converse is not true.

**Problem II.**

In every model where $Q$-points are known not to exist,
rapid ultrafilters do not exist either.

# AP-sets and van der Waerden ideal

**Definition.**

A set $A \subseteq \omega$ is called an AP-set if it contains arbitrary long arithmetic progressions.

Sets which are not AP-sets form a proper ideal on $\omega$.
It is van der Waerden ideal $\mathcal{W}$.

The van der Waerden ideal $\mathcal{W}$ is $F_\sigma$-ideal, not a $P$-ideal.

# Ultrafilters disjoint from $\mathcal{W}$

**Theorem 1.**

($\text{MA}_{\text{ctble}}$) There is a $P$-point $\mathcal{U}$ such that $\mathcal{U} \cap \mathcal{W} = \emptyset$.

**Theorem 2.**

($\text{MA}_{\text{ctble}}$) There is a rapid ultrafilter $\mathcal{U}$ such that $\mathcal{U} \cap \mathcal{W} = \emptyset$.

**Corollary 3.**

($\text{MA}_{\text{ctble}}$) There is a rapid $P$-point $\mathcal{U}$ such that $\mathcal{U} \cap \mathcal{W} = \emptyset$.

# Ultrafilters intersecting $\mathcal{W}$

**Lemma 4.**

Every $Q$-point has a nonempty intersection with the ideal $\mathcal{W}$.

**Proof of Lemma 4.**

1. Let $\omega = \bigcup_{n\in\omega} I_n$ where $I_n = [2^n, 2^{n+1})$.
2. $\exists U_0$ in the ultrafilter such that $|U_0 \cap I_n| \leq 1$ for every $n$.
3. Either $U_1 = \bigcup_{n\ odd} I_n$ or $U_2 = \bigcup_{n\ even} I_n$ is in the ultrafilter.
4. The set $U = U_0 \cap U_i$ is in $\mathcal{W}$.

# $\mathcal{W}$-ultrafilters

**Definition.**

An ultrafilter $\mathcal{U} \in \omega^*$ is called

- a weak $\mathcal{W}$-ultrafilter if for every finite-to-one $f : \omega \to \omega$ there exists $U \in \mathcal{U}$ such that $f[U] \in \mathcal{W}$.
- an $\mathcal{W}$-ultrafilter if for every $f : \omega \to \omega$ there exists $U \in \mathcal{U}$ such that $f[U] \in \mathcal{W}$.

Every $\mathcal{W}$-ultrafilter is a weak $\mathcal{W}$-ultrafilter.

Every weak $\mathcal{W}$-ultrafilter has a nonempty intersection with the van der Waerden ideal.

# $Q$-points and $\mathcal{W}$-ultrafilters

**Lemma 5.**

Every $Q$-point is a weak $\mathcal{W}$-ultrafilter.

**Proposition 6.**

($\mathrm{MA}_{\mathrm{ctble}}$) There is a $Q$-point which is not a $\mathcal{W}$-ultrafilter.

($\mathrm{MA}_{\mathrm{ctble}}$) For every tall ideal $\mathcal{I}$ there is a $Q$-point which is not an $\mathcal{I}$-ultrafilter.

# $\mathcal{W}$-ultrafilters and other ultrafilters

**Theorem 7.**

($MA_{ctble}$) There is a $\mathcal{W}$-ultrafilter which is not a $Q$-point.

**Question A.**

Does there (consistently) exist a $\mathcal{W}$-ultrafilter which is not a rapid ultrafilter?

**Theorem 8.**

($MA_{ctble}$) There is a $\mathcal{W}$-ultrafilter which is not a $P$-point.

# Algebraic structure on $\beta\mathbb{N}$

The addition $+$ on $\mathbb{N}$ extends in a natural way to $(\beta\mathbb{N}, +)$.

Definition.

An ultrafilter $p$ is called idempotent if $p + p = p$.

Neither $P$-points nor $Q$-points are idempotents.

In fact: $p + q$ is never a $P$-point or $Q$-point.

What about the rapid ultrafilters?
Does there consistently exist a rapid ultrafilter which is idempotent?

# Idempotents in $(\beta\mathbb{N}, +)$

Proposition (Blass, Krautzberger).

Every strongly summable ultrafilter is rapid.

Strongly summable ultrafilters are idempotent, but far from being minimal idempotents.

If $p$ is a minimal idempotent, then every $A \in p$ is an AP-set.

Are there consistently any rapid minimal idempotents?

# Rapid minimal idempotents

**Theorem (Krautzberger)**

If there is a rapid ultrafilter then there exists a rapid ultrafilter which is a minimal idempotent.

**The idea of the proof:**

1. If $p$ is a rapid ultrafilter then every $q \in \mathbb{N}^* + p$ is rapid.
2. $\mathbb{N}^* + p$ is a left ideal, thus intersects a minimal ideal.
3. There are many rapid minimal idempotents.

**Corollary**

If there is a rapid ultrafilter then there exists a rapid ultrafilter which contains only AP-sets.

# References

J. Flašková, The relation of rapid ultrafilters and $Q$-points to van der Waerden ideal, *to appear*. arXiv:1004.1879v2

A. Miller, There are no $Q$-points in Laver's model for the Borel conjecture, *Proc. Amer. Math. Soc.* **78** (1980), 498 – 502.

S. Shelah, *Proper forcing*, Lecture Notes in Mathematics, 940. Springer-Verlag, Berlin-New York, 1982.