Nowhere dense sets corresponding to summable ideals

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Čech-Stone compactification of ω

Definition.

Čech-Stone compactification of ω is a compact topological space $\beta\omega$ such that:

- ω is a dense subspace of $\beta\omega$
- every (continuous) function $f:\omega \longrightarrow [0,1]$ can be extended to a continuous function $\beta f:\beta\omega \longrightarrow [0,1]$

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$\omega^* = \beta\omega \setminus \omega$ is called remainder of $\beta\omega$

Ultrafilters on ω

Definition.

A family $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is called a filter on ω if:

- $\mathcal{F} \neq \emptyset$ and $\emptyset \notin \mathcal{F}$
- if $F_1, F_2 \in \mathcal{F}$ then $F_1 \cap F_2 \in \mathcal{F}$
- if $F \in \mathcal{F}$ and $F \subseteq G \subseteq X$ then $G \in \mathcal{F}$.

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If moreover $\mathcal{F}$ satisfies

- for every $M \subseteq \omega$ either $M \in \mathcal{F}$ or $\omega \setminus M \in \mathcal{F}$

then $\mathcal{F}$ is called an ultrafilter.

Topology on ω^*

Points in $\beta\omega$ may be identified with ultrafilters on ω:

points in $\omega \leftrightarrow$ fixed ultrafilters

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Some more facts about ω^*:

- zero-dimensional
- cardinality $2^\mathfrak{c}$
- dense-in-itself

Ideals on ω

Definition.

An ideal on ω is a family $\mathcal{I} \subseteq \mathcal{P}(\omega)$ such that:

- $\mathcal{I} \neq \mathcal{P}(\omega)$ and $\emptyset \in \mathcal{I}$
- if $A_1, A_2 \in \mathcal{I}$ then $A_1 \cup A_2 \in \mathcal{I}$
- if $A \in \mathcal{I}$ and $B \subseteq A \subseteq \omega$ then $B \in \mathcal{I}$.

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Examples: $\mathrm{Fin} =$ finite subsets of ω

$$\mathcal{I}_{1/n} = \{A \subseteq \mathbb{N} : \sum_{a \in A} \frac{1}{a} < \infty\}$$

$$\mathcal{Z}_0 = \{A \subseteq \mathbb{N} : \limsup_{n \to \infty} \frac{|A \cap n|}{n} = 0\}$$

Summable ideals

Definition.

Given a function $g : \omega \to [0, \infty)$ such that $\sum_{n \in \omega} g(n) = \infty$ then the family

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We will consider only tall summable ideals.

Ideals and open sets

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An ideal $\mathcal{I}$ on ω is tall (dense) if for every infinite set $A \subseteq \omega$ there exists $B \in [A]^\omega$ such that $B \in \mathcal{I}$.

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A summable ideal is tall if and only if $\lim_{n\to\infty} g(n) = 0$.

To every ideal $\mathcal{I}$ on ω assign an open set

$$\sigma(\mathcal{I}) = \bigcup\{A^* : A \in \mathcal{I}\}$$

and a closed set

$$\delta(\mathcal{I}) = \omega^* \setminus \sigma(\mathcal{I})$$

Ideals and (nowhere) dense sets

$$\sigma(\mathcal{I}) = \{\mathcal{U} \in \omega^* : \mathcal{U} \cap \mathcal{I} \neq \emptyset\}$$
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Lemma.

For an ideal $\mathcal{I}$ on ω the following are equivalent:

- $\mathcal{I}$ is tall
- $\sigma(\mathcal{I})$ is dense in ω^*
- $\delta(\mathcal{I})$ is nowhere dense in ω^*

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Lemma.

For an ideal $\mathcal{I}$ on ω the following are equivalent:

- $\mathcal{I}$ is tall
- $\sigma(\mathcal{I})$ is dense in ω^*
- $\delta(\mathcal{I})$ is nowhere dense in ω^*

If $\mathcal{I} \subseteq \mathcal{J}$ then $\sigma(\mathcal{I}) \subseteq \sigma(\mathcal{J})$ and $\delta(\mathcal{I}) \supseteq \delta(\mathcal{J})$.

The problem

Problem S.4. (van Douwen [1978])

Is it true in ZFC that $\bigcup_{\pi\in S_\omega}\beta\pi[A]\neq\omega^*$ whenever $A\subseteq\omega^*$ is nowhere dense? What if $A=\delta(\mathcal{Z}_0)$ or $A=\delta(\mathcal{I}_{1/n})$?

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Problem 235. (Hart, van Mill [1990])

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Theorem (Gryzlov)

For $A = \delta(\mathcal{Z}_0)$ we have $\bigcup_{\pi \in S_\omega} \beta\pi[A] \neq \omega^*$.

Results

Theorem 1.

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If $\mathcal{U} \in \omega^* \setminus \bigcup_{\pi \in S_\omega} \beta\pi[A]$ then for all $\pi \in S_\omega$ there exists set $U \in \mathcal{U}$ with $\pi[U] \in \mathcal{I}_{1/n}$.

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Definition A.

An ultrafilter $\mathcal{U}$ on ω is called a summable ultrafilter if for every one-to-one function $f : \omega \longrightarrow \mathbb{N}$ there exists $U \in \mathcal{U}$ such that $f[U] \in \mathcal{I}_{1/n}$.

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Theorem 1*.

Summable ultrafilters exist in ZFC.

Results

Definition B.

An ultrafilter $\mathcal{U}$ on ω is called a g-summable ultrafilter if for every one-to-one function $f : \omega \longrightarrow \mathbb{N}$ there exists $U \in \mathcal{U}$ such that $f[U] \in \mathcal{I}_g$.

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Definition B.

An ultrafilter $\mathcal{U}$ on ω is called a g-summable ultrafilter if for every one-to-one function $f : \omega \to \mathbb{N}$ there exists $U \in \mathcal{U}$ such that $f[U] \in \mathcal{I}_g$.

Corollary 2.

If $g : \omega \to [0, \infty)$ satisfies $\frac{1}{n} \gg g(n)$ then g-summable ultrafilters exist in ZFC.

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Corollary 2.

If $g : \omega \to [0, \infty)$ satisfies $\frac{1}{n} \gg g(n)$ then g-summable ultrafilters exist in ZFC.

Theorem 3.

If $g(n) = \frac{\ln^p n}{n}$, $p \in \omega$, then g-summable ultrafilters exist in ZFC (i.e. $\delta(\mathcal{I}_g)$ solves Problem 235.).

More results and questions

Theorem 4.

(MA$_{\text{ctble}}$) If I_g is a summable ideal and $A = \delta(\mathcal{I}_g)$ then $\bigcup_{\pi \in S_\omega} \beta\pi[A] \neq \omega^*$.

More results and questions

Theorem 4.

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Question.

Do g-summable ultrafilters exist in ZFC for every tall summable ideal $\mathcal{I}_g$? What if $g(n) = \frac{1}{\sqrt{n}}$ or $\frac{1}{\ln n}$?

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Theorem 4.

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Question.

Do g-summable ultrafilters exist in ZFC for every tall summable ideal $\mathcal{I}_g$? What if $g(n) = \frac{1}{\sqrt{n}}$ or $\frac{1}{\ln n}$?

Question.

Is there a g-summable ultrafilter which is not an h-summable ultrafilter if I_g and I_h are two incomparable summable ultrafilters?

Construction

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Definition.

A family $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is called

- a k-linked family if $F_1 \cap \ldots \cap F_k$ is infinite whenever $F_i \in \mathcal{F}$, $i \leq k$.
- a centered system if $\mathcal{F}$ is k-linked for every k i.e., if any finite subfamily of $\mathcal{F}$ has an infinite intersection.

Construction

We say that $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is a summable family if for every one-to-one function $f : \omega \to \mathbb{N}$ there is $A \in \mathcal{F}$ such that $f[A] \in \mathcal{I}_{1/n}$.

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We say that $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is a summable family if for every one-to-one function $f : \omega \longrightarrow \mathbb{N}$ there is $A \in \mathcal{F}$ such that $f[A] \in \mathcal{I}_{1/n}$.

Proposition 5.

For every $k \in \mathbb{N}$ there exists a summable k-linked family $\mathcal{F}_k \subseteq \mathcal{P}(\omega)$.

Construction

Lemma 6.

If $\mathcal{F}_k \subseteq \mathcal{P}(\omega)$ is a k-linked family then

$$\mathcal{F} = \{F \subseteq \omega : (\forall k)(\exists U^k \in \mathcal{F}_k)\, U^k \subseteq^* F\}$$

is a centered system.

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Lemma 6.

If $\mathcal{F}_k \subseteq \mathcal{P}(\omega)$ is a k-linked family then

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is a centered system.

If every $\mathcal{F}_k$ is summable then $\mathcal{F}$ is summable.

More generally, if $\mathcal{I}$ is a P-ideal and for every one-to-one function $f \in {}^{\omega}\mathbb{N}$ and for every $k \in \mathbb{N}$ there exists $U^k \in \mathcal{F}_k$ such that $f[U^k] \in \mathcal{I}$ then there exists $U \in \mathcal{F}$ such that $f[U] \in \mathcal{I}$.

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