The generic existence of certain $\mathcal{I}$-ultrafilters

Jörg Brendle[1] Jana Flašková[2]

[1]Graduate School of Engineering
Kobe University

[2]Department of Mathematics
University of West Bohemia in Pilsen

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Ultrafilters on ω

Definition.

$\mathcal{U} \subseteq \mathcal{P}(\omega)$ is an ultrafilter if

- $\mathcal{U} \neq \emptyset$ and $\emptyset \notin \mathcal{U}$
- if $U_1, U_2 \in \mathcal{U}$ then $U_1 \cap U_2 \in \mathcal{U}$
- if $U \in \mathcal{U}$ and $U \subseteq V \subseteq \omega$ then $V \in \mathcal{U}$.
- for every $M \subseteq \omega$ either M or $\omega \setminus M$ belongs to $\mathcal{U}$

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Example. fixed (or principal) ultrafilter $\{A \subseteq \omega : n \in A\}$

Ultrafilters on ω

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A free ultrafilter $\mathcal{U}$ is called a *P*-point if for all partitions of ω, $\{R_i : i \in \omega\}$, either for some i, $R_i \in \mathcal{U}$, or $(\exists U \in \mathcal{U})\ (\forall i \in \omega)$ $|U \cap R_i| < \omega$.

- Assuming CH or MA *P*-points exist.
- Shelah proved that consistently there may be no *P*-points.

Generic existence of ultrafilters

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A class $\mathcal{C}$ of ultrafilters exists generically if every filter base of size less than $\mathfrak{c}$ can be extended to an ultrafilter belonging to $\mathcal{C}$.

Given a class of ultrafilters $\mathcal{C}$ let $\mathfrak{ge}(\mathcal{C})$ denote the minimal cardinality of a filter base which cannot be extended to an ultrafilter from $\mathcal{C}$.

Obviously, ultrafilters from $\mathcal{C}$ exist generically if and only $\mathfrak{ge}(\mathcal{C}) = \mathfrak{c}$.

Some examples

Theorem (Ketonen).

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Theorem (Canjar). $\mathfrak{ge}(\text{selective ultfs}) = \mathrm{cov}(\mathcal{M})$

Theorem (Brendle). $\mathfrak{ge}(\text{nowhere dense ultfs}) = \mathrm{cof}(\mathcal{M})$

$\mathcal{I}$-ultrafilters

Definition. (Baumgartner)

Let $\mathcal{I}$ be a family of subsets of a set X such that $\mathcal{I}$ contains all singletons and is closed under subsets.

An ultrafilter $\mathcal{U}$ on ω is called an $\mathcal{I}$-ultrafilter if for every $f : \omega \to X$ there exists $U \in \mathcal{U}$ such that $f[U] \in \mathcal{I}$.

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Example. P-points are $\mathcal{I}$-ultrafilters in case of

- $X = 2^\omega$ and $\mathcal{I}$ are finite and converging sequences
- $X = \omega \times \omega$ and $\mathcal{I} = \mathrm{Fin} \times \mathrm{Fin}$

Generic existence of $\mathcal{I}$-ultrafilters

We write $\mathfrak{ge}(\mathcal{I})$ instead of $\mathfrak{ge}(\mathcal{I}\text{-ultrafilters})$.

Ketonen's result in this notation: $\mathfrak{ge}(\mathrm{Fin} \times \mathrm{Fin}) = \mathfrak{d}$.

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$\mathfrak{ge}(\mathcal{I})$ denotes the minimal cardinality of a filter base which cannot be extended to an $\mathcal{I}$-ultrafilter.

Lemma.

$\mathfrak{ge}(\mathcal{I}) =$
$\min\{|\mathcal{F}| : \mathcal{F} \text{ filter base}, \mathcal{F} \subseteq \mathcal{I}^+ \wedge (\forall I \in \mathcal{I})(\exists F \in \mathcal{F})|I \cap F| < \omega\}$

Cofinality of ideals

The cofinality of an ideal $\mathcal{I}$ on ω is defined as

$$\mathrm{cof}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I}, (\forall I \in \mathcal{I})(\exists A \in \mathcal{A})\ I \subseteq A\}$$

More generally, we define for $\mathcal{I} \subseteq \mathcal{J}$

$$\mathrm{cof}(\mathcal{I}, \mathcal{J}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{J} \text{ and } (\forall I \in \mathcal{I})(\exists J \in \mathcal{A}) I \subseteq J\}$$

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Lemma (Brendle).

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Uniformity of ideals

$$\mathrm{non}^*(\mathcal{I}) = \min\{|\mathcal{X}| : \mathcal{X} \subseteq [\omega]^\omega, (\forall I \in \mathcal{I})(\exists X \in \mathcal{X})\, |I \cap X| < \omega\}$$

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Lemma.

$$\mathrm{non}^*(\mathcal{I}) \leq \mathfrak{ge}(\mathcal{I}) \leq \mathrm{cof}(\mathcal{I})$$

$\mathcal{Z}$-ultrafilters

$$\mathcal{Z} = \{A \subseteq \mathbb{N} : \limsup_{n\to\infty} \frac{|A \cap n|}{n} = 0\}$$

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Fremlin proved $\mathrm{cof}(\mathcal{Z}) = \mathrm{cof}(\mathcal{N})$.

Theorem.

It is consistent with ZFC that $\mathfrak{ge}(\mathcal{Z}) < \mathrm{cof}(\mathcal{N})$.

$\mathcal{Z}$-ultrafilters

Theorem (Hernández-Hernández, Hrušák).

$$\min\{\mathfrak{d}, \mathrm{cov}(\mathcal{N})\} \leq \mathrm{non}^*(\mathcal{Z}) \leq \max\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$$

$\mathrm{cov}(\mathcal{M}) \leq \mathrm{non}^*(\mathcal{Z})$ holds in ZFC

$\mathfrak{d} \leq \mathrm{cof}(\mathcal{M}) < \mathrm{non}^*(\mathcal{Z})$ holds in dual Hechler model

$\mathrm{cof}(\mathcal{M}) > \mathrm{non}^*(\mathcal{Z})$ holds in random model

It is an open question whether $\mathfrak{d} \leq \mathrm{non}^*(\mathcal{Z})$ holds in ZFC.

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Theorem (Hernández-Hernández, Hrušák).

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Proposition. $\mathfrak{d} \leq \mathfrak{ge}(\mathcal{Z})$

$\mathcal{I}_{1/n}$-ultrafilters

$$\mathcal{I}_{1/n} = \{A \subseteq \mathbb{N} : \sum_{a \in A} \frac{1}{a} < \infty\}$$

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$\mathrm{non}^*(\mathcal{I}_{1/n}) \leq \mathfrak{ge}(\mathcal{I}_{1/n}) \leq \mathrm{cof}(\mathcal{I}_{1/n})$

$\mathfrak{ge}(\mathcal{I}_{1/n}) \leq \mathfrak{ge}(\mathcal{Z})$

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$\mathfrak{ge}(\mathcal{I}_{1/n}) \leq \mathfrak{ge}(\mathcal{Z})$

Theorem.

$$\mathrm{cov}(\mathcal{N}) \leq \mathfrak{ge}(\mathcal{I}_{1/n}) \leq \mathfrak{ge}(\mathcal{Z})$$

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