Ultrafilters and two ideals on ω

J. Flašková

Charles University, Faculty of Mathematics and Physics, Prague, Czech Republic.

Abstract. Two classes of ultrafilters on natural numbers are presented which are determined by the summable ideal and the density ideal. The relationship between these two classes is shown and also the relationship to the well-known class of P-points. Under some additional set-theoretic assumptions (Continuum Hypothesis or Martin's Axiom for countable posets) we construct several examples of these ultrafilters.

Introduction

The notion of $\mathscr{I}$-ultrafilter was introduced in *Baumgartner* [1995]: Let $\mathscr{I}$ be a family of subsets of a set X such that $\mathscr{I}$ contains all singletons and is closed under subsets. Given a free ultrafilter $\mathscr{U}$ on ω, we say that $\mathscr{U}$ is an *$\mathscr{I}$-ultrafilter* if for any $F : \omega \to X$ there is $A \in \mathscr{U}$ such that $F[A] \in \mathscr{I}$.

Baumgartner defined in his article discrete ultrafilters, scattered ultrafilters, measure zero ultrafilters and nowhere dense ultrafilters which he obtained by taking $X = 2^\omega$, the Cantor set, and $\mathscr{I}$ the collection of discrete sets, scattered sets, sets with closure of measure zero, nowhere dense sets respectively. If we let $\mathscr{I}$ be the collection of sets with countable closure then we obtain countably closed ultrafilters which were introduced in *Brendle* [1999]. Yet another class of $\mathscr{I}$-ultrafilters was introduced in *Barney* [2003] by taking $\mathscr{I}$ to be the sets with σ-compact closure. All these classes of ultrafilters are proved to be pairwise distinct under some additional set-theoretic assumptions (Continuum Hypothesis or some form of Martin's Axiom). It seems that some additional set-theoretic assumptions cannot be avoided completely when speaking about $\mathscr{I}$-ultrafilters because *Shelah* [1998] proved that it is consistent with ZFC that there are no nowhere dense ultrafilters, which implies that the existence of any of these ultrafilters (being a subclass of nowhere dense ultrafilters) is not provable in ZFC.

Another example of $\mathscr{I}$-ultrafilters are ordinal ultrafilters which were introduced also in *Baumgartner* [1995] by taking $X = \omega_1$ and $\mathscr{I} = \{A \subseteq \omega_1 : A \text{ has order type } \leq \alpha\}$ for some (indecomposable) ordinal α.

Although in general the ultrafilters defined for $X = 2^\omega$ and the ultrafilters defined for $X = \omega_1$ do not have much in common, P-points can be described in both ways for a suitable collection $\mathscr{I}$. If $X = 2^\omega$ then P-points are precisely the $\mathscr{I}$-ultrafilters for $\mathscr{I}$ consisting of all finite and converging sequences, if $X = \omega_1$ then P-points are precisely the $\mathscr{I}$-ultrafilters for $\mathscr{I} = \{A \subseteq \omega_1 : A \text{ has order type } \leq \omega\}$.

In this article we define two special classes of $\mathscr{I}$-ultrafilters for $X = \omega$ and $\mathscr{I}$ the summable and the density ideal on ω which we refer to as (S)-ultrafilters and (H)-ultrafilters. The double characterization of P-points in terms of $\mathscr{I}$-ultrafilters inspired our investigation of the relationship between P-points and the class of (S)-ultrafilters and (H)-ultrafilters.

Preliminaries and definitions

Given $A, B \subseteq \omega$ we write $A \subseteq^* B$ if $A \setminus B$ is finite and we use the symbol ${}^\omega\omega$ to denote the set of all functions from ω to ω.

Ultrafilters and ideals on ω

A free ultrafilter $\mathscr{U}$ is called a *P-point* if for all partitions of ω, $\{R_i : i \in \omega\}$, either for some i, $R_i \in \mathscr{U}$, or $(\exists U \in \mathscr{U})$ $(\forall i \in \omega)$ $|U \cap R_i| < \omega$. Equivalent combinatorial description is:

a free ultrafilter $\mathscr{U}$ is a P-point if and only if whenever $U_n \in \mathscr{U}$, $n \in \omega$, there is $U \in \mathscr{U}$ such that $U \subseteq^* U_n$ for each n.

A free ultrafilter $\mathscr{U}$ is called a *selective ultrafilter* if for all partitions of ω, $\{R_i : i \in \omega\}$, either for some i, $R_i \in \mathscr{U}$, or $(\exists U \in \mathscr{U})$ $(\forall i \in \omega)$ $|U \cap R_i| \leq 1$. In the proof of Proposition 6. we will profit from the following equivalent characterization of selective ultrafilters: if $\mathscr{U}$ is a selective ultrafilter on a countable set then for every $f \in {}^\omega\omega$ there is $U \in \mathscr{U}$ such that $f \restriction U$ is either one-to-one or constant (see *Comfort, Negrepontis* [1974]).

Family $\mathscr{I}$ from the definition of an $\mathscr{I}$-ultrafilter need not be an ideal in general. However, $\mathscr{I}$-ultrafilters and $\langle\mathscr{I}\rangle$-ultrafilters coincide, where $\langle\mathscr{I}\rangle$ is the ideal generated by $\mathscr{I}$. Therefore without loss of generality only ideals can be chosen as $\mathscr{I}$. Observe that if $\mathscr{I} \subseteq \mathscr{J}$ then any $\mathscr{I}$-ultrafilter is a $\mathscr{J}$-ultrafilter.

The summable and the density ideal

We say that $A \subseteq \omega$ is an (S)-*set* if $\sum_{a \in A} \frac{1}{a} < +\infty$ and we call A an (H)-*set* if $d^*(A) = \limsup_n \frac{|A \cap n|}{n} = 0$, i. e. A has asymptotic density zero.

Every (S)-set has asymptotic density zero. The set of all prime numbers is an example of an (H)-set that is not an (S)-set. It is known that (S)-sets and (H)-sets form an ideal on natural numbers. We will denote the corresponding ideal by (S) and (H) respectively. They are also known as the summable ideal and the density ideal (introduced in a more general setting in *Farah* where the author uses a different notation).

Notice that $A \subseteq^* B$ implies $d^*(A) \leq d^*(B)$.

(S)-ultrafilters and (H)-ultrafilters

The main goal of this section is to show that the existence of (S)-ultrafilters and (H)-ultrafilters is consistent with ZFC and these classes of ultrafilters do not coicide.

The existence of (S)-ultrafilters under Martin's Axiom follows from Proposition 6. in the following section where even more is proved. The proof, however, makes use of a former result by Booth concerning the existence of selective ultrafilters. Therefore we decided to present in this section a short direct proof that (S)-ultrafilters exist if Continuum Hypothesis holds.

Similarly, it follows from Proposition 4. and 5. in the following section that it is sufficient to assume Martin's Axiom for countable posets to construct an (H)-ultrafilter which is not (S)-ultrafilter. However, we construct in Proposition 3. such an ultrafilter assuming Continuum Hypothesis because the construction is straightforward and clear.

Proposition 1. *(CH) There is an* (S)*-ultrafilter.*

Proof. Enumerate ${}^\omega\omega = \{f_\alpha : \alpha < \omega_1\}$. By transfinite induction on $\alpha < \omega_1$ we construct countable filter bases $\mathscr{F}_\alpha$ satisfying

(i) $\mathscr{F}_0$ is the Fréchet filter
(ii) $\mathscr{F}_\alpha \subseteq \mathscr{F}_\beta$ whenever $\alpha \leq \beta$
(iii) $\mathscr{F}_\gamma = \bigcup_{\alpha<\gamma} \mathscr{F}_\alpha$ for γ limit
(iv) $(\forall\alpha)$ $(\exists F \in \mathscr{F}_{\alpha+1})$ $f_\alpha[F] \in (S)$

Suppose we have already constructed $\mathscr{F}_\alpha$. If there exists a set $F \in \mathscr{F}_\alpha$ such that $f_\alpha[F] \in (S)$ then put $\mathscr{F}_{\alpha+1} = \mathscr{F}_\alpha$. If $f_\alpha[F] \notin (S)$ (in particular, $f_\alpha[F]$ is infinite) for every $F \in \mathscr{F}_\alpha$ then enumerate $\mathscr{F}_\alpha = \{F_n : n \in \omega\}$ and construct by induction set $U = \{u_n : n \in \omega\}$ which we can add to the filter base:

Choose arbitrary $u_0 \in F_0$ such that $f_\alpha(u_0) > 0$ (such an element exists since $f_\alpha[F_0]$ is infinite). If $u_0, u_1, \ldots, u_{k-1}$ are already known we can choose $u_k \in \bigcap_{i \leq k} F_i$ so that $f_\alpha(u_k) > 2 \cdot f_\alpha(u_{k-1})$ (this is again possible because $f_\alpha[\bigcap_{i \leq k} F_i]$ is infinite).

For all $n \in \omega$ we have $U \subseteq^* F_n$ and it is easy to check that $f_\alpha[U] \in (S)$:

$$\sum_{a \in f_\alpha[U]} \frac{1}{a} = \sum_{k \in \omega} \frac{1}{f_\alpha(u_k)} \leq \frac{1}{f_\alpha(u_0)} \cdot \sum_{k \in \omega} \frac{1}{2^k} \leq 2 \cdot \frac{1}{f_\alpha(u_0)} < +\infty$$

To complete the induction step let $\mathscr{F}_{\alpha+1}$ be the countable filter base generated by $\mathscr{F}_\alpha$ and the set U.

It is obvious that any ultrafilter which extends the filter base $\mathscr{F} = \bigcup_{\alpha<\omega_1} \mathscr{F}_\alpha$ is an (S)-ultrafilter because of condition (iv). □

Since $(S) \subseteq (H)$ we get immediately from the definition of $\mathscr{I}$-ultrafilter the following

Fact 2. *Every (S)-ultrafilter is an (H)-ultrafilter.*

Proposition 3. *(CH) There is an (H)-ultrafilter which is not (S)-ultrafilter.*

Proof. Enumerate ${}^\omega\omega = \{f_\alpha : \alpha < \omega_1\}$. By transfinite induction on $\alpha < \omega_1$ we construct countable filter bases $\mathscr{F}_\alpha$ satisfying

(i) $\mathscr{F}_0$ is the Fréchet filter
(ii) $\mathscr{F}_\alpha \subseteq \mathscr{F}_\beta$ whenever $\alpha \leq \beta$
(iii) $\mathscr{F}_\gamma = \bigcup_{\alpha<\gamma} \mathscr{F}_\alpha$ for γ limit
(iv) $(\forall\alpha)$ $(\forall F \in \mathscr{F}_\alpha)$ $\sum_{a \in F} \frac{1}{a} = +\infty$, i. e. $F \notin (S)$
(v) $(\forall\alpha)$ $(\exists F \in \mathscr{F}_{\alpha+1})$ $f_\alpha[F] \in (H)$

Suppose we have already constructed $\mathscr{F}_\alpha$. If there exists a set $F \in \mathscr{F}_\alpha$ such that $f_\alpha[F] \in (H)$ then put $\mathscr{F}_{\alpha+1} = \mathscr{F}_\alpha$. If $f_\alpha[F] \notin (H)$ (in particular, $f_\alpha[F] \notin (S)$) for every $F \in \mathscr{F}_\alpha$ then enumerate $\mathscr{F}_\alpha = \{F_n : n \geq 1\}$. By induction we will construct an increasing sequence of natural numbers $\langle a_k \rangle_{k \in \mathbb{N}}$ which will help us to construct a set U which we can add to the filter base:

Put $a_0 = 0$. Since $F_1 \notin (S)$ we may choose $a_1 > a_0$ such that $\sum_{a \in F_1 \cap f_\alpha^{-1}[a_0, a_1)} \frac{1}{a} > 1$. Assume we already know a_i for $i = 1, 2, \ldots, k-1$. Since $\bigcap_{i \leq k} F_i \notin (S)$ and $f[\bigcap_{i \leq k} F_i] \notin (S)$ the set $\bigcap_{i \leq k} F_i \setminus f_\alpha^{-1}[0, a_{k-1})$ does not belong to (S). Hence we can choose $a_k \in \omega$ such that $a_k \geq a_{k-1} + k^3$ and

$$\sum_{a \in \bigcap_{i \leq k} F_i \cap f_\alpha^{-1}[a_{k-1}, a_k)} \frac{1}{a} > k \ .$$

Now we decompose the kth interval $[a_{k-1}, a_k)$ into k disjoint sets modulo k and denote $E_i^{(k)} = \{n \in [a_{k-1}, a_k) : (\exists d \in \omega)\ n = d \cdot k + i\}$ for $i = 0, 1, \ldots, k-1$. We get

$$\sum_{a \in \bigcap_{j \leq k} F_j \cap f_\alpha^{-1}[E_i^{(k)}]} \frac{1}{a} > 1 \quad \text{for some } i < k \ .$$

Choose such a set $E_i^{(k)}$ and denote it by A_k. Let $U = \bigcup_{k=1}^\infty \left(f_\alpha^{-1}[A_k] \cap \bigcap_{j \leq k} F_j \right)$. It remains to check that

- $(\forall F \in \mathscr{F}_\alpha)$ $U \cap F \notin (S)$

For every $F \in \mathscr{F}_\alpha$ we have $U \cap F \supseteq \bigcup_{k=j}^\infty \left(f_\alpha^{-1}[A_k] \cap \bigcap_{i \leq k} F_i \right)$ for some $j \geq 1$. It follows that $\sum_{a \in U \cap F} \frac{1}{a} \geq \sum_{k=j}^\infty 1 = +\infty$.

- $f_\alpha[U] \in (H)$

Since $f_\alpha[U] \subseteq \bigcup_{k=1}^\infty f_\alpha[f_\alpha^{-1}[A_k]] = \bigcup_{k=1}^\infty A_k$ it is sufficient to show that the set $A = \bigcup_{k=1}^\infty A_k$ has asymptotic density zero. The latter follows from the fact that $A \subseteq^* A^n$ and $d^*(A^n) < \frac{2}{n}$ for every n where $A^n = \bigcup_{k=n}^\infty A_k$.

To complete the induction step let $\mathscr{F}_{\alpha+1}$ be the countable filter base generated by $\mathscr{F}_\alpha$ and the set U.

Because of condition (iv) the filter base $\mathscr{F} = \bigcup_{\alpha<\omega_1} \mathscr{F}_\alpha$ can be extended into an ultrafilter disjoint from (S) which cannot be an (S)-ultrafilter and it is obvious that any ultrafilter extending $\mathscr{F}$ is an (H)-ultrafilter because of condition (v). □

Connections to *P*-points

We investigate the relationship of (S)-ultrafilters and (H)-ultrafilters to P-points in this section. We prove that any P-point is an (H)-ultrafilter and that there is no inclusion between the class of P-points and (S)-ultrafilters. Assuming Martin's Axiom for countable posets we construct a P-point which is not an (S)-ultrafilter and an (S)-ultrafilter which is not a P-point.

Proposition 4. *Every P-point is an (H)-ultrafilter.*

Proof. Let $\mathscr{U}$ be a P-point and let $f \in {}^\omega\omega$ be an arbitrary function. Our aim is to find $U \in \mathscr{U}$ such that $f[U] \in (H)$.

Take arbitrary $U_0 \in \mathscr{U}$. If $f[U_0] \in (H)$ then set $U = U_0$. Otherwise, we will proceed by induction. Suppose we already know $U_i \in \mathscr{U}$, $i = 0, 1, \dots, k-1$, such that $U_i \subseteq U_{i-1}$ for $i > 0$ and $0 < d^*(f[U_i]) \leq \frac{1}{2^i} \cdot d^*(f[U_0])$ for every $i < k$. Enumerate $f[U_{k-1}] = \{u_n^{k-1} : n \in \omega\}$. Since $\mathscr{U}$ is an ultrafilter either $f^{-1}[\{u_{2n}^{k-1} : n \in \omega\}] \cap U_{k-1}$ or $f^{-1}[\{u_{2n+1}^{k-1} : n \in \omega\}] \cap U_{k-1}$ belongs to $\mathscr{U}$. Denote this set by U_k. If $d^*(f[U_k]) = 0$ then let $U = U_k$. If $f[U_k] \notin (H)$ then we may continue the induction because $0 < d^*(f[U_k]) \leq \frac{1}{2} \cdot d^*(f[U_{k-1}]) \leq \frac{1}{2^k} \cdot d^*(f[U_0])$.

If we obtain an infinite sequence of sets $U_n \in \mathscr{U}$ such that $U_n \supseteq U_{n+1}$ and $0 < d^*(f[U_n]) \leq \frac{1}{2^n} \cdot d^*(f[U_0])$ for every $n \in \omega$ then since $\mathscr{U}$ is a P-point there is $U \in \mathscr{U}$ such that $U \subseteq^* U_n$ for every $n \in \omega$. For this set we have $f[U] \subseteq^* f[U_n]$ for every $n \in \omega$. It follows from $\lim_{n\to\infty} d^*(f[U_n]) = 0$ that $d^*(f[U]) = 0$, i. e. $f[U] \in (H)$. □

Proposition 5. *(MA_{ctble}) There is a P-point which is not an (S)-ultrafilter.*

Proof. Enumerate all infinite partitions of ω as $\{\mathscr{R}_\alpha : \alpha < \mathfrak{c}\}$. By transfinite induction on $\alpha < \mathfrak{c}$ we will construct filter bases $\mathscr{F}_\alpha$, $\alpha < \mathfrak{c}$, so that the following conditions are satisfied:

(i) $\mathscr{F}_0$ is the Fréchet filter
(ii) $\mathscr{F}_\alpha \subseteq \mathscr{F}_\beta$ whenever $\alpha \leq \beta$
(iii) $\mathscr{F}_\gamma = \bigcup_{\alpha<\gamma} \mathscr{F}_\alpha$ for γ limit
(iv) $(\forall\alpha)$ $|\mathscr{F}_\alpha| \leq |\alpha| \cdot \omega$
(v) $(\forall\alpha)$ $(\forall F \in \mathscr{F}_\alpha)$ $\sum_{a\in F} \frac{1}{a} = +\infty$
(vi) $(\forall\alpha)$ $(\exists F \in \mathscr{F}_{\alpha+1})$ such that either $(\exists R_n^\alpha \in \mathscr{R}_\alpha)$ $F \subseteq R_n^\alpha$ or $(\forall R_n^\alpha \in \mathscr{R}_\alpha)$ $|F \cap R_n^\alpha| < \omega$

Induction step: Suppose we already know $\mathscr{F}_\alpha$ and we want to construct $\mathscr{F}_{\alpha+1}$.

Case A. $(\exists K \in [\omega]^{<\omega})$ $(\forall F \in \mathscr{F}_\alpha)$ $F \cap \bigcup_{n\in K} R_n^\alpha \notin (S)$

For some $n_0 \in K$ the filter base generated by $R_{n_0}^\alpha$ and $\mathscr{F}_\alpha$ does not meet (S). Hence $\mathscr{F}_{\alpha+1}$ may be obtained as the filter base generated by $\mathscr{F}_\alpha$ and the set $R_{n_0}^\alpha$. Because otherwise, there would be for every $n \in K$ a set $F_n \in \mathscr{F}_\alpha$ such that $F_n \cap R_n^\alpha \in (S)$ and we would have $\bigcap_{n\in K} F_n \cap \bigcup_{n\in K} R_n^\alpha \in (S)$ – a contradiction to the assumption of *Case A*.

Case B. $(\forall K \in [\omega]^{<\omega})$ $(\exists F_K \in \mathscr{F}_\alpha)$ $F_K \cap \bigcup_{n\in K} R_n^\alpha \in (S)$

We consider a countable poset $P = \{(K, n) \in [\omega]^{<\omega} \times \omega : K \subseteq \bigcup_{i\leq n} R_i^\alpha,\ K \cap R_n^\alpha \neq \emptyset\}$ with ordering given by $(K, n) <_P (L, m)$ if $K \supset L$, $\min(K \setminus L) > \max L$, $n > m$ and $(K \setminus L) \cap \bigcup_{i\leq m} R_i^\alpha = \emptyset$. Enumerate $\mathscr{F}_\alpha = \{F_\lambda : \lambda \in \Lambda_\alpha\}$ and define $D_{\lambda,k} = \{(K, n) \in P : \sum_{a\in K\cap F_\lambda} \frac{1}{a} > k\}$ and $D_j = \{(K, n) \in P : n \geq j\}$.

Claim: $D_{\lambda,k}$ is dense in P for every $\lambda \in \Lambda_\alpha$ and $k \in \omega$; D_j is dense in P for every $j \in \omega$.

Take $(L, m) \in P$ arbitrary. According to the assumptions there is $F_m \in \mathscr{F}_\alpha$ such that $F_m \cap \bigcup_{i\leq m} R_i^\alpha \in (S)$. It follows that $(F_m \cap F_\lambda) \setminus (\bigcup_{i\leq m} R_i^\alpha \cup [0, \max L]) \notin (S)$. Therefore we can choose a finite set $L' \subseteq (F_m \cap F_\lambda) \setminus (\bigcup_{i\leq m} R_i^\alpha \cup [0, \max L])$ such that $\sum_{a\in L'} \frac{1}{a} > k$. Let $n = \max\{i : L' \cap R_i^\alpha \neq \emptyset\}$ and $K = L \cup L'$. Obviously, $(K, n) <_P (L, m)$ and $(K, n) \in D_{\lambda,k}$. So $D_{\lambda,k}$ is dense. For any $j > m$ we can choose arbitrary $r \in \omega \setminus (\bigcup_{i<j} R_i^\alpha \cup [0, \max L])$. Let $K' = L \cup \{r\}$. If $r \in R_l^\alpha$ then $(K', l) <_P (L, m)$ and $(K', l) \in D_j$. So D_j is dense.

The family $\mathscr{D} = \{D_{\lambda,k} : \lambda \in \Lambda_\alpha, k \in \omega\} \cup \{D_j : j \in \omega\}$ consists of dense sets in P and $|\mathscr{D}| < \mathfrak{c}$. Since we assume $\mathrm{MA}_{\mathrm{ctble}}$ there is a $\mathscr{D}$-generic filter $\mathcal{G}$.

Let $U = \bigcup\{K : (\exists n)(K, n) \in \mathcal{G}\}$. It remains to check that:

• $(\forall F_\lambda \in \mathscr{F}_\alpha) \sum_{a\in U\cap F_\lambda} \frac{1}{a} = +\infty$

We have $U \cap F_\lambda \notin (S)$ for every λ because for every $k \in \omega$ there exists $(K, n) \in \mathcal{G} \cap D_{\lambda,k}$ and we get $\sum_{a\in U\cap F_\lambda} \frac{1}{a} \geq \sum_{a\in K\cap F_\lambda} \frac{1}{a} > k$.

• $(\forall R_n^\alpha \in \mathscr{R}_\alpha)\ |U \cap R_n^\alpha| < \omega$

Take $(K_n, j_n) \in \mathcal{G} \cap D_n$ where $j_n = \min\{j : (\exists K \in [\omega]^{<\omega})(K, j) \in \mathcal{G} \cap D_n\}$. Now observe that for $(K, m) \in \mathcal{G}$ we have $K \cap R_n^\alpha = \emptyset$ if $m < n$ and that $K \cap R_n^\alpha = K_n \cap R_n^\alpha$ if $m \geq j_n$. To see the latter consider $(L, m') \in \mathcal{G}$ such that $(L, m') <_P (K, m)$ and $(L, m') <_P (K_n, j_n)$ (such a condition exists because $\mathcal{G}$ is a filter) for which we get $L \cap R_n^\alpha = K \cap R_n^\alpha$ and $L \cap R_n^\alpha = K_n \cap R_n^\alpha$. It follows that $U \cap R_n^\alpha = K_n \cap R_n^\alpha$ is finite.

To complete the induction step let $\mathscr{F}_{\alpha+1}$ be the filter base generated by $\mathscr{F}_\alpha$ and U.

It is obvious that every ultrafilter which extends $\mathscr{F} = \bigcup_{\alpha<\mathfrak{c}} \mathscr{F}_\alpha$ is a P-point. Because of condition (v) there exists an ultrafilter extending $\mathscr{F}$ which extends the dual filter of (S) (therefore is not an (S)-ultrafilter). □

Proposition 6. *(MA_{ctble}) There is an (S)-ultrafilter which is not a P-point.*

Proof. It is known that under $\mathrm{MA}_{\mathrm{ctble}}$ selective ultrafilters exist (see *Booth* [1970]). We will show that the square of a selective ultrafilter is an (S)-ultrafilter which is not a P-point.

Let us recall the definition of product of ultrafilters (see e. g. *Comfort, Negrepontis* [1974]): If $\mathscr{U}$ and $\mathscr{V}$ are ultrafilters on ω then $\mathscr{U} \cdot \mathscr{V} = \{A \subseteq \omega \times \omega : \{n : \{m : \langle n, m\rangle \in A\} \in \mathscr{V}\} \in \mathscr{U}\}$ is an ultrafilter on $\omega \times \omega$. By the square of ultrafilter $\mathscr{U}$ we mean the ultrafilter $\mathscr{U} \cdot \mathscr{U}$.

Notice that the partition $\{\{n\} \times \omega : n \in \omega\}$ of $\omega \times \omega$ witnesses the fact that no product of free ultrafilters on ω is a P-point. Hence to complete the proof of Proposition 6. it is sufficient to check that if $\mathscr{U}$ is a selective ultrafilter on ω then $\mathscr{U} \cdot \mathscr{U}$ is an (S)-ultrafilter, i. e. for every $f \in {}^{\omega\times\omega}\omega$ there is $U \in \mathscr{U} \cdot \mathscr{U}$ such that $f[U] \in (S)$.

Let $\mathscr{U}$ be a selective ultrafilter on ω and let $f : \omega \times \omega \to \omega$ be an arbitrary function. For every $n \in \omega$ we define $f_n : \omega \to \omega$ by $f_n(m) = f(\langle n, m\rangle)$. Since $\mathscr{U}$ is selective there is $U_n \in \mathscr{U}$ for every $n \in \omega$ such that $f_n \restriction U_n$ is either one-to-one or constant. Let $I = \{n : f_n \restriction U_n$ is constant$\}$ and $J = \{n : f_n \restriction U_n$ is one-to-one$\}$.

Case A. If $I \in \mathscr{U}$ then $V = \bigcup\{\{n\} \times U_n : n \in I\} \in \mathscr{U} \cdot \mathscr{U}$ and $f[V] = \bigcup\{f_n[U_n] : n \in \omega\}$. If $f[V] \in (S)$ then V is the required set in $\mathscr{U} \cdot \mathscr{U}$. If $f[V] \notin (S)$ then consider $\pi_1 : \omega \to \omega$ defined by $\{\pi_1(n)\} = f_n[U_n]$ for every n. There exists $V_0 \in \mathscr{U}$ such that $|\pi_1[V_0] \cap [2^n, 2^{n+1})| \leq 1$ for every $n \in \omega$ (such a set exists since $\pi_1(\mathscr{U})$ is a selective ultrafilter). Let $U = \bigcup\{\{n\} \times U_n : n \in I \cap V_0\}$. Obviously, $U \in \mathscr{U} \cdot \mathscr{U}$ and it is not difficult to check that

$$\sum_{u\in f[U]} \frac{1}{u} \leq \sum_{u\in\pi_1[V_0]} \frac{1}{u} < +\infty$$

Case B. If $J \in \mathscr{U}$ then for every $n \in J$ there is $V_n \in \mathscr{U}$ such that $f_n[V_n] \cap [0, 2^{n+1}) = \emptyset$ and $|f_n[V_n] \cap [2^k, 2^{k+1})| \leq 1$ for every $k > n$. It follows that $\sum_{u\in f_n[V_n]} \frac{1}{u} \leq \frac{1}{2^n}$. Let $U = \bigcup\{\{n\} \times (U_n \cap V_n) : n \in J\}$. It is obvious that $U \in \mathscr{U} \cdot \mathscr{U}$ and it is easy to verify that

$$\sum_{u\in f[U]} \frac{1}{u} \leq \sum_{n\in\omega} \sum_{u\in f_n[U_n\cap V_n]} \frac{1}{u} \leq \sum_{n\in\omega} \frac{1}{2^n} < +\infty$$

□

Corollary 7. *(MA_{ctble}) There is an (H)-ultrafilter which is not a P-point.*

If Martin's Axiom for countable posets holds then there are both (S)-ultrafilters and (H)-ultrafilters. We constructed examples which are not P-points and might therefore exist also in models where no P-points exist. However, we were not able to construct even (H)-ultrafilters in ZFC, so the folowing question remains open.

Open problem 8. Do (H)-ultrafilters (and (S)-ultrafilters) exist in ZFC?

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