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IDEALS AND SEQUENTIALLY COMPACT SPACES

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ABSTRACT. We say that a topological space X is an $\mathcal{I}_{1/n}$ -space if for every sequence $\langle x_n \rangle_{n \in \mathbb{N}}$ in X there exists a converging subsequence $\langle x_{n_k} \rangle_{k \in \mathbb{N}}$ with $\sum_{k \in \omega} \frac{1}{n_k} = \infty$. Every $\mathcal{I}_{1/n}$ -space is sequentially compact, but not every sequentially compact space is $\mathcal{I}_{1/n}$ -space.

Assuming Martin's axiom for σ -centered posets we construct a van der Waerden space that is not an $\mathcal{I}_{1/n}$ -space and an $\mathcal{I}_{1/n}$ -space that is not Hindman.

1. INTRODUCTION

A Hausdorff topological space X is *sequentially compact* if for every sequence $\langle x_n \rangle_{n \in \mathbb{N}}$ in X there exists a converging subsequence $\langle x_{n_k} \rangle_{k \in \mathbb{N}}$. Is it possible to require that the converging subsequence has some additional properties? One could for example require that the set of indices of the converging subsequence is large with respect to some ideal on ω . In particular, if we consider the van der Waerden ideal consisting of subsets of natural numbers which are not AP-sets (a set of natural numbers is called an AP-set if it contains arithmetic progressions of arbitrary length) then we obtain the following notion introduced by Kojman in [2]:

Definition 1.1. A topological space X is called *van der Waerden* if for every sequence $\langle x_n \rangle_{n \in \mathbb{N}}$ in X there exists a converging subsequence $\langle x_{n_k} \rangle_{k \in \mathbb{N}}$ so that $\{n_k : k \in \mathbb{N}\}$ is an AP-set.

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Every van der Waerden space is according to the definition sequentially compact and Kojman proved in [2] that the converse is not true.

Before we give a definition of another subclass of sequentially compact spaces let us recall that a set $A \subseteq \mathbb{N}$ is an *IP-set* if there exists an infinite set $B \subseteq \mathbb{N}$ such that $FS(B) \subseteq A$ where $FS(B)$ denotes the set of all finite sums of elements of B . A sequence $\langle x_n \rangle_{n \in FS(B)}$ in a topological space X *IP-converges* to a point $x \in X$ if for every neighborhood U of the point x there exists $m \in \mathbb{N}$ so that $\{x_n : n \in FS(D \setminus m)\} \subseteq U$.

Definition 1.2. A topological space X is called *Hindman* if for every sequence $\langle x_n \rangle_{n \in \mathbb{N}}$ in X there exists an infinite set $D \subseteq \mathbb{N}$ such that $\langle x_n \rangle_{n \in FS(D)}$ IP-converges to some $x \in X$.

Again, every Hindman space is from definition sequentially compact and in [3] Kojman proved that there exists a sequentially compact space which is not a Hindman space.

Kojman proved that compact metric spaces are both van der Waerden and Hindman space and he raised the question whether the two subclasses of sequentially compact spaces coincide [3]. In [4], however, he constructed together with Shelah assuming CH a van der Waerden space which is not Hindman. Jones [1] obtained the same result assuming MA and he posed the question whether it is consistent that there is a Hindman space which is not van der Waerden.

The main goal of this article is to strengthen the result of Jones and give a partial answer to his question. In order to do so we will generalize the definition of a van der Waerden space by replacing the van der Waerden ideal by another suitable (possibly smaller) ideal on natural numbers.

1.1. Ideals on natural numbers. Let us recall that a family $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ is an *ideal* on ω if it 1) contains the empty set, 2) with every set contains all its subsets and 3) contains the union of any two sets from \mathcal{I} .

According to van der Waerden Theorem sets which are not AP-sets form an ideal which we refer to as van der Waerden ideal. Hindman Theorem implies that sets which are not IP-sets form an ideal called here Hindman ideal. Another example of an ideal is the summable ideal $\mathcal{I}_{1/n} = \{A \subset \mathbb{N} : \sum_{a \in A} \frac{1}{a} < \infty\}$.

These three ideals differ in many ways. For example, the summable ideal is (unlike the other two) a P -ideal, i.e. whenever A_n , $n \in \mathbb{N}$, are sets from the ideal there exists A in the ideal such that $|A_n \setminus A| < \omega$ for every n .

If we identify a subset of \mathbb{N} with its characteristic function we may regard ideals as subsets of the Cantor set $2^{\mathbb{N}}$. Hence Borel hierarchy on the Cantor set induces Borel hierarchy on ideals and so we may speak about Borel ideals or analytic ideals.

In the sequel the following characterization of F_σ -ideals due to Mazur [5] will be important: For every F_σ -ideal \mathcal{I} there is a lower semicontinuous submeasure $\varphi : \mathcal{P}(\mathbb{N}) \rightarrow [0, \infty]$ so that $\mathcal{I} = \text{Fin}(\varphi) = \{A \subseteq \mathbb{N} : \varphi(A) < \infty\}$. Remember that a submeasure φ is called *lower semicontinuous* (*lsc* in short) if $\varphi(A) = \lim_{n \rightarrow \infty} \varphi(A \cap n)$.

Both van der Waerden and summable ideals $\mathcal{I}_{1/n}$ are F_σ -ideals, whereas the Hindman ideal is of higher Borel complexity.

1.2. Ψ -spaces. A Hausdorff, compact, sequentially compact and separable space which is first-countable at all points but one, which is not van der Waerden (resp. Hindman), was constructed in [2] and [3] respectively. Let us recall here the definition of a Ψ -space which is crucial for both examples and also for examples in this paper.

Given a maximal almost disjoint (MAD) family \mathcal{A} of infinite subsets of \mathbb{N} we define the space $\Psi(\mathcal{A})$ as follows: $\mathbb{N} \cup \{p_A : A \in \mathcal{A}\}$ is the underlying set. Every point in \mathbb{N} is isolated and every point p_A has neighborhood base of sets $\{p_A\} \cup A \setminus F$ where F is a finite subset of A . The space $\Psi(\mathcal{A})$ is regular, first countable and separable.

2. \mathcal{I} -SPACES

We may define a new subclass of sequentially compact spaces if we replace van der Waerden ideal in Definition 1.1 by an arbitrary ideal \mathcal{I} on ω . Unfortunately, it may happen that the corresponding class of topological spaces contains only finite spaces (see [3] for Hindman ideal).

It turns out that infinite \mathcal{I} -spaces according to the definition exist for every F_σ -ideal \mathcal{I} that contains all finite sets. In particular, we obtain the following definition for the summable ideal $\mathcal{I}_{1/n}$.

Definition 2.1. A topological space X is called an $\mathcal{I}_{1/n}$ -space if for every sequence $\langle x_n \rangle_{n \in \mathbb{N}}$ in X there exists a converging subsequence

$\langle x_{n_k} \rangle_{k \in \mathbb{N}}$ with $\sum_{k \in \mathbb{N}} \frac{1}{n_k} = \infty$, i.e. $\{n_k : k \in \mathbb{N}\}$ is not in the summable ideal $\mathcal{I}_{1/n}$.

As van der Waerden or Hindman spaces, $\mathcal{I}_{1/n}$ -spaces form a proper subclass of sequentially compact spaces.

Proposition 2.2. *There exists a Hausdorff, compact, sequentially compact, separable space which is first-countable at all points but one, which is not $\mathcal{I}_{1/n}$ -space.*

Proof. Consider the one-point compactification of the space $\Psi(\mathcal{A})$ where \mathcal{A} is a maximal almost disjoint family consisting exclusively of sets from $\mathcal{I}_{1/n}$. \square

Notice that Proposition 2.2 remains true also if $\mathcal{I}_{1/n}$ is replaced by an arbitrary F_σ -ideal.

Kojman formulated in [2] a sufficient condition on a Hausdorff space X to be van der Waerden (and Hindman):

(*) The closure of every countable set in X is compact and first-countable.

It turns out that it is also sufficient for X to be $\mathcal{I}_{1/n}$ -space (or \mathcal{I} -space for arbitrary F_σ -ideal \mathcal{I}). We give here only the proof for $\mathcal{I}_{1/n}$ -spaces, the more general case of arbitrary F_σ -ideal is left to the reader since it is a straightforward modification of the presented proof (using Mazur's characterization of F_σ -ideals).

Theorem 2.3. *If a Hausdorff space X satisfies the following condition*

(*) *The closure of every countable set in X is compact and first-countable.*

then X is an $\mathcal{I}_{1/n}$ -space.

Proof. Suppose that $\langle x_n \rangle_{n \in \mathbb{N}} \subseteq X$ is given. Let \mathcal{U} be an ultrafilter which extends the dual filter of $\mathcal{I}_{1/n}$. Let $D = \text{cl}_X \{x_n : n \in \mathbb{N}\}$ and define an ultrafilter \mathcal{V} over D by $A \in \mathcal{V}$ if and only if $\{n : x_n \in A\} \in \mathcal{U}$. Since D is compact (and Hausdorff), there is a unique point $x \in D$ such that $x \in \bigcap \mathcal{V}$ (every neighborhood of x belongs to \mathcal{V}). There is also a decreasing neighborhood base $\langle U_k : k \in \mathbb{N} \rangle$ at x because D is first-countable. Notice that $\{n : x_n \in U_k\} \in \mathcal{U}$ for every $k \in \mathbb{N}$ and remember that \mathcal{U} consists of sets not in $\mathcal{I}_{1/n}$. So $\sum_{x_n \in U_k} \frac{1}{n} = \infty$. We can choose finite sets $C_k \subseteq \{n : x_n \in U_k\}$

so that $\sum_{n \in C_k} \frac{1}{n} \geq k$. It is obvious that $B = \bigcup_{k \in \mathbb{N}} C_k$ is not in $\mathcal{I}_{1/n}$ and $\langle x_n \rangle_{n \in B}$ converges to x . \square

3. VAN DER WAERDEN SPACES NEED NOT BE $\mathcal{I}_{1/n}$ -SPACES

Erdős-Turán conjecture states that every set $A \notin \mathcal{I}_{1/n}$ is an AP-set. Hence if Erdős-Turán conjecture is true then every $\mathcal{I}_{1/n}$ -space is van der Waerden. In this section we prove that it is consistent with ZFC that the class of van der Waerden spaces is strictly larger than the class of $\mathcal{I}_{1/n}$ -spaces. To do so we will need a rather technical Lemma 3.1 and a MAD family with special properties which is constructed in Proposition 3.2 assuming Martin's axiom for σ -centered posets.

Lemma 3.1. *Let $A \subseteq \mathbb{N}$ be an AP-set, i.e. A contains arithmetic progressions of arbitrary length, and let $f : \mathbb{N} \rightarrow \mathbb{N}$. There exists an AP-set $C \subseteq A$ such that*

- (1) *either f is constant on C*
- (2) *or f is finite-to-one on C and $f[C] \in \mathcal{I}_{1/n}$.*

Proof. If there exists a finite set $M \subseteq \mathbb{N}$ such that $f^{-1}[M] \cap A$ is an AP set then $f^{-1}(m) \cap A$ is an AP set for some $m \in M$ (because of van der Waerden Theorem) and conclusion (1) holds.

If $A \cap f^{-1}[M]$ is not AP for every finite set $M \subseteq \mathbb{N}$ then we construct by induction an AP set $C \subseteq A$ for which conclusion (2) holds.

We construct for every $n \in \mathbb{N}$ (finite) sets $C_n \subseteq A$ such that

- i) $C_1 = \{\min A\}$
- ii) $(\forall n \in \mathbb{N}) C_n$ is an arithmetic progression of length n .
- iii) $(\forall n \in \mathbb{N}) \min f[C_{n+1}] > 2^{n+1} \cdot \max f[C_n]$

Assume we have already defined C_i as required for $i = 1, 2, \dots, n$. Since $A \setminus f^{-1}[0, 2^{n+1} \cdot \max f[C_n]]$ is an AP set it contains an arithmetic progression of length $n+1$. Define this subset of A as C_{n+1} . It is easy to see that properties ii) and iii) are satisfied.

Finally, let $C = \bigcup_{n \in \mathbb{N}} C_n$. The set $C \subseteq A$ is obviously an AP-set and it follows from the construction that the function f is finite-to-one on C . So it remains to check that $f[C] \in \mathcal{I}_{1/n}$.

To this end notice that

$$\sum_{c \in f[C_{n+1}]} \frac{1}{c} \leq \frac{|C_{n+1}|}{2^{n+1} \cdot \max f[C_n]} \leq \frac{n+1}{2^{n+1}}.$$

Hence

$$\sum_{c \in f[C]} \frac{1}{c} \leq \sum_{n=0}^{\infty} \sum_{c \in f[C_{n+1}]} \frac{1}{c} \leq \sum_{n=0}^{\infty} \frac{n+1}{2^{n+1}} < \infty$$

and $f[C] \in \mathcal{I}_{1/n}$. \square

Proposition 3.2. (*MA $_{\sigma}$ -centered*) *There exists a maximal almost disjoint family $\mathcal{A} \subseteq \mathcal{I}_{1/n}$ so that for every AP-set $B \subseteq \mathbb{N}$ and every finite-to-one function $f : B \rightarrow \mathbb{N}$ there exists an AP-set $C \subseteq B$ and $A \in \mathcal{A}$ so that $f[C] \subseteq A$.*

Proof. Let us enumerate as $\{\langle f_{\alpha}, B_{\alpha} \rangle : \omega \leq \alpha \leq 2^{\omega}\}$ all pairs $\langle f, B \rangle$ where $B \subseteq \mathbb{N}$ is an AP-set and $f : B \rightarrow \mathbb{N}$ a finite-to-one function. By transfinite induction on $\alpha < 2^{\omega}$ we construct almost disjoint families $\mathcal{A}_{\alpha} \subseteq \mathcal{I}_{1/n}$, $\omega \leq \alpha < 2^{\omega}$, so that the following conditions are satisfied:

- (i) $\mathcal{A}_{\omega} = \{A_n : n \in \mathbb{N}\} \subseteq \mathcal{I}_{1/n}$ is a partition of \mathbb{N}
 - (ii) $\mathcal{A}_{\alpha} \subseteq \mathcal{A}_{\beta}$ whenever $\alpha \leq \beta$
 - (iii) $\mathcal{A}_{\gamma} = \bigcup_{\alpha < \gamma} \mathcal{A}_{\alpha}$ for γ limit
 - (iv) $(\forall \alpha) |\mathcal{A}_{\alpha}| \leq |\alpha|$
 - (v) $(\forall \alpha) (\exists A \in \mathcal{A}_{\alpha+1}) (\exists C \subseteq B_{\alpha}) C$ is an AP-set and $f_{\alpha}[C] \subseteq A$
- Suppose we already know \mathcal{A}_{α} .

Case A. $(\exists A \in \mathcal{A}_{\alpha}) f_{\alpha}^{-1}[A]$ is an AP-set

Let $C = f_{\alpha}^{-1}[A]$ and $\mathcal{A}_{\alpha+1} = \mathcal{A}_{\alpha}$.

Case B. $(\forall A \in \mathcal{A}_{\alpha}) f_{\alpha}^{-1}[A]$ is not an AP-set

Enumerate \mathcal{A}_{α} as $\{A_{\delta} : \delta \in I_{\alpha}\}$ where I_{α} is an index set of size less or equal to $|\alpha|$. Let $P = [B_{\alpha}]^{<\omega} \times [I_{\alpha}]^{<\omega}$ and define order \leq_P by $\langle K, D \rangle \leq_P \langle L, E \rangle$ if $K \supseteq L$ and $D \supseteq E$ and $K \setminus L \subseteq B_{\alpha} \setminus f_{\alpha}^{-1}[\bigcup_{\varepsilon \in E} A_{\varepsilon}]$. It is easy to see that (P, \leq_P) is a σ -centered poset. For $k \in \omega$ and $\delta \in I_{\alpha}$ define $D_k = \{\langle K, D \rangle \in P : K \text{ contains arithmetic progression of length } k\}$ and $D_{\delta} = \{\langle K, D \rangle \in P : \delta \in D\}$.

Claim 1: D_k is dense in (P, \leq_P) for every $k \in \omega$.

Proof of Claim 1. Consider $\langle L, E \rangle \in P$ arbitrary. If L contains arithmetic progression of length k then $\langle L, E \rangle \in D_k$. Otherwise, remember that the set $F_{\alpha} = B_{\alpha} \setminus f_{\alpha}^{-1}[\bigcup_{\varepsilon \in E} A_{\varepsilon}]$ is an AP-set, so it contains arithmetic progression $M \subseteq F_{\alpha}$ of length k . Put

$K = L \cup M$. Obviously, $\langle K, E \rangle \in D_k$ and $\langle K, E \rangle \leq_P \langle L, E \rangle$. Hence the set D_k is dense in P . \square

Claim 2: D_δ are dense in (P, \leq_P) for every $\delta \in I_\alpha$.

Proof of Claim 2. Consider $\langle L, E \rangle \in P$ arbitrary. If $\delta \in E$ then $\langle L, E \rangle \in D_\delta$. Otherwise, set $D = E \cup \{\delta\}$. Obviously, $\langle L, D \rangle \in D_\delta$ and $\langle L, D \rangle \leq_P \langle L, E \rangle$, thus the set D_δ is dense. \square

The family $\mathcal{D} = \{D_k : k \in \omega\} \cup \{D_\delta : \delta \in I_\alpha\}$ consists of dense subsets of P and $|\mathcal{D}| < 2^\omega$. If Martin's Axiom for σ -centered posets holds there is a \mathcal{D} -generic filter \mathcal{G} . Let $H = \bigcup \{K : (\exists D) \langle K, D \rangle \in \mathcal{G}\}$. The set H is an AP-set and $f_\alpha[H]$ is almost disjoint with A_α for every $\alpha \in I_\alpha$.

- H is an AP-set

For every $k \in \omega$ there exists $\langle K, D \rangle \in \mathcal{G} \cap D_k$ and $H \supseteq K$ contains arithmetic progression of length k .

- $(\forall A \in \mathcal{A}_\alpha) |f_\alpha[H] \cap A| < \omega$

Fix $\langle K, D \rangle \in \mathcal{G} \cap D_\delta$. For arbitrary $\langle L, E \rangle \in \mathcal{G}$ there is $\langle L', E' \rangle$ such that $\langle L', E' \rangle \leq_P \langle L, E \rangle$ and $\langle L', E' \rangle \leq_P \langle K, D \rangle$ (because \mathcal{G} is a filter). Notice that $f_\alpha[L] \cap A_\delta \subseteq f_\alpha[L'] \cap A_\delta = f_\alpha[K] \cap A_\delta$. It follows that $f_\alpha[H] \cap A_\delta = f_\alpha[K] \cap A_\delta$ is finite.

Now, apply Lemma 3.1 to set H and function f_α which is finite-to-one. Thus conclusion (2) holds and there exists an AP-set $C \subseteq H$ such that $f_\alpha[C] \in \mathcal{I}_{1/n}$.

To complete the induction step let $\mathcal{A}_{\alpha+1} = \mathcal{A}_\alpha \cup \{f_\alpha[C]\}$.

The family $\mathcal{A} = \bigcup_{\alpha < 2^\omega} \mathcal{A}_\alpha \subseteq \mathcal{I}_{1/n}$ is an almost disjoint family. To check that \mathcal{A} is maximal let $M \subseteq \mathbb{N}$ be given and let $f : \mathbb{N} \rightarrow M$ be the increasing enumeration of M . Since there is an AP-set $C \subseteq \mathbb{N}$ and $A \in \mathcal{A}$ such that $f[C] \subseteq A$ it is clear that $M \cap A$ is infinite. \square

Now we can conclude this section with the construction of a Hausdorff, compact, sequentially compact and separable space that is first-countable at all points but one, which is not $\mathcal{I}_{1/n}$ -space.

Theorem 3.3. (*MA $_{\sigma}$ -centered*) *There exists a compact, separable van der Waerden space that is not an $\mathcal{I}_{1/n}$ -space.*

Proof. Let $\mathcal{A} \subseteq \mathcal{I}_{1/n}$ be as in Proposition 3.2 and let X be the one-point compactification of $\Psi(\mathcal{A})$. It was noticed in Proposition 2.2 that the space X is sequentially compact but not $\mathcal{I}_{1/n}$ -space.

We have yet to see that X is van der Waerden. Suppose $f : \mathbb{N} \rightarrow X$ is given. Let $g : f[\mathbb{N}] \rightarrow \mathbb{N}$ be one-to-one. By Lemma 3.1 we can find an AP-set $B \subseteq \mathbb{N}$ so that $(g \circ f) \upharpoonright B$ is constant or finite-to-one, and hence $f \upharpoonright B$ is constant or finite-to-one. In the former case, the sequence $\langle f(n) \rangle_{n \in B}$ is constant, and therefore converges. So assume that $f \upharpoonright B$ is finite-to-one. Since either $B \cap f^{-1}[\mathbb{N}]$ or $B \setminus f^{-1}[\mathbb{N}]$ is AP, we may assume, by shrinking B to some AP-subset, that either $f[B] \subseteq \mathbb{N}$ or $f[B] \subseteq X \setminus (\mathbb{N} \cup \{p\})$. In the former case, there is some $A \in \mathcal{A}$ and an AP-set $C \subseteq B$ such that $f[C] \subseteq A$. Since $f \upharpoonright B$ is finite-to-one, $\langle f(n) \rangle_{n \in C}$ converges to p_A . In the latter case, we claim that the sequence $\langle f(n) \rangle_{n \in B}$ converges to p . To see this, let Z be a compact subset of $\Psi(\mathcal{A})$ such that $X \setminus Z$ is a basic neighborhood of p . Then $Z \setminus \mathbb{N}$ is finite so, since $f \upharpoonright B$ is finite-to-one, $\langle f(n) \rangle_{n \in B}$ is eventually in $X \setminus Z$. \square

Remark. It is not difficult to see that all propositions in this section remain true if the summable ideal $\mathcal{I}_{1/n}$ is replaced by an arbitrary F_σ ideal \mathcal{I} which is also a P -ideal.

4. $\mathcal{I}_{1/n}$ -SPACE WHICH IS NOT HINDMAN

Since it is consistent with ZFC that the class of $\mathcal{I}_{1/n}$ -spaces is strictly smaller than the class of van der Waerden spaces we may (consistently) strengthen Jones' result that there is a van der Waerden space which is not Hindman if we construct an $\mathcal{I}_{1/n}$ -space which is not Hindman.

Definition 4.1. We say that set $A \subseteq \mathbb{N}$ has property (SC), in short, A is an (SC)-set, if $\lim_{n \rightarrow \infty} (a_{n+1} - a_n) = \infty$ where $\{a_n : n \in \mathbb{N}\}$ is the increasing enumeration of A .

It is easy to see that every (SC)-set belongs to the Hindman ideal.

Lemma 4.2. Let $A \subseteq \mathbb{N}$ be an $\mathcal{I}_{1/n}$ -positive set, i.e. $\sum_{a \in A} \frac{1}{a} = \infty$, and let $f : \mathbb{N} \rightarrow \mathbb{N}$. There exists an $\mathcal{I}_{1/n}$ -positive set $C \subseteq A$ such that

- (1) either f is constant on C
- (2) or f is finite-to-one on C and $f[C]$ is an (SC)-set.

Proof. If there there exists a finite set $M \subseteq \mathbb{N}$ such that $f^{-1}[M] \cap A$ is an $\mathcal{I}_{1/n}$ -positive set then $f^{-1}(m) \cap A$ is $\mathcal{I}_{1/n}$ -positive for some $m \in$

M (because $\mathcal{I}_{1/n}$ -positive sets are partition regular) and conclusion (1) holds.

If $A \cap f^{-1}[M]$ is not $\mathcal{I}_{1/n}$ -positive for every finite set $M \subseteq \mathbb{N}$ then we construct by induction an $\mathcal{I}_{1/n}$ -positive set $C \subseteq A$ for which conclusion (2) holds. For every $n \in \mathbb{N}$ we construct (finite) sets $C_n \subseteq A$ such that

- i) $C_1 = \{\min A\}$
- ii) $(\forall n \in \mathbb{N}) \min f[C_{n+1}] \geq 2 \cdot \max f[C_n]$.
- iii) $(\forall n \in \mathbb{N}) (\forall c, d \in f[C_n]) |c - d| \geq n$.
- iv) $(\forall n \in \mathbb{N}) \sum_{c \in f[C_n]} \frac{1}{c} \geq n$

Assume we have already defined C_i as required for $i = 1, 2, \dots, n$. Remember that $A_n = A \setminus f^{-1}[0, 2 \cdot \max f[C_n]]$ is an $\mathcal{I}_{1/n}$ -positive subset of A . Fix an increasing enumeration $f[A_n] = \{a_k : k \in \omega\}$. For every $0 \leq r < n + 1$ define $B_r = \{a_{i(n+1)+r} : i \in \omega\}$. The set $f^{-1}[B_r]$ is $\mathcal{I}_{1/n}$ -positive for some r , so there exists a finite set $C_{n+1} \subseteq f^{-1}[B_r]$ such that $\sum_{j \in C_{n+1}} \frac{1}{j} \geq n + 1$. Observe that $|c - d| \geq n + 1$ whenever c and d are distinct elements of $f[C_{n+1}]$. Obviously, condition ii) is also fulfilled.

Finally, let $C = \bigcup_{n \in \mathbb{N}} C_n$. It follows from the construction that $C \subseteq A$ is $\mathcal{I}_{1/n}$ -positive, the function f is finite-to-one on C and if $\{c_n : n \in \mathbb{N}\}$ is an increasing enumeration of $f[C]$ then $\lim_{n \rightarrow \infty} c_{n+1} - c_n = \infty$. \square

Proposition 4.3. (*MA $_{\sigma}$ -centered*) *There exists a maximal almost disjoint family \mathcal{A} consisting of (SC)-sets so that for every $\mathcal{I}_{1/n}$ -positive set $B \subseteq \mathbb{N}$ and every finite-to-one function $f : B \rightarrow \mathbb{N}$ there exists an $\mathcal{I}_{1/n}$ -positive set $C \subseteq B$ and $A \in \mathcal{A}$ so that $f[C] \subseteq A$.*

Proof. Let us enumerate as $\{\langle f_\alpha, B_\alpha \rangle : \omega \leq \alpha \leq 2^\omega\}$ all pairs $\langle f, B \rangle$ where $B \subseteq \mathbb{N}$ is an $\mathcal{I}_{1/n}$ -positive set and $f : B \rightarrow \mathbb{N}$ a finite-to-one function. By transfinite induction on $\alpha < 2^\omega$ we construct almost disjoint families \mathcal{A}_α , $\omega \leq \alpha < 2^\omega$, (consisting of (SC)-sets) so that the following conditions are satisfied:

- (i) \mathcal{A}_ω is an infinite partition of \mathbb{N}
 - (ii) $\mathcal{A}_\alpha \subseteq \mathcal{A}_\beta$ whenever $\alpha \leq \beta$
 - (iii) $\mathcal{A}_\gamma = \bigcup_{\alpha < \gamma} \mathcal{A}_\alpha$ for γ limit
 - (iv) $(\forall \alpha) |\mathcal{A}_\alpha| \leq |\alpha|$
 - (v) $(\forall \alpha) (\exists A \in \mathcal{A}_{\alpha+1}) (\exists C \subseteq B_\alpha) f_\alpha[C] \subseteq A$
- Suppose we already know \mathcal{A}_α .

Case A. $(\exists A \in \mathcal{A}_\alpha)$ $f_\alpha^{-1}[A]$ is $\mathcal{I}_{1/n}$ -positive

Let $C = f_\alpha^{-1}[A]$ and $\mathcal{A}_{\alpha+1} = \mathcal{A}_\alpha$.

Case B. $(\forall A \in \mathcal{A}_\alpha)$ $f_\alpha^{-1}[A] \in \mathcal{I}_{1/n}$

Enumerate \mathcal{A}_α as $\{A_\delta : \delta \in I_\alpha\}$ where I_α is an index set of size less or equal $|\alpha|$. Let $P = [B_\alpha]^{<\omega} \times [I_\alpha]^{<\omega}$ and define order \leq_P by $\langle K, D \rangle \leq_P \langle L, E \rangle$ if $K \supseteq L$ and $D \supseteq E$ and $K \setminus L \subseteq B_\alpha \setminus f_\alpha^{-1}[\bigcup_{\varepsilon \in E} A_\varepsilon]$. It is easy to see that (P, \leq_P) is a σ -centered poset. For $k \in \mathbb{N}$ and $\delta \in I_\alpha$ define $D_k = \{\langle K, D \rangle \in P : \sum_{a \in K} \frac{1}{a} \geq k\}$ and $D_\delta = \{\langle K, D \rangle \in P : \delta \in D\}$.

Claim 1: D_k is dense in (P, \leq_P) for every $k \in \mathbb{N}$.

Proof of Claim 1. Consider $\langle L, E \rangle \in P$ arbitrary. If $\sum_{a \in L} \frac{1}{a} \geq k$ then $\langle L, E \rangle \in D_k$. Otherwise, remember that the set $F_\alpha = B_\alpha \setminus f_\alpha^{-1}[\bigcup_{\varepsilon \in E} A_\varepsilon]$ is $\mathcal{I}_{1/n}$ -positive, so there exists a finite set $M \subseteq F_\alpha$ such that $\sum_{a \in M} \frac{1}{a} \geq k$. Put $K = L \cup M$. Obviously, $\langle K, E \rangle \in D_k$ and $\langle K, E \rangle \leq_P \langle L, E \rangle$. Hence the set D_k is dense in P . \square

Claim 2: D_δ are dense in (P, \leq_P) for every $\delta \in I_\alpha$.

Proof of Claim 2. Consider $\langle L, E \rangle \in P$ arbitrary. If $\delta \in E$ then $\langle L, E \rangle \in D_\delta$. Otherwise, set $D = E \cup \{\delta\}$. Obviously, $\langle L, D \rangle \in D_\delta$ and $\langle L, D \rangle \leq_P \langle L, E \rangle$, thus the set D_δ is dense. \square

The family $\mathcal{D} = \{D_k : k \in \mathbb{N}\} \cup \{D_\delta : \delta \in I_\alpha\}$ consists of dense subsets of P and $|\mathcal{D}| < 2^\omega$. According to Martin's axiom for σ -centered posets there is a \mathcal{D} -generic filter \mathcal{G} . Let $H = \bigcup \{K : (\exists D) \langle K, D \rangle \in \mathcal{G}\}$. The set H is $\mathcal{I}_{1/n}$ -positive and $f_\alpha[H]$ is almost disjoint with A_α for every $\alpha \in I_\alpha$.

• H is an $\mathcal{I}_{1/n}$ -positive set (in particular, H is infinite)

For every $k \in \mathbb{N}$ there exists $\langle K, D \rangle \in \mathcal{G} \cap D_k$ and $\sum_{a \in H} \frac{1}{a} \geq \sum_{a \in K} \frac{1}{a} \geq k$.

• $(\forall A \in \mathcal{A}_\alpha)$ $|f_\alpha[H] \cap A| < \omega$

Fix $\langle K, D \rangle \in \mathcal{G} \cap D_\delta$. For arbitrary $\langle L, E \rangle \in \mathcal{G}$ there is $\langle L', E' \rangle$ such that $\langle L', E' \rangle \leq_P \langle L, E \rangle$ and $\langle L', E' \rangle \leq_P \langle K, D \rangle$ (because \mathcal{G} is a filter). Notice that $f_\alpha[L] \cap A_\delta \subseteq f_\alpha[L'] \cap A_\delta = f_\alpha[K] \cap A_\delta$. It follows that $f_\alpha[H] \cap A_\delta = f_\alpha[K] \cap A_\delta$ is finite.

Now, apply Lemma 4.2 to set H and function f_α which is finite-to-one. Thus conclusion (2) holds and there exists an $\mathcal{I}_{1/n}$ -positive set $C \subseteq H$ such that $f_\alpha[C]$ is an (SC) -set.

To complete the induction step let $\mathcal{A}_{\alpha+1} = \mathcal{A}_\alpha \cup \{f_\alpha[C]\}$.

The family $\mathcal{A} = \bigcup_{\alpha < 2^\omega} \mathcal{A}_\alpha$ is an almost disjoint family of (SC) -sets. To check that \mathcal{A} is maximal let $M \subseteq \mathbb{N}$ be given and let $f : \mathbb{N} \rightarrow M$ be the increasing enumeration of M . Since there is an $\mathcal{I}_{1/n}$ -positive set $C \subseteq \mathbb{N}$ and $A \in \mathcal{A}$ such that $f[C] \subseteq A$ it is clear that $M \cap A$ is infinite. \square

Theorem 4.4. (*MA $_\sigma$ -centered*) *There exists a compact, separable $\mathcal{I}_{1/n}$ -space that is not Hindman.*

Proof. Let \mathcal{A} be a MAD family as in Proposition 4.3 and let X be the one-point compactification of $\Psi(\mathcal{A})$. It was shown in [3] and [4] that the space X is sequentially compact but not Hindman.

We have yet to see that X is an $\mathcal{I}_{1/n}$ -space. Suppose $f : \mathbb{N} \rightarrow X$ is given. Let $g : f[\mathbb{N}] \rightarrow \mathbb{N}$ be one-to-one. By Lemma 4.2 we can find an $\mathcal{I}_{1/n}$ -positive set $B \subseteq \mathbb{N}$ so that $(g \circ f) \upharpoonright B$ is constant or finite-to-one, and hence $f \upharpoonright B$ is constant or finite-to-one. In the former case, the sequence $\langle f(n) \rangle_{n \in B}$ is constant, and therefore converges. So assume that $f \upharpoonright B$ is finite-to-one. Since either $B \cap f^{-1}[\mathbb{N}]$ or $B \setminus f^{-1}[\mathbb{N}]$ is $\mathcal{I}_{1/n}$ -positive, we may assume, by shrinking B to some $\mathcal{I}_{1/n}$ -positive subset, that either $f[B] \subseteq \mathbb{N}$ or $f[B] \subseteq X \setminus (\mathbb{N} \cup \{p\})$. In the former case, there is some $A \in \mathcal{A}$ and an $\mathcal{I}_{1/n}$ -positive set $C \subseteq B$ such that $f[C] \subseteq A$. Since $f \upharpoonright B$ is finite-to-one, $\langle f(n) \rangle_{n \in C}$ converges to p_A . In the latter case, we claim that the sequence $\langle f(n) \rangle_{n \in B}$ converges to p . To see this, let Z be a compact subset of $\Psi(\mathcal{A})$ such that $X \setminus Z$ is a basic neighborhood of p . Then $Z \setminus \mathbb{N}$ is finite so, since $f \upharpoonright B$ is finite-to-one, $\langle f(n) \rangle_{n \in B}$ is eventually in $X \setminus Z$. \square

Remark. All the proofs in section 4 can be modified for an arbitrary F_σ -ideal \mathcal{I} .

5. $\mathcal{I}_{1/n}$ -SPACES AND IP-RICH SETS

We say that $A \subseteq \mathbb{N}$ is an *ip-rich set* if $(\forall k \in \mathbb{N}) (\exists D \subseteq \mathbb{N}) |D| = k$ and $FS(D) \subseteq A$. It is known that ip-rich sets form an ideal and we denote it as \mathcal{I}_{ipr} . Since it is an F_σ -ideal we may replace $\mathcal{I}_{1/n}$ in Definition 2.1 by \mathcal{I}_{ipr} and obtain a new subclass of sequentially compact spaces. We know from section 2 that infinite \mathcal{I}_{ipr} -spaces exist.

It is obvious that every IP-set is ip-rich, so every Hindman space is \mathcal{I}_{ipr} -space. The converse is (consistently) not true (it follows from the generalization of Theorem 4.4 to the F_σ -ideal \mathcal{I}_{ipr}).

We construct in the following an \mathcal{I}_{ipr} -space that is not an $\mathcal{I}_{1/n}$ -space, which might be considered as a very inaccurate approximation of the desired Hindman not van der Waerden space. The construction follows the same pattern as the constructions in sections 3 and 4 (and as in the proof of Kojman and Shelah).

Lemma 5.1. *Let $A \subseteq \mathbb{N}$ be an ip-rich set and let $f : \mathbb{N} \rightarrow \mathbb{N}$. There exists an ip-rich set $C \subseteq A$ such that*

- (1) *either f is constant on C*
- (2) *or f is finite-to-one on C and $\sum_{c \in f[C]} \frac{1}{c} < \infty$.*

In other words, $f[C] \in \mathcal{I}_{1/n}$.

Proof. If there exists a finite set $M \subseteq \mathbb{N}$ such that $f^{-1}[M] \cap A$ is an ip-rich set then $f^{-1}(m) \cap A$ is an ip-rich set for some $m \in M$ (because ip-rich sets are partition regular) and conclusion (1) holds.

If $A \cap f^{-1}[M]$ is not ip-rich for every finite set $M \subseteq \mathbb{N}$ then we construct by induction an ip-rich set $C \subseteq A$ for which conclusion (2) holds.

We construct for every $n \in \mathbb{N}$ (finite) sets $C_n \subseteq A$ and $D_n \subseteq A$ such that

- i) $C_1 = D_1 = \{\min A\}$
- ii) $(\forall n \in \mathbb{N}) |D_n| = n$ and $C_n = FS(D_n)$
- iii) $(\forall n \in \mathbb{N}) \min f[C_{n+1}] > 2^{2n+1} \cdot \max f[C_n]$

Assume we have already defined C_i and D_i as required for $i = 1, 2, \dots, n$. The set $A \cap f^{-1}[0, 2^{2n+1} \cdot \max f[C_n]]$ is not ip-rich. Hence $A \setminus f^{-1}[0, 2^{2n+1} \cdot \max f[C_n]]$ is ip-rich and there exists a finite set $D_{n+1} \subseteq A \setminus f^{-1}[0, 2^{2n+1} \cdot \max f[C_n]]$ such that $|D_{n+1}| = n + 1$ and $FS(D_{n+1}) \subseteq A \setminus f^{-1}[0, 2^{2n+1} \cdot \max f[C_n]]$. Put $C_{n+1} = FS(D_{n+1})$. It is easy to see that properties ii) and iii) are satisfied.

Finally, let $C = \bigcup_{n \in \mathbb{N}} C_n$. The set $C \subseteq A$ is obviously ip-rich and it follows from the construction that the function f is finite-to-one on C . So it remains to check that $f[C] \in \mathcal{I}_{1/n}$.

To this end notice that

$$\sum_{c \in f[C_{n+1}]} \frac{1}{c} \leq \frac{|C_{n+1}|}{2^{2n+1} \cdot \max f[C_n]} \leq \frac{2^{n+1}}{2^{2n+1} \cdot \max f[C_n]} \leq \frac{1}{2^n}.$$

So we get

$$\sum_{c \in f[C]} \frac{1}{c} \leq \sum_{n \in \omega} \sum_{c \in f[C_{n+1}]} \frac{1}{c} \leq \sum_{n \in \omega} \frac{1}{2^n} < \infty$$

and $f[C] \in \mathcal{I}_{1/n}$. \square

Proposition 5.2. (*MA $_{\sigma}$ -centered*) *There exists a maximal almost disjoint family $\mathcal{A} \subseteq \mathcal{I}_{1/n}$ so that for every ip-rich set $B \subseteq \mathbb{N}$ and every finite-to-one function $f : B \rightarrow \mathbb{N}$ there exists an ip-rich set $C \subseteq B$ and $A \in \mathcal{A}$ so that $f[C] \subseteq A$.*

Proof. Let us enumerate as $\{\langle f_{\alpha}, B_{\alpha} \rangle : \omega \leq \alpha \leq 2^{\omega}\}$ all pairs $\langle f, B \rangle$ where $B \subseteq \mathbb{N}$ is an ip-rich set and $f : B \rightarrow \mathbb{N}$ a finite-to-one function. By transfinite induction on $\alpha < 2^{\omega}$ we construct almost disjoint families $\mathcal{A}_{\alpha} \subseteq \mathcal{I}_{1/n}$, $\omega \leq \alpha < 2^{\omega}$, so that the following conditions are satisfied:

- (i) $\mathcal{A}_{\omega} = \{A_n : n \in \mathbb{N}\} \subseteq \mathcal{I}_{1/n}$ is a partition of \mathbb{N}
- (ii) $\mathcal{A}_{\alpha} \subseteq \mathcal{A}_{\beta}$ whenever $\alpha \leq \beta$
- (iii) $\mathcal{A}_{\gamma} = \bigcup_{\alpha < \gamma} \mathcal{A}_{\alpha}$ for γ limit
- (iv) $(\forall \alpha) |\mathcal{A}_{\alpha}| \leq |\alpha|$
- (v) $(\forall \alpha) (\exists A \in \mathcal{A}_{\alpha+1}) (\exists C \subseteq B_{\alpha})$ C is ip-rich and $f_{\alpha}[C] \subseteq A$

Suppose we already know \mathcal{A}_{α} .

Case A. $(\exists A \in \mathcal{A}_{\alpha}) f_{\alpha}^{-1}[A]$ is ip-rich

Let $C = f_{\alpha}^{-1}[A]$ and $\mathcal{A}_{\alpha+1} = \mathcal{A}_{\alpha}$.

Case B. $(\forall A \in \mathcal{A}_{\alpha}) f_{\alpha}^{-1}[A]$ is not ip-rich

Enumerate \mathcal{A}_{α} as $\{A_{\delta} : \delta \in I_{\alpha}\}$ where I_{α} is an index set of size less or equal to $|\alpha|$. Let $P = [B_{\alpha}]^{<\omega} \times [I_{\alpha}]^{<\omega}$ and define order \leq_P by $\langle K, D \rangle \leq_P \langle L, E \rangle$ if $K \supseteq L$ and $D \supseteq E$ and $K \setminus L \subseteq B_{\alpha} \setminus f_{\alpha}^{-1}[\bigcup_{\varepsilon \in E} A_{\varepsilon}]$. It is easy to see that (P, \leq_P) is a σ -centered poset. For $k \in \mathbb{N}$ and $\delta \in I_{\alpha}$ define $D_k = \{\langle K, D \rangle \in P : (\exists M \in [\mathbb{N}]^{<\omega}) |M| = k \text{ and } K \supseteq FS(M)\}$ and $D_{\delta} = \{\langle K, D \rangle \in P : \delta \in D\}$.

Claim 1: D_k is dense in (P, \leq_P) for every $k \in \mathbb{N}$.

Proof of Claim 1. Consider $\langle L, E \rangle \in P$ arbitrary. If L contains finite sums of a set with k elements then $\langle L, E \rangle \in D_k$. Otherwise, remember that the set $F_{\alpha} = B_{\alpha} \setminus f_{\alpha}^{-1}[\bigcup_{\varepsilon \in E} A_{\varepsilon}]$ is ip-rich, so it contains $FS(M)$ for some set M such that $|M| = k$. Put $K = L \cup FS(M)$. Obviously, $\langle K, E \rangle \in D_k$ and $\langle K, E \rangle \leq_P \langle L, E \rangle$. Hence the set D_k is dense in P . \square

Claim 2: D_δ are dense in (P, \leq_P) for every $\delta \in I_\alpha$.

Proof of Claim 2. Consider $\langle L, E \rangle \in P$ arbitrary. If $\delta \in E$ then $\langle L, E \rangle \in D_\delta$. Otherwise, set $D = E \cup \{\delta\}$. Obviously, $\langle L, D \rangle \in D_\delta$ and $\langle L, D \rangle \leq_P \langle L, E \rangle$, thus the set D_δ is dense. \square

The family $\mathcal{D} = \{D_k : k \in \mathbb{N}\} \cup \{D_\delta : \delta \in I_\alpha\}$ consists of dense subsets of P and $|\mathcal{D}| < 2^\omega$. If Martin's Axiom for σ -centered posets holds there is a \mathcal{D} -generic filter \mathcal{G} . Let $H = \bigcup \{K : (\exists D) \langle K, D \rangle \in \mathcal{G}\}$. The set H is ip-rich and $f_\alpha[H]$ is almost disjoint with A_α for every $\alpha \in I_\alpha$.

- H is an ip-rich set

For every $k \in \omega$ there exists $\langle K, D \rangle \in \mathcal{G} \cap D_k$ and $H \supseteq K \supseteq FS(M)$ where $|M| = k$.

- $(\forall A \in \mathcal{A}_\alpha) |f_\alpha[H] \cap A| < \omega$

Fix $\langle K, D \rangle \in \mathcal{G} \cap D_\delta$. For arbitrary $\langle L, E \rangle \in \mathcal{G}$ there is $\langle L', E' \rangle$ such that $\langle L', E' \rangle \leq_P \langle L, E \rangle$ and $\langle L', E' \rangle \leq_P \langle K, D \rangle$ (because \mathcal{G} is a filter). Notice that $f_\alpha[L] \cap A_\delta \subseteq f_\alpha[L'] \cap A_\delta = f_\alpha[K] \cap A_\delta$. It follows that $f_\alpha[H] \cap A_\delta = f_\alpha[K] \cap A_\delta$ is finite.

Now, apply Lemma 5.1 to set H and function f_α which is finite-to-one. Thus conclusion (2) holds and there exists an ip-rich set $C \subseteq H$ such that $f_\alpha[C] \in \mathcal{I}_{1/n}$.

To complete the induction step let $\mathcal{A}_{\alpha+1} = \mathcal{A}_\alpha \cup \{f_\alpha[C]\}$.

The family $\mathcal{A} = \bigcup_{\alpha < 2^\omega} \mathcal{A}_\alpha \subseteq \mathcal{I}_{1/n}$ is an almost disjoint family. To check that \mathcal{A} is maximal let $M \subseteq \mathbb{N}$ be given and let $f : \mathbb{N} \rightarrow M$ be the increasing enumeration of M . Since there is an ip-rich set $C \subseteq \mathbb{N}$ and $A \in \mathcal{A}$ such that $f[C] \subseteq A$ it is clear that $M \cap A$ is infinite. \square

Theorem 5.3. (*MA $_{\sigma}$ -centered*) *There exists a compact, separable topological space \mathcal{I}_{ipr} -space X that is not an $\mathcal{I}_{1/n}$ -space.*

Proof. Let $\mathcal{A} \subseteq \mathcal{I}_{1/n}$ be as in Proposition 5.2 and let X be the one-point compactification of $\Psi(\mathcal{A})$. It was noticed in Proposition 2.2 that the space X is sequentially compact but not $\mathcal{I}_{1/n}$ -space.

We have yet to verify that X is an \mathcal{I}_{ipr} -space. Suppose $f : \mathbb{N} \rightarrow X$ is given. Let $g : f[\mathbb{N}] \rightarrow \mathbb{N}$ be one-to-one. By Lemma 5.1 we can find an ip-rich set $B \subseteq \mathbb{N}$ so that $(g \circ f) \upharpoonright B$ is constant or finite-to-one, and hence $f \upharpoonright B$ is constant or finite-to-one. In the former case, the sequence $\langle f(n) \rangle_{n \in B}$ is constant, and therefore converges.

So assume that $f \upharpoonright B$ is finite-to-one. Since either $B \cap f^{-1}[\mathbb{N}]$ or $B \setminus f^{-1}[\mathbb{N}]$ is ip-rich, we may assume, by shrinking B to some ip-rich subset, that either $f[B] \subseteq \mathbb{N}$ or $f[B] \subseteq X \setminus (\mathbb{N} \cup \{p\})$. In the former case, there is some $A \in \mathcal{A}$ and an ip-rich set $C \subseteq B$ such that $f[C] \subseteq A$. Since $f \upharpoonright B$ is finite-to-one, $\langle f(n) \rangle_{n \in C}$ converges to p_A . In the latter case, we claim that the sequence $\langle f(n) \rangle_{n \in B}$ converges to p . To see this, let Z be a compact subset of $\Psi(\mathcal{A})$ such that $X \setminus Z$ is a basic neighborhood of p . Then $Z \setminus \mathbb{N}$ is finite so, since $f \upharpoonright B$ is finite-to-one, $\langle f(n) \rangle_{n \in B}$ is eventually in $X \setminus Z$. \square

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