Description of some ultrafilters via $\mathcal{I}$-ultrafilters

Jana Flašková

University of West Bohemia, Department of Mathematics

Abstract

We give a description of several classes of ultrafilters on ω in terms of $\mathcal{I}$-ultrafilters. For example, we characterize Q-points as weak thin ultrafilters and P-points as $\mathrm{Fin} \times \mathrm{Fin}$-ultrafilters.

1 Introduction

The definition of $\mathcal{I}$-ultrafilter was given by Baumgartner: Let $\mathcal{I}$ be a family of subsets of a set X such that $\mathcal{I}$ contains all singletons and is closed under subsets. Given an ultrafilter $\mathcal{U}$ on ω, we say that $\mathcal{U}$ is an *$\mathcal{I}$-ultrafilter* if for every function $F : \omega \to X$ there is $A \in \mathcal{U}$ such that $F[A] \in \mathcal{I}$.

If only finite-to-one functions F are considered then we refer to the corresponding ultrafilters as *weak $\mathcal{I}$-ultrafilters*. If only one-to-one functions F are considered then we refer to the corresponding ultrafilters as *$\mathcal{I}$-friendly ultrafilters*.

Obviously, every $\mathcal{I}$-ultrafilter is a weak $\mathcal{I}$-ultrafilter and every weak $\mathcal{I}$-ultrafilter is an $\mathcal{I}$-friendly ultrafilter. The reverse implications in general need not be true.

In [1], Baumgartner studied $\mathcal{I}$-ultrafilters for $X = 2^\omega$ and $X = \omega_1$. He proved that P-points can be described as $\mathcal{I}$-ultrafilters: If $X = 2^\omega$ then P-points are precisely the $\mathcal{I}$-ultrafilters for $\mathcal{I}$ consisting of all finite and converging sequences, if $X = \omega_1$ then P-points are precisely the $\mathcal{I}$-ultrafilters for $\mathcal{I} = \{A \subseteq \omega_1 : A \text{ has order type} \leq \omega\}$.

We study $\mathcal{I}$-ultrafilters, weak $\mathcal{I}$-ultrafilters and $\mathcal{I}$-friendly ultrafilters in the setting $X = \omega$ (or another countable set) and $\mathcal{I}$ is an ideal on X which contains all finite subsets of ω. We will use these concepts to describe rapid ultrafilters, Q-points, P-points and selective ultrafilters. Every class of ultrafilters is discussed in one section of the paper.

In this introduction we will recall some definitions used in the subsequent sections (however, notions used only in one particular section are recalled at the beginning of the respective section). We also mention some tools we used to translate some known results in terms of $\mathcal{I}$-ultrafilters.

Since we deal with ultrafilters on ω let us first recall two (quasi)orderings defined on ω^*, the remainder of the Čech-Stone compactification of ω:

Rudin-Keisler order $\leq_{RK}$. For $\mathcal{U}$, $\mathcal{V} \in \beta\omega$ we write $\mathcal{U} \leq_{RK} \mathcal{V}$ iff there exists a function $f : \omega \to \omega$ such that $(\forall U \in \mathcal{U})\ f^{-1}[U] \in \mathcal{V}$.

Rudin-Blass order $\leq_{RB}$. For $\mathcal{U}$, $\mathcal{V} \in \beta\omega$ we write $\mathcal{U} \leq_{RB} \mathcal{V}$ iff there is a finite-to-one function $f : \omega \to \omega$ such that $(\forall U \in \mathcal{U})\ f^{-1}[U] \in \mathcal{V}$.

Katětov order $\leq_K$ is an extension of the Rudin-Keisler order to arbitrary filters or ideals. We write $\mathcal{F} \leq_K \mathcal{G}$ if there exists a function $f : \omega \to \omega$ such that $f^{-1}[U] \in \mathcal{G}$ for every $U \in \mathcal{F}$. It is easy to check that $\mathcal{F} \leq_K \mathcal{G}$ if and only if $\mathcal{F}^* \leq_K \mathcal{G}^*$ (where $\mathcal{F}^*$ denotes the dual ideal to filter $\mathcal{F}$ or the dual filter to ideal $\mathcal{F}$ according to the situation).

Katětov-Blass order $\leq_{KB}$ is an extension of the Rudin-Blass order to arbitrary filters or ideals. We write $\mathcal{F} \leq_{KB} \mathcal{G}$ if there exists a finite-to-one function $f : \omega \to \omega$ such that $f^{-1}[U] \in \mathcal{G}$ for every $U \in \mathcal{F}$.

Some known results may be restated in terms of $\mathcal{I}$-ultrafilters using the following lemma from [3]:

Lemma 1.1. *Let $\mathcal{I}$ be an ideal on ω. For ultrafilter $\mathcal{U} \in \omega^*$ the following assertions are equivalent:*

(i) $\mathcal{U}$ is an $\mathcal{I}$-ultrafilter

(ii) $\mathcal{V} \not\leq_{RK} \mathcal{U}$ for every ultrafilter $\mathcal{V} \supseteq \mathcal{I}^*$

(iii) $\mathcal{I}^* \not\leq_K \mathcal{U}$

Proof. $(i) \Rightarrow (ii)$ Assume there exists an ultrafilter $\mathcal{V} \supseteq \mathcal{I}^*$ such that $\mathcal{V} \leq_{RK} \mathcal{U}$. Since there exists $f : \omega \to \omega$ such that $f[U] \in \mathcal{V}$ for every $U \in \mathcal{U}$ we have $f[U] \notin \mathcal{I}$ for every $U \in \mathcal{U}$. Hence $\mathcal{U}$ is not an I-ultrafilter.

$(ii) \Rightarrow (iii)$ If $\mathcal{I}^* \leq_K \mathcal{U}$ then there is a function $f : \omega \to \omega$ such that $f^{-1}[A] \in \mathcal{U}$ for every $A \in \mathcal{I}^*$. Put $\mathcal{V} = \{B \subseteq \omega : (\exists U \in \mathcal{U}) f[U] \subseteq B\}$. It is easy to see that $\mathcal{I}^* \subseteq \mathcal{V}$, $\mathcal{V}$ is an ultrafilter and $\mathcal{V} \leq_{RK} \mathcal{U}$.

$(iii) \Rightarrow (i)$ If $\mathcal{I}^* \not\leq_K \mathcal{U}$ then for every $f : \omega \to \omega$ there is $A \in \mathcal{I}^*$ such that $f^{-1}[A] \notin \mathcal{U}$. Since $\mathcal{U}$ is an ultrafilter we get $\omega \setminus f^{-1}[A] \in \mathcal{U}$ and we have also $f[\omega \setminus f^{-1}[A]] \subseteq \omega \setminus A \in \mathcal{I}$. Hence $\mathcal{U}$ is an $\mathcal{I}$-ultrafilter. □

A weak $\mathcal{I}$-ultrafilter can be characterized analogously, it is sufficient to replace in Lemma 1.1 $\not\leq_{RK}$ by $\not\leq_{RB}$ and $\not\leq_K$ by $\not\leq_{KB}$.

Lemma 1.2. *Let $\mathcal{I}$ be an ideal on ω. For ultrafilter $\mathcal{U} \in \omega^*$ the following are equivalent:*

(i) $\mathcal{U}$ is a weak $\mathcal{I}$-ultrafilter

(ii) $\mathcal{V} \not\leq_{RB} \mathcal{U}$ for every ultrafilter $\mathcal{V} \supseteq \mathcal{I}^*$

(iii) $\mathcal{I}^* \not\leq_{KB} \mathcal{U}$ $\square$

As an immediate consequence of preceding lemmas we can generalize the obvious fact that if $\mathcal{I} \subseteq \mathcal{J}$ then every $\mathcal{I}$-ultrafilter is a $\mathcal{J}$-ultrafilter.

Observation 1.3. *Assume $\mathcal{I}$, $\mathcal{J}$ are two ideals on ω.*

- *If $\mathcal{I} \leq_K \mathcal{J}$ then every $\mathcal{I}$-ultrafilter is a $\mathcal{J}$-ultrafilter.*
- *If $\mathcal{I} \leq_{KB} \mathcal{J}$ then every weak $\mathcal{I}$-ultrafilter is a weak $\mathcal{J}$-ultrafilter.*
- *If there is a one-to-one function $f : \omega \to \omega$ such that $f^{-1}[A] \in \mathcal{J}$ for every $A \in \mathcal{I}$ then every $\mathcal{I}$-friendly ultrafilter is a $\mathcal{J}$-friendly ultrafilter.*

An ideal $\mathcal{I}$ on ω is called *tall (dense)* if for every infinite $A \subseteq \omega$ there exists infinite $B \subseteq A$ such that $B \in \mathcal{I}$.

2 Rapid ultrafilters

Definition 2.1. An ultrafilter $\mathcal{U} \in \omega^*$ is called a *rapid ultrafilter* if the enumeration functions of its sets form a dominating family in $({}^{\omega}\omega, \leq^*)$.

For an alternative description of rapid ultrafilters summable ultrafilters are useful. Whenever $f : \omega \to (0, \infty)$ is a function such that $\sum_{n\in\omega} f(n) = \infty$ then the family $\mathcal{I}_f = \{A \subseteq \omega : \sum_{a\in A} f(a) < +\infty\}$ is an ideal on ω and it is called a *summable ideal*. It is known (see e.g. [2]) that every summable ideal is an F_σ P-ideal and that a summable ideal $\mathcal{I}_f$ is tall if and only if $f \in c_0$ where $c_0 = \{f : \omega \to [0, \infty) | \lim_{n\to\infty} f(n) = 0\}$.

The connection between rapid ultrafilters and summable ideals was first mentioned by Vojtáš in [9]:

Theorem 2.2 (Vojtáš)**.** *For $\mathcal{U} \in \omega^*$ the following are equivalent:*

1) $\mathcal{U}$ is rapid

2) $(\forall f \in c_0)(\exists U \in \mathcal{U})$ $U \in \mathcal{I}_f$ i.e. $\mathcal{U} \cap \mathcal{I}_f \neq \emptyset$ for each summable ideal $\mathcal{I}_f$

Another characterization of rapid ultrafilters is due to Hrušák [5]:

Theorem 2.3 (Hrušák)**.** *For $\mathcal{U} \in \omega^*$ the following are equivalent:*

1) $\mathcal{U}$ is rapid

2) $\mathcal{I} \not\leq_{KB} \mathcal{U}^*$ for every tall analytic P-ideal $\mathcal{I}$

3) $\mathcal{I} \not\leq_{KB} \mathcal{U}^*$ *for every tall* F_σ *P-ideal* $\mathcal{I}$

4) $\mathcal{I} \not\leq_{KB} \mathcal{U}^*$ *for every tall summable ideal* $\mathcal{I}$

Both results are included in the following theorem:

Theorem 2.4. *For an ultrafilter* $\mathcal{U} \in \omega^*$ *the following are equivalent:*

1) $\mathcal{U}$ *is rapid*

2) $\mathcal{U}$ *is a weak* $\mathcal{I}$*-ultrafilter for every tall analytic P-ideal* $\mathcal{I}$

3) $\mathcal{U}$ *is a weak* $\mathcal{I}$*-ultrafilter for every tall* F_σ *P-ideal* $\mathcal{I}$

4) $\mathcal{U}$ *is a weak* $\mathcal{I}$*-ultrafilter for every tall summable ideal* $\mathcal{I}$

5) $\mathcal{U}$ *is an* $\mathcal{I}$*-friendly ultrafilter for every tall analytic P-ideal* $\mathcal{I}$

6) $\mathcal{U}$ *is an* $\mathcal{I}$*-friendly ultrafilter for every tall* F_σ *P-ideal* $\mathcal{I}$

7) $\mathcal{U}$ *is an* $\mathcal{I}$*-friendly ultrafilter for every tall summable ideal* $\mathcal{I}$

8) $\mathcal{U} \cap \mathcal{I} \neq \emptyset$ *for every tall analytic P-ideal* $\mathcal{I}$

9) $\mathcal{U} \cap \mathcal{I} \neq \emptyset$ *for every tall* F_σ *P-ideal* $\mathcal{I}$

10) $\mathcal{U} \cap \mathcal{I} \neq \emptyset$ *for every tall summable ideal* $\mathcal{I}$

Proof. It is sufficient to prove 1) $\Rightarrow$ 2) and 10) $\Rightarrow$ 1).

1) $\Rightarrow$ 2) Assume $\mathcal{U}$ is a rapid ultrafilter and $\mathcal{I}$ is a tall analytic P-ideal. By Solecki's characterization in [8] there exists a lsc submeasure φ such that $\mathcal{I} = \mathrm{Exh}(\varphi)$. It is known that $\mathcal{I}$ is tall if and only $\lim_{n\to\infty} \varphi(\{n\}) = 0$.

Fix a strictly increasing sequence $\langle n_k \rangle_{k\in\omega}$ such that for every $n \geq n_k$ one has $\varphi(\{n\}) < \frac{1}{2^k}$. Given a finite-to-one function $f : \omega \to \omega$ define recursively $h : \omega \to \omega$ by $h(0) = n_0$ and $h(i+1) = 1 + \max\{h(i), \max f^{-1}[0, n_{i+1}]\}$. Since $\mathcal{U}$ is a rapid ultrafilter, there exists $U \in \mathcal{U}$ such that $h \leq^* e_U$. Hence there is $i_0 \in \omega$ such that for every $i \geq i_0$ the i-th element of the set $u_i \geq h(i)$. According to the definition of $h(i)$ we have $f(u_i) \geq n_i$ for every $i \geq i_0$ and

$$\varphi(f[U] \setminus [0, \max\{f(u_j) : j < i\}]) \leq \sum_{j\geq i} \varphi(\{f(u_j)\}) \leq \sum_{j\geq i} \frac{1}{2^j} = \frac{1}{2^{i-1}}$$

and it follows that $f[U] \in \mathrm{Exh}(\varphi) = \mathcal{I}$ and $\mathcal{U}$ is a weak $\mathcal{I}$-ultrafilter.

10) $\Rightarrow$ 1). Assume for contrary that $\mathcal{U} \in \omega^*$ is not rapid. Then there exists a function $h : \omega \to \omega$ such that $h \not\leq^* e_U$ for every $U \in \mathcal{U}$. We may

assume that h is strictly increasing (and $h(n+1) \geq h(n)+n+1$). Define $f \in c_0$ by $f(k)=1$ if $k \leq h(0)$ and $f(k)=\frac{1}{n+1}$ if $k \in (h(n), h(n+1)]$, $n \in \omega$. We will show that $\mathcal{U} \cap \mathcal{I}_f = \emptyset$.

For every $U \in \mathcal{U}$ there are infinitely many n with $|U \cap [0, h(n)]| \geq n$. Hence we can choose a sequence $\langle n_i \rangle_{i\in\omega}$ such that $|U \cap [0, h(n_i)]| \geq n_i$ and $n_{i+1} \geq 2n_i$ for every $i \in \omega$. We get

$$\sum_{u\in U} f(u) \geq \sum_{i\in\omega} \sum_{u\in(h(n_i),h(n_{i+1})]} f(u) \geq \sum_{i\in\omega} (n_{i+1}-n_i)\frac{1}{n_{i+1}} \geq \sum_{i\in\omega} \frac{n_{i+1}}{2} \cdot \frac{1}{n_{i+1}}$$

and the set U does not belong to $\mathcal{I}_f$. □

3 Q-points

Definition 3.1. An ultrafilter $\mathcal{U} \in \omega^*$ is called a *Q-point* if for every partition $\{Q_n : n \in \omega\}$ of ω into finite sets there exists $U \in \mathcal{U}$ such that $|U \cap Q_n| \leq 1$ for every $n \in \omega$.

One alternative description of Q-points was presented by Hrušák in his talk [5] at set theory seminar in Prague. In accordance with [4] let us recall that family $\mathcal{ED} = \{A \subseteq \omega \times \omega : (\exists m)(\exists n)(\forall k \geq n)\{l : \langle k, l\rangle \in A\} \leq m\}$ is an ideal on $\omega \times \omega$ generated by graphs of functions and vertical sections and its restriction on the set $\mathbb{L} = \{\langle k, l\rangle \in \omega \times \omega : l \leq k\}$ is denoted by $\mathcal{ED}_{fin}$.

Theorem 3.2 (Hrušák)**.** *For an ultrafilter $\mathcal{U} \in \omega^*$ the following assertions are equivalent:*

1) $\mathcal{U}$ is Q-point

2) $\mathcal{ED}_{fin} \not\leq_{KB} \mathcal{U}^*$

We will translate this result in terms of weak $\mathcal{I}$-ultrafilters making use of Lemma 1.2 and provide some other equivalent descriptions for Q-points as well. For this we have to introduce two more ideals on ω.

Let A be a subset of ω with an increasing enumeration $A = \{a_n : n \in \omega\}$. We say that A is *thin* if $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 0$ and we call A *almost thin* if $\limsup_{n\to\infty} \frac{a_n}{a_{n+1}} < 1$. As $\mathcal{T}$ we denote the ideal generated by all finite and thin sets, as $\mathcal{A}$ we denote the ideal generated by all finite and almost thin sets.

It follows from definition that $\mathcal{T} \subseteq \mathcal{A}$. Both ideals are comparable with $\mathcal{ED}_{fin}$ in Katětov-Blass order.

Lemma 3.3. $\mathcal{T} \leq_{KB} \mathcal{ED}_{fin} \leq_{KB} \mathcal{A}$.

Proof. Let $\mathcal{K} = \{A \subseteq \omega : (\exists p \in \omega)\, (\forall n \in \omega) |A \cap [2^n, 2^{n+1})| \leq p\}$. It is easy to see that $\mathcal{T} \subseteq \mathcal{K} \subseteq \mathcal{A}$. Define $f : \mathbb{L} \to \omega$ by $\langle n, k\rangle \mapsto 2^n + k$. Check that for every $A \in \mathcal{K}$ we have $f^{-1}[A] \in \mathcal{ED}_{fin}$, so $\mathcal{K} \leq_{KB} \mathcal{ED}_{fin}$. Since $\leq_{KB}$ is a transitive relation we get also $\mathcal{T} \leq_{KB} \mathcal{ED}_{fin}$.

Now, define $g : \omega \to \mathbb{L}$ by $m \mapsto \langle 2^n, k\rangle$ where n is the unique integer for which $m \in [2^n, 2^{n+1})$ and $m = 2^n + k$. It is not difficult to check that for every $A \in \mathcal{ED}_{fin}$ the preimage $f^{-1}[A]$ is almost contained in some set from $\mathcal{K}$. However, $\mathcal{A}$ is an ideal generated by finite sets and almost thin sets and $\mathcal{K} \subseteq \mathcal{A}$, so $f^{-1}[A] \in \mathcal{A}$ and we get $\mathcal{ED}_{fin} \leq_{KB} \mathcal{A}$. □

Theorem 3.4. *For an ultrafilter $\mathcal{U} \in \omega^*$ the following assertions are equivalent:*

1) $\mathcal{U}$ is a Q-point

2) $\mathcal{U}$ is a weak $\mathcal{T}$-ultrafilter

3) $\mathcal{U}$ is a weak $\mathcal{ED}_{fin}$-ultrafilter

4) $\mathcal{U}$ is a weak $\mathcal{A}$-ultrafilter

5) $\mathcal{U}$ is a $\mathcal{T}$-friendly ultrafilter

6) $\mathcal{U}$ is an $\mathcal{ED}_{fin}$-friendly ultrafilter

7) $\mathcal{U}$ is an $\mathcal{A}$-friendly ultrafilter

Proof. Now, it is sufficient to prove 1) ⇒ 2) and 7) ⇒ 1).

1) ⇒ 2) Let $f : \omega \to \omega$ be an arbitrary finite-to-one function. Then $\{f^{-1}[n!, (n+1)!) : n \in \omega\}$ is a partition of ω into finite sets. Either $U_1 = \bigcup_{n \text{ odd}} f^{-1}[n!, (n+1)!)$ or $U_2 = \bigcup_{n \text{ even}} f^{-1}[n!, (n+1)!)$ belongs to $\mathcal{U}$ because $\mathcal{U}$ is an ultrafilter. Since $\mathcal{U}$ is a Q-point there exists $U_3 \in \mathcal{U}$ such that $|U_3 \cap f^{-1}[n!, (n+1)!)| \leq 1$ for every $n \in \omega$. Now, assume that $U_1 \in \mathcal{U}$ and put $U = U_1 \cap U_3$. It is not difficult to check that $f[U]$ is a thin set (note that $\frac{u_n}{u_{n+1}} \leq \frac{(m+1)!}{(m+2)!} = \frac{1}{m+2}$ for some $m \geq n$).

7) ⇒ 1) Let $\{Q_n : n \in \omega\}$ be a partition of ω into finite sets. Put $k_n = |Q_n| - 1$, enumerate $Q_n = \{q_i^n : i = 0, \ldots, k_n\}$ and define a strictly increasing function $f : \omega \to \omega$ as follows: $f(q_0^0) = 0$, $f(q_0^{n+1}) = (n+2) \cdot \max\{f(q_{k_n}^n), k_{n+1}\}$ for $n \in \omega$ and $f(q_i^n) = f(q_0^n) + i$ for $i \leq k_n$, $n \in \omega$. Since $\mathcal{U}$ is an $\mathcal{A}$-friendly ultrafilter there exists $U \in \mathcal{U}$ such that $f[U] = \{v_m : m \in \omega\}$ is an almost thin set, i.e. $\limsup_{m\to\infty} \frac{v_m}{v_{m+1}} = \delta < 1$.

Claim: $|U \cap Q_n| \leq 1$ *for all but finitely many* n

Proof. First observe that according to the assumptions there exists $m_1 \in \omega$ such that $\frac{v_m}{v_{m+1}} < \frac{1+\delta}{2}$ for every $m \geq m_1$. Notice that there exists $m_2 \in \omega$ such that $\frac{m+1}{m+2} \geq \frac{1+\delta}{2}$ for every $m \geq m_2$. Now, assume for the contrary that there are infinitely many $n \in \omega$ with $|U \cap Q_n| \geq 2$. Then there exists $n \geq m_2$ such that $n \geq \max\{f^{-1}(v_m) : m < m_1\}$ and $|U \cap Q_n| \geq 2$. Consider two distinct elements $u_1, u_2 \in U \cap Q_n$, $u_1 < u_2$. Then $f(u_1) = v_m$ for some $m \geq m_1$ and $f(u_2) = v_n$ for some $n \geq m+1$. We get $\frac{v_m}{v_{m+1}} \geq \frac{v_m}{v_n} = \frac{f(u_1)}{f(u_2)} \geq \frac{f(q_0^n)}{f(q_0^n)+k_n} \geq \frac{(n+1)\cdot M}{(n+1)\cdot M+M} = \frac{n+1}{n+2}$ where $M = \max\{f(q_{k_{n-1}}^{n-1}), k_n\}$. But $\frac{n+1}{n+2} \geq \frac{1+\delta}{2}$ — a contradiction. □

Since $\mathcal{U}$ is a uniform ultrafilter it follows from the claim that there exists $V \in \mathcal{U}$ such that $V \subseteq U$, $U \setminus V$ is finite and $|V \cap Q_n| \leq 1$ for every n. Thus $\mathcal{U}$ is a Q-point. □

A different characterization of Q-points can be found in [9] where Vojtáš defined for this purpose the following ideals on ω: Assume $\mathcal{R}$ is a partition of ω into finite sets and put $\mathcal{I}_{\mathcal{R}} = \{A \subseteq \omega : (\exists k \in \omega)(\forall R \in \mathcal{R})|R \cap A| \leq k\}$. Denote by $\mathbf{R}$ the set of all such partitions $\mathcal{R} \subseteq [\omega]^{<\omega}$.

Theorem 3.5 (Vojtáš)**.** *For an ultrafilter* $\mathcal{U} \in \omega^*$ *the following assertions are equivalent:*

1) $\mathcal{U}$ *is a* Q*-point*

2) $(\forall \mathcal{R} \in \mathbf{R})$ $(\exists U \in \mathcal{U})$ $(\exists k \in \omega)$ $(\forall R \in \mathcal{R})$ $|U \cap R| \leq k$*, i.e.* $U \in \mathcal{I}_{\mathcal{R}}$*.*

Applying lemma 1.2 we may translate and extend this characterization.

Theorem 3.6. *For an ultrafilter* $\mathcal{U} \in \omega^*$ *the following are equivalent:*

1) $\mathcal{U}$ *is a* Q*-point*

2) $\mathcal{U}$ *is a weak* $\mathcal{I}_{\mathcal{R}}$*-ultrafilter for every* $\mathcal{R} \in \mathbf{R}$

3) $\mathcal{U}$ *is an* $\mathcal{I}_{\mathcal{R}}$*-friendly ultrafilter for every* $\mathcal{R} \in \mathbf{R}$

4) $\mathcal{U} \cap \mathcal{I}_{\mathcal{R}} \neq \emptyset$ *for every* $\mathcal{R} \in \mathbf{R}$

Proof. We have to prove 1) $\Rightarrow$ 2) and 4) $\Rightarrow$ 1).

1) $\Rightarrow$ 2) Assume $f : \omega \to \omega$ is a finite-to-one function and $\mathcal{R}$ is a partition of ω into finite sets. Put $\mathcal{R}_f = \{f^{-1}[R] : R \in \mathcal{R}\}$. It is again a partition of ω into finite sets. If $\mathcal{U}$ is a Q-point then there exists $U \in \mathcal{U}$ such that

$|U \cap Q| \leq 1$ for every $Q \in \mathcal{R}_f$. So we get $|f[U] \cap R| \leq 1$ for every $R \in \mathcal{R}$, i.e. $U \in \mathcal{I}_{\mathcal{R}}$, and $\mathcal{U}$ is a weak $\mathcal{I}_{\mathcal{R}}$-ultrafilter.

4) $\Rightarrow$ 1) According to the assumption for a arbitrary $\mathcal{R} \in \mathbf{R}$ there exists $U \in \mathcal{U}$ and $k \in \omega$ such that $|U \cap R| \leq k$ for every $R \in \mathcal{R}$. We can split U into k sets in such a way that each of them intersects every $R \in \mathcal{R}$ in at most one point. Since $\mathcal{U}$ is an ultrafilter, one of those sets belongs to $\mathcal{U}$. Hence $\mathcal{U}$ contains a selector for arbitrary partition $\mathcal{R} \in \mathbf{R}$, i.e. $\mathcal{U}$ is a Q-point. □

4 P-points

Definition 4.1. An ultrafilter $\mathcal{U} \in \omega^*$ is called a *P-point* if for all partitions of ω, $\{R_i : i \in \omega\}$, either for some i, $R_i \in \mathcal{U}$, or $(\exists U \in \mathcal{U})$ $(\forall i \in \omega)$ $|U \cap R_i| < \omega$.

A new description of P-points mentioned Hrušák in [5] using the ideal $\mathrm{Fin} \times \mathrm{Fin} = \{A \subseteq \omega \times \omega : \{n : \{m : \langle n, m\rangle \in A\} \notin \mathrm{Fin}\} \in \mathrm{Fin}\}$.

Theorem 4.2 (Hrušák)**.** *For an ultrafilter $\mathcal{U} \in \omega^*$ the following assertions are equivalent:*

1) $\mathcal{U}$ is a P-point

2) $\mathrm{Fin} \times \mathrm{Fin} \not\leq_K \mathcal{U}^*$

Lemma 1.1 provides a translation of the previous theorem in terms of $\mathcal{I}$-ultrafilters.

Theorem 4.3. *For an ultrafilter $\mathcal{U} \in \omega^*$ the following are equivalent:*

1) $\mathcal{U}$ is a P-point

2) $\mathcal{U}$ is a $\mathrm{Fin} \times \mathrm{Fin}$*-ultrafilter*

3) $\mathcal{U}$ is a weak $\mathrm{Fin} \times \mathrm{Fin}$*-ultrafilter*

4) $\mathcal{U}$ is a $\mathrm{Fin} \times \mathrm{Fin}$*-friendly ultrafilter*

Proof. We will prove only 1) $\Rightarrow$ 2) and 4) $\Rightarrow$ 1).

1) $\Rightarrow$ 2) It is not difficult to check that every P-point is a $\mathrm{Fin} \times \mathrm{Fin}$-ultrafilter: For every $f : \omega \to \omega \times \omega$ consider partition $\{f^{-1}[\{n\} \times \omega] : n \in \omega\}$ of ω. Since $\mathcal{U}$ is a P-point then there exists either $U \in \mathcal{U}$ such that $U = f^{-1}[\{n\} \times \omega]$ for some $n \in \omega$ (and hence $f[U] = \{n\} \times \omega \in \mathrm{Fin} \times \mathrm{Fin}$) or $|U \cap f^{-1}[\{n\} \times \omega]| < \omega$ for every $n \in \omega$ (and so $f[U] \in \mathrm{Fin} \times \mathrm{Fin}$ again).

4) $\Rightarrow$ 1) Let $\{R_n : n \in \omega\}$ be a partition of ω into infinite sets. Define a one-to-one function $f : \omega \to \omega \times \omega$ by $f(m) = \langle n, k\rangle$ if m is kth element of set R_n. Since $\mathcal{U}$ is a $\mathrm{Fin} \times \mathrm{Fin}$-ultrafilter there exists $U \in \mathcal{U}$ such that $f[U] \in \mathrm{Fin} \times \mathrm{Fin}$. So, $f[U] \subseteq V_1 \cup V_2$ where $V_1 \subseteq [0, n_0] \times \omega$ for some $n_0 \in \omega$ and $|V_2 \cap \{n\} \times \omega| < \omega$ for every $n \in \omega$. Either $f^{-1}[V_1] \in \mathcal{U}$ (and then $R_n \in \mathcal{U}$ for some $n \leq n_0$) or $f^{-1}[V_2] \in \mathcal{U}$ (and $|f^{-1}[V_2] \cap R_n| < \omega$ for every $n \in \omega$). $\square$

For a different characterization of P-points it is important to say that conv denotes the ideal generated by convergent sequences of rational numbers from $[0, 1]$ because P-points may be described as conv-ultrafilters, which follows from the following theorem:

Theorem 4.4 (Hrušák [5])**.** *For an ultrafilter $\mathcal{U} \in \omega^*$ the following assertions are equivalent:*

1) $\mathcal{U}$ is a P-point

2) $\mathsf{conv} \not\leq_K \mathcal{U}^*$

Let us conclude this section with a general combinatorial description of P-points.

Theorem 4.5 (Hrušák, Thümmel [6])**.** *$\mathcal{U}$ is a P-point if and only if for every tall analytic ideal $\mathcal{I}$ such that $\mathcal{U} \cap \mathcal{I} = \emptyset$ there is an F_σ ideal $\mathcal{J} \supseteq \mathcal{I}$ such that $\mathcal{U} \cap \mathcal{J} = \emptyset$.*

5 Selective ultrafilters

Definition 5.1. A free ultrafilter $\mathcal{U}$ is called a *selective ultrafilter* (or a Ramsey ultrafilter) if for all partitions of ω, $\{R_i : i \in \omega\}$, either for some i, $R_i \in \mathcal{U}$, or $(\exists U \in \mathcal{U})\ (\forall i \in \omega)\ |U \cap R_i| \leq 1$.

Several different descriptions of selective ultrafilters were presented by Hrušák in [5]. Here is one of them with ideal $\mathcal{ED}$ (for definition see section 3 of this paper).

Theorem 5.2 (Hrušák)**.** *For an ultrafilter $\mathcal{U} \in \omega^*$ the following assertions are equivalent:*

1) $\mathcal{U}$ is selective

2) $\mathcal{ED} \not\leq_K \mathcal{U}^*$

We will prove the translation of the theorem in terms of $\mathcal{I}$-ultrafilters.

Theorem 5.3. *For an ultrafilter $\mathcal{U} \in \omega^*$ the following are equivalent:*

1) $\mathcal{U}$ is selective

2) $\mathcal{U}$ is an $\mathcal{ED}$-ultrafilter

3) $\mathcal{U}$ is a weak $\mathcal{ED}$-ultrafilter

4) $\mathcal{U}$ is an $\mathcal{ED}$-friendly ultrafilter

Proof. We will prove 1) $\Rightarrow$ 2) and 4) $\Rightarrow$ 1) because all the other implications follow from definition.

1) $\Rightarrow$ 2) Assume $\mathcal{U}$ is a selective ultrafilter and $f : \omega \to \omega \times \omega$ an arbitrary function. Consider partition $\{f^{-1}[\{n\} \times \omega] : n \in \omega\}$. Since $\mathcal{U}$ is selective there exists $U \in \mathcal{U}$ such that either $U = f^{-1}[\{n\} \times \omega]$ for some $n \in \omega$ and then $f[U] = \{n\} \times \omega \in \mathcal{ED}$, or $|U \cap f^{-1}[\{n\} \times \omega]| \leq 1$ for every $n \in \omega$ and then $f[U] \in \mathcal{ED}$ again.

4) $\Rightarrow$ 1) Let $\{R_n : n \in \omega\}$ be a partition of ω into infinite sets. Define a one-to-one function $f : \omega \to \omega \times \omega$ by $f(m) = \langle n, k\rangle$ if m is kth element of set R_n. Since $\mathcal{U}$ is an $\mathcal{ED}$-friendly ultrafilter there exists $U \in \mathcal{U}$ such that $f[U] \subseteq V_1 \cup V_2$ where $V_1 \subseteq [0, n_0] \times \omega$ for some $n_0 \in \omega$ and there exists $m \in \omega$ such that $|V_2 \cap \{n\} \times \omega| \leq m$ for every $n \in \omega$. Either $f^{-1}[V_1] \in \mathcal{U}$ or $f^{-1}[V_2] \in \mathcal{U}$. If $f^{-1}[V_1] \in \mathcal{U}$ then $R_n \in \mathcal{U}$ for some $n \leq n_0$. If $f^{-1}[V_2] \in \mathcal{U}$ then there exists $\tilde{U} \in \mathcal{U}$ such that $\tilde{U} \subseteq f^{-1}[V_2]$ and $|\tilde{U} \cap R_n| \leq 1$ for every $n \in \omega$. $\square$

Here is another characterization of selective ultrafilters:

Theorem 5.4 (Hrušák)**.** *For an ultrafilter $\mathcal{U} \in \omega^*$ the following assertions are equivalent:*

1) $\mathcal{U}$ is selective

2) $\mathcal{I} \not\leq_K \mathcal{U}^*$ for every tall F_σ ideal $\mathcal{I}$

Again, we will prove its counterpart in the language of $\mathcal{I}$-ultrafilters.

Theorem 5.5. *For an ultrafilter $\mathcal{U} \in \omega^*$ the following are equivalent:*

1) $\mathcal{U}$ is selective

2) $\mathcal{U}$ is an $\mathcal{I}$-ultrafilter for every tall F_σ ideal $\mathcal{I}$

3) $\mathcal{U}$ is a weak $\mathcal{I}$-ultrafilter for every tall F_σ ideal $\mathcal{I}$

4) $\mathcal{U}$ is an $\mathcal{I}$-friendly ultrafilter for every tall F_σ ideal $\mathcal{I}$

5) $\mathcal{U} \cap \mathcal{I} \neq \emptyset$ for every tall F_σ ideal $\mathcal{I}$

Proof. 1) $\Rightarrow$ 2) Assume $\mathcal{I}$ is a tall F_σ ideal. Mazur in [7] proved that there exists a lsc submeasure φ such that $\mathcal{I} = \mathrm{Fin}(\varphi)$. It is easy to see that $\mathcal{I}$ is tall if and only $\lim_{n\to\infty} \varphi(\{n\}) = 0$. So we may fix a strictly increasing sequence $\langle n_k \rangle_{k\in\omega}$ such that for every $n \geq n_k$ one has $\varphi(\{n\}) < \frac{1}{2^k}$.

Let $f : \omega \to \omega$ be an arbitrary function. Now, consider the partition $\{f^{-1}[n_{k-1}, n_k) : k \in \omega\}$ (where $n_{-1} = 0$). Since $\mathcal{U}$ is a selective ultrafilter either there exists $k \in \omega$ such that $f^{-1}[n_{k-1}, n_k) = U_1 \in \mathcal{U}$, or there exists $U_2 \in \mathcal{U}$ such that $|U_2 \cap f^{-1}[n_{k-1}, n_k)| \leq 1$ for every $k \in \omega$. Observe that $f[U_1]$ is finite, hence $f[U_1] \in \mathcal{I}$, and for U_2 we get

$$\varphi(f[U_2]) \leq \max\{\varphi\{n\} : n < n_0\} + \sum_{k\in\omega} \frac{1}{2^k} < \infty$$

i.e. $f[U_2] \in \mathrm{Fin}(\varphi) = \mathcal{I}$. It follows that $\mathcal{U}$ is an $\mathcal{I}$-ultrafilter.

5) $\Rightarrow$ 1) Let $\mathcal{R} = \{R_n : n \in \omega\}$ be a partition of ω into infinite sets. Define an ideal $\mathcal{I}$ on ω by $A \in \mathcal{I}$ if either there exists $m \in \omega$ such that $A \subseteq \bigcup_{n\leq m} R_n$ or there exists $m \in \omega$ such that $|A \cap R_n| \leq m$ for every $n \in \omega$. It is not difficult to check that such an ideal is a tall F_σ ideal.

Now, assume $\mathcal{U}$ is an ultrafilter, $\mathcal{U} \cap \mathcal{I} \neq \emptyset$ and $U \in \mathcal{U} \cap \mathcal{I}$. If U is covered by a finite union of sets from $\mathcal{R}$ then $R_n \in \mathcal{U}$ for some $n \in \omega$. If $|U \cap R_n| \leq m$ for every $R_n \in \mathcal{R}$ then there is $\tilde{U} \subseteq U$ such that $|\tilde{U} \cap R_n| \leq 1$ for every $n \in \omega$. It follows that $\mathcal{U}$ is a selective ultrafilter. $\square$

To complete the overview of characterizations of selective ultrafilters, we present without proofs some more mentioned by Hrušák in [5].

Theorem 5.6 (Hrušák)**.** *For an ultrafilter $\mathcal{U} \in \omega^*$ the following assertions are equivalent:*

1) $\mathcal{U}$ is selective

2) $\mathcal{I} \not\leq_K \mathcal{U}^*$ for every tall Borel ideal $\mathcal{I}$

3) $\mathcal{I} \not\leq_K \mathcal{U}^*$ for every tall analytic ideal $\mathcal{I}$

It is also possible to describe selective ultrafilters as $\mathcal{R}$-ultrafilters where $\mathcal{R}$ is the Random Graph Ideal generated by homogeneous subsets of the countable Random Graph (for more deatils see [6]).

Theorem 5.7 (Hrušák)**.** *For an ultrafilter $\mathcal{U} \in \omega^*$ the following assertions are equivalent:*

1) $\mathcal{U}$ is selective

2) $\mathcal{R} \not\leq_K \mathcal{U}^*$

References

[1] Baumgartner, J., Ultrafilters on ω, *J. Symbolic Logic* **60**, no. 2, 624–639, 1995.

[2] Farah, I., Analytic Quotients: Theory of Liftings for Quotients over Analytic Ideals on the Integers, *Mem. Amer. Math. Soc.* **148**, no. 702, 2000.

[3] Flašková, J., Ultrafilters and small sets, Ph.D. Thesis, Charles University, Prague, 2006.

[4] Hernández-Hernández, F., Hrušák, M., Cardinal invariants of analytic P-ideals, *Can. J. Math.* **59**, no. 3, 575–595, 2007.

[5] Hrušák, M., talk in Prague Set theory seminar, August 2007.

[6] Hrušák, M., Thümmel, E., Ramsey type properties of ideals ona countable set, *preprint.*

[7] Mazur, K., F_σ-ideals and $\omega_1\omega_1^*$-gaps in the Boolean algebra $\mathcal{P}(\omega)/I$, *Fund. Math.* **138**, 103 – 111, 1991.

[8] Solecki, S., Analytic ideals and their applications, *Ann. Pure and Applied Logic* **99**, 51 – 72, 1999.

[9] Vojtáš, P., On ω^* and absolutely divergent series, *Topology Proceedings* **19**, 335 – 348, 1994.