RESEARCH STATEMENT

I wrote my master thesis "Compactification and connectedness" under the supervision of prof. Petr Simon from Charles University, Prague. As the title suggests the topic of the thesis was from general topology. During my Ph.D. studies my mathematical interest changed towards set theory, especially infinite combinatorics and descriptive set theory. I finished my Ph.D. thesis "Ultrafilters and small sets" again under the supervision of prof. Petr Simon.

Ultrafilters play an important role in set theory and topology, but also in many other areas of mathematics. Ultrafilters are used in forcing theory to construct a generic extension over a ground model, from model theory comes the ultraproduct construction which is important in algebra. Ultrafilters may be viewed as quantifiers or as some convergence operators for compact Hausdorff spaces. The most typical way of introducing an ultrafilter, however, is the set-theoretic definition as a non-empty family of subsets of a given set that is closed under supersets, finite intersections and for every set contains either the set itself or its complement; or the topological view of ultrafilters as points in the Čech-Stone compactification.

The attention of many mathematicians has been attracted in particular by ultrafilters on the set ω of natural numbers which are topologically viewed as points in the Čech-Stone compactification $\beta\omega$. This is one of the most important topological spaces and the long list of open problems on $\beta\omega$ in Open Problems in Topology [6] shows that it is still not very well understood in many respects.

Past research

For me the most intriguing topic concerning ultrafilters on natural numbers have been two attempts to connect ultrafilters with families of „small" sets – I-ultrafilters and 0-points.

The key notion for my Ph.D. thesis was the concept of an I-ultrafilter which was introduced by Baumgartner [1]: Let I be a family of subsets of a given set X, such that I contains all singletons in X and is closed under subsets (i.e. with every set it contains also its subsets). A free ultrafilter $\mathcal{U}$ on natural numbers is called I-ultrafilter, if for every mapping $f : \omega \to X$ there exists a set $U \in \mathcal{U}$ such that $f[U] \in I$.

Further, two weakenings of the notion have been also investigated: A free ultrafilter $\mathcal{U}$ on natural numbers is called weak I-ultrafilter (resp. I-friendly ultrafilter), if for every finite-to-one (resp. one-to-one) mapping $f : \omega \to X$ there exists $U \in \mathcal{U}$ such that $f[U] \in I$. The latter notion generalizes the definition of 0-points which was given by Gryzlov [5]: A free ultrafilter $\mathcal{U}$ on ω is called a 0-point if for every one-to-one function $f : \omega \to \omega$ there exists a set $U \in \mathcal{U}$ such that $f[U] \in Z_0$ where $Z_0 = \{A \subseteq \omega : \limsup_{n\to\infty} \frac{|A\cap(n+1)|}{n+1} = 0\}$ is the ideal of sets with asymptotic density zero.

I studied in my Ph.D. thesis I-ultrafilters in the setting $X = \omega$ and I is an ideal on ω or another family of „small" subsets of natural numbers that contains finite sets and is closed under subsets. I considered as I the ideal Z_0, the summable ideal $I_{1/n} = \{A \subseteq \omega : \sum_{n \in A} 1/(n+1) < \infty\}$, or the family of (almost) thin sets or (SC)-sets (we call an increasingly enumerated set $A = \{a_n : n \in \omega\}$ an (SC)-set if $\lim_{n\to\infty} a_{n+1} - a_n = +\infty$). I proved that the existence of corresponding I-ultrafilters is consistent with ZFC, investigated sums and products of such ultrafilters and studied their relations to other well-known classes of ultrafilters among others to P-points or Q-points.

In the past, I obtained a **new description of Q-points** in terms of I-ultrafilters which provides a new view of Q-points as weak thin ultrafilters. Closely related with this result is the fact that **thin ultrafilters and almost thin ultrafilters coincide**, which means that even for distinct families I and J (thin and almost thin sets in this case) the corresponding classes of I-ultrafilters may coincide.

Assuming Martin's Axiom I constructed **a hereditarily rapid ultrafilter that is not a Q-point**. This strengthens the result of Bukovský and Copláková (rapid ultrafilters need not be Q-points) and provides better understanding of the relation between Q-points and rapid ultrafilters, which is important for the eventual construction of a model where rapid ultrafilters exist, but no Q-points (till now no such model is known).

Baumgartner showed that P-points can be described as I-ultrafilters in two ways: If $X = 2^\omega$ then P-points are precisely the I-ultrafilters for I consisting of all finite and converging sequences, if $X = \omega_1$ then P-points are precisely the I-ultrafilters for $I = \{A \subseteq \omega_1 : A$ has order type $\leq \omega\}$. While I was trying to find an analogous description of P-points in the case $X = \omega$, I obtained some partial (negative) answers: (assuming Martin's Axiom) **P-points cannot be described as I-ultrafilters for any $\mathbf{F_\sigma}$-ideal or tall P-ideal I**. The reason why P-points cannot be characterized as I-ultrafilters for a tall P-ideal I is the fact that **I-ultrafilters are closed under products for every tall P-ideal I**, which is an interesting fact on its own.

I proved that **there exists in ZFC an $\mathbf{I_{1/n}}$-friendly ultrafilter** for the summable ideal which improves Gryzlov's result concerning the existence of 0-points. An important consequence of Gryzlov's result is that the set $A = \{\mathcal{U} \in \omega^* : Z_0^* \subseteq \mathcal{U}\}$, consisting of all ultrafilters that extend the dual filter to Z_0, is a non-trivial ZFC example of a nowhere dense subset of the remainder of the Čech-Stone compactification of ω, which fulfills the requirement from the question 235 in reference [7]: For what nowhere dense sets $A \subseteq \omega^*$ do we have $\bigcup_{\pi \in S_\omega} \pi[A] \neq \omega^*$? In fact, Gryzlov's example (inspired by a question of van Douwen [4]) has been for a long time the only known ZFC example of such a nowhere dense set. My result covers another part of van Douwen's question and implies that $A = \{\mathcal{U} \in \omega^* : I_{1/n}^* \subseteq \mathcal{U}\}$ is another ZFC example of a nowhere dense subset of ω^* with the property from the question 235.

Present and future research

I pursued the investigation after defending my thesis and was able to strengthen some results from it and obtained also some new results.

In summer 2008 I spent three weeks in Japan where I started scientific collaboration with prof. Jörg Brendle who studied I-ultrafilters in the past in [3] and I expect that this fruitful collaboration will continue in the future regardless in which place in the world I will be. We are both interested in questions about I-ultrafilters. Here are some of the problems I would like to solve together with him or separately:

- Do I-ultrafilters exist in ZFC if I is the ideal of sets with asymptotic density zero? What about the summable ideal, i.e. the ideal consisting of subsets of natural numbers for which $\sum_{n \in A} \frac{1}{n} < \infty$?
- Is there an ultrafilter $\mathcal{U}$ on ω such that for every one-to-one function $f : \omega \to \omega$ there exists $U \in \mathcal{U}$ with $\sum_{n \in f[U]} \frac{1}{\sqrt{n}} < \infty$? Or even $\sum_{n \in f[U]} \frac{1}{\ln n} < \infty$?

Another topic I am especially interested in are products and sums of ultrafilters. Czech mathematicians, especially Frolík and Katětov, contributed significantly to this area and it is a matter of honor for me to participate in this stream of mathematical research. I am working on a paper about products and sums of I-ultrafilters. For a large class of ideals the product of I-ultrafilters is again an I-ultrafilter, for other this is not true and the question is open for some other ideals, e.g. van der Waerden ideal $\mathcal{W}$, which consists of subsets of ω that do not contain arithmetic progressions of arbitrary length.

- Is it true that the product of two $\mathcal{W}$-ultrafilters is a $\mathcal{W}$-ultrafilter?

I expect that I will continue research in general topology in the future. There are two areas I am particularly interested in: Firstly, there are some unsolved questions about connected compactifications and compactifications with connected remainder. Secondly, I would like to investigate some special subclasses of the class of all sequentially compact spaces, which I have already started to do in one of my papers. One of the questions from the first group of problems is

- Has the square of the Sorgenfrey line a connected compactification?

Although there are no set theorists or topologists at University of West Bohemia where I work, I found opportunity to grow mathematically – as long as my teaching duties allowed me I attended regularly set theory seminar in Prague and once or twice a year research seminar in KGRC Wien. I enjoyed also participating at research and education seminars of the geometry group or the group of discrete mathematics in our department.

My Ph.D. thesis and all preprints are available at
`http://home.zcu.cz/~flaskova/english`

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