Rapid ultrafilters and summable ideals

Jana Flašková

Department of Mathematics
University of West Bohemia in Pilsen

Trends in Set Theory
11. 7. 2012

Rapid ultrafilters

Definition.

A free ultrafilter $\mathcal{U}$ on ω is called rapid if the enumeration functions of its sets form a dominating family in $(\omega^\omega, \leq^*)$.

Rapid ultrafilters

Definition.

A free ultrafilter $\mathcal{U}$ on ω is called rapid if the enumeration functions of its sets form a dominating family in $(\omega^\omega, \leq^*)$.

Theorem (Booth?).

(CH) Rapid ultrafilters exist.

Theorem (Miller).

In Laver's model there are no rapid ultrafilters.

Summable ideals

Definition.

Given a function $g : \mathbb{N} \to [0, \infty)$ such that $\sum_{n \in \mathbb{N}} g(n) = +\infty$ then the family

$$\mathcal{I}_g = \{A \subseteq \mathbb{N} : \sum_{a \in A} g(a) < +\infty\}$$

is a proper ideal which we call summable ideal determined by function g.

A summable ideal is tall (dense) if and only if $\lim_{n \to \infty} g(n) = 0$.

Characterization of rapid ultrafilters

Theorem (Vojtáš).

The following are equivalent for an ultrafilter $\mathcal{U} \in \omega^*$:

- $\mathcal{U}$ is rapid
- $\mathcal{U} \cap \mathcal{I}_g \neq \emptyset$ for every tall summable ideal $\mathcal{I}_g$

Characterization of rapid ultrafilters

Theorem (Vojtáš).

The following are equivalent for an ultrafilter $\mathcal{U} \in \omega^*$:

- $\mathcal{U}$ is rapid
- $\mathcal{U} \cap \mathcal{I}_g \neq \emptyset$ for every tall summable ideal $\mathcal{I}_g$

One can add two more equivalent conditions:

- $(\forall f : \omega \to \mathbb{N}$ one-to-one$)$ $(\exists U \in \mathcal{U})$ such that $f[U] \in \mathcal{I}_g$ for every tall summable ideal $\mathcal{I}_g$

Characterization of rapid ultrafilters

Theorem (Vojtáš).

The following are equivalent for an ultrafilter $\mathcal{U} \in \omega^*$:

- $\mathcal{U}$ is rapid
- $\mathcal{U} \cap \mathcal{I}_g \neq \emptyset$ for every tall summable ideal $\mathcal{I}_g$

One can add two more equivalent conditions:

- $(\forall f : \omega \to \mathbb{N}$ one-to-one$)$ $(\exists U \in \mathcal{U})$ such that $f[U] \in \mathcal{I}_g$ for every tall summable ideal $\mathcal{I}_g$
- $(\forall f : \omega \to \mathbb{N}$ finite-to-one$)$ $(\exists U \in \mathcal{U})$ such that $f[U] \in \mathcal{I}_g$ for every tall summable ideal $\mathcal{I}_g$ (= $\mathcal{U}$ is a weak $\mathcal{I}_g$-ultrafilter for every $\mathcal{I}_g$)

$\mathcal{I}_g$-ultrafilters

Definition.

An ultrafilter $\mathcal{U} \in \omega^*$ is called an $\mathcal{I}_g$-ultrafilter if for every $f : \omega \to \mathbb{N}$ there exists $U \in \mathcal{U}$ such that $f[U] \in \mathcal{I}_g$.

$\mathcal{I}_g$-ultrafilters

Definition.

An ultrafilter $\mathcal{U} \in \omega^*$ is called an $\mathcal{I}_g$-ultrafilter if for every $f : \omega \to \mathbb{N}$ there exists $U \in \mathcal{U}$ such that $f[U] \in \mathcal{I}_g$.

It is not known whether $\mathcal{I}_g$-ultrafilters exist in ZFC.

$\mathcal{I}_g$-ultrafilters

Definition.

An ultrafilter $\mathcal{U} \in \omega^*$ is called an $\mathcal{I}_g$-ultrafilter if for every $f : \omega \to \mathbb{N}$ there exists $U \in \mathcal{U}$ such that $f[U] \in \mathcal{I}_g$.

It is not known whether $\mathcal{I}_g$-ultrafilters exist in ZFC.

Theorem 1.

(MA_{ctble}) There exists $\mathcal{U} \in \omega^*$ such that $\mathcal{U}$ is an $\mathcal{I}_g$-ultrafilter for every tall summable ideal $\mathcal{I}_g$.

Rapid ultrafilters vs. $\mathcal{I}_g$-ultrafilters

Rapid ultrafilters need not be $\mathcal{I}_g$-ultrafilters.

Rapid ultrafilters vs. $\mathcal{I}_g$-ultrafilters

Rapid ultrafilters need not be $\mathcal{I}_g$-ultrafilters.

Theorem 2.

(MA_{ctble}) There is a rapid ultrafilter which is not an $\mathcal{I}_g$-ultrafilter for any summable ideal $\mathcal{I}_g$.

$\mathcal{I}_g$-ultrafilters vs. rapid ultrafilters

If $\mathcal{U} \in \omega^*$ is an $\mathcal{I}_g$-ultrafilter for every tall summable ideal $\mathcal{I}_g$ then $\mathcal{U}$ is a rapid ultrafilter.

$\mathcal{I}_g$-ultrafilters vs. rapid ultrafilters

If $\mathcal{U} \in \omega^*$ is an $\mathcal{I}_g$-ultrafilter for every tall summable ideal $\mathcal{I}_g$ then $\mathcal{U}$ is a rapid ultrafilter.

Theorem 3.

($\mathrm{MA}_{\mathrm{ctble}}$) There is an $\mathcal{I}_{\frac{1}{n}}$-ultrafilter which is not a rapid ultrafilter.

$\mathcal{I}_g$-ultrafilters vs. rapid ultrafilters

If $\mathcal{U} \in \omega^*$ is an $\mathcal{I}_g$-ultrafilter for every tall summable ideal $\mathcal{I}_g$ then $\mathcal{U}$ is a rapid ultrafilter.

Theorem 3.

($\mathrm{MA}_{\mathrm{ctble}}$) There is an $\mathcal{I}_{\frac{1}{n}}$-ultrafilter which is not a rapid ultrafilter.

Theorem 4.

($\mathrm{MA}_{\sigma-\mathrm{centered}}$) For every tall summable ideal $\mathcal{I}_g$ there is an $\mathcal{I}_g$-ultrafilter which is not rapid.

Possible extension and its limits

Is it possible that an ultrafilter is an $\mathcal{I}_g$-ultrafilter for “many" tall summable ideals simultaneously and still not a rapid ultrafilter?

Possible extension and its limits

Is it possible that an ultrafilter is an $\mathcal{I}_g$-ultrafilter for “many" tall summable ideals simultaneously and still not a rapid ultrafilter?

Proposition 5.

There is a family $\mathcal{D}$ of tall summable ideals such that $|\mathcal{D}| = \mathfrak{d}$ and an ultrafilter $\mathcal{U} \in \omega^*$ is rapid if and only if it has a nonempty intersection with every tall summable ideal in $\mathcal{D}$.

Possible extension and its limits

Proposition 6.

If $\mathcal{D}$ is a family of tall summable ideals and $|\mathcal{D}| < \mathfrak{b}$ then there exists a tall summable ideal $\mathcal{I}_g$ such that $\mathcal{I}_g \subseteq \mathcal{I}_h$ for every $\mathcal{I}_h \in \mathcal{D}$.

Possible extension and its limits

Proposition 6.

If $\mathcal{D}$ is a family of tall summable ideals and $|\mathcal{D}| < \mathfrak{b}$ then there exists a tall summable ideal $\mathcal{I}_g$ such that $\mathcal{I}_g \subseteq \mathcal{I}_h$ for every $\mathcal{I}_h \in \mathcal{D}$.

Corollary 7.

$(MA_{\sigma-\text{centered}})$ If $\mathcal{D}$ is a family of tall summable ideals and $|\mathcal{D}| < \mathfrak{d}$ then there exists an ultrafilter $\mathcal{U} \in \omega^*$ such that $\mathcal{U}$ is an $\mathcal{I}$-ultrafilter for every $\mathcal{I} \in \mathcal{D}$, but $\mathcal{U}$ is not a rapid ultrafilter.

$\mathcal{I}_g$-ultrafilters need not be rapid

Proof of Theorem 4.

Theorem 4.

($MA_{\sigma-\text{centered}}$) For every tall summable ideal $\mathcal{I}_g$ there is an $\mathcal{I}_g$-ultrafilter which is not rapid.

$\mathcal{I}_g$-ultrafilters need not be rapid

Proof of Theorem 4.

Theorem 4.

($MA_{\sigma-\text{centered}}$) For every tall summable ideal $\mathcal{I}_g$ there is an $\mathcal{I}_g$-ultrafilter which is not rapid.

Proposition 4a.

For every tall summable ideal $\mathcal{I}_g$ there is a tall summable ideal $\mathcal{I}_h$ such that $\mathcal{I}_g \not\leq_K \mathcal{I}_h$.

$\mathcal{I}_g$-ultrafilters need not be rapid

Proof of Theorem 4.

Theorem 4.

($MA_{\sigma-\text{centered}}$) For every tall summable ideal $\mathcal{I}_g$ there is an $\mathcal{I}_g$-ultrafilter which is not rapid.

Proposition 4a.

For every tall summable ideal $\mathcal{I}_g$ there is a tall summable ideal $\mathcal{I}_h$ such that $\mathcal{I}_g \not\leq_K \mathcal{I}_h$.

Theorem 4b.

($MA_{\sigma-\text{centered}}$) For arbitrary tall summable ideals $\mathcal{I}_g$ and $\mathcal{I}_h$ such that $\mathcal{I}_g \not\leq_K \mathcal{I}_h$ there is an $\mathcal{I}_g$-ultrafilter $\mathcal{U}$ with $\mathcal{U} \cap \mathcal{I}_h = \emptyset$.

$\mathcal{I}_g$-ultrafilters need not be rapid

Proof of Theorem 4. — more details

Theorem 4b.

($\text{MA}_{\sigma-\text{centered}}$) For arbitrary tall summable ideals $\mathcal{I}_g$ and $\mathcal{I}_h$ such that $\mathcal{I}_g \not\leq_K \mathcal{I}_h$ there is an $\mathcal{I}_g$-ultrafilter $\mathcal{U}$ with $\mathcal{U} \cap \mathcal{I}_h = \emptyset$.

$\mathcal{I}_g$-ultrafilters need not be rapid

Proof of Theorem 4. — more details

Theorem 4b.

(MA$_{\sigma-\text{centered}}$) For arbitrary tall summable ideals $\mathcal{I}_g$ and $\mathcal{I}_h$ such that $\mathcal{I}_g \not\leq_K \mathcal{I}_h$ there is an $\mathcal{I}_g$-ultrafilter $\mathcal{U}$ with $\mathcal{U} \cap \mathcal{I}_h = \emptyset$.

1. Enumerate all functions in ${}^\omega\omega$ as $\{f_\alpha : \alpha < \mathfrak{c}\}$.
2. For $\alpha < \mathfrak{c}$ construct filter bases $\mathcal{F}_\alpha$ such that for every $\alpha < \mathfrak{c}$:
   - (i) $\mathcal{F}_0$ is the Fréchet filter
   - (ii) $\mathcal{F}_\alpha \supseteq \mathcal{F}_\beta$ whenever $\alpha \geq \beta$
   - (iii) $\mathcal{F}_\gamma = \bigcup_{\alpha<\gamma} \mathcal{F}_\alpha$, for γ limit
   - (iv) $(\forall\alpha)\ |\mathcal{F}_\alpha| \leq |\alpha + 1| \cdot \omega$
   - (v) $(\forall\alpha)\ \mathcal{F}_\alpha \cap \mathcal{I}_h = \emptyset$
   - (vi) $(\forall\alpha)\ (\exists F \in \mathcal{F}_{\alpha+1})\ f_\alpha[F] \in \mathcal{I}_g$

$\mathcal{I}_g$-ultrafilters need not be rapid

Proof of Theorem 4. — more details

Lemma 4c.

($MA_{\sigma-\text{centered}}$) Assume $\mathcal{I}_g$ and $\mathcal{I}_h$ are two tall summable ideals such that $\mathcal{I}_g \not\leq_K \mathcal{I}_h$. Assume $\mathcal{F}$ is a filter base with $|\mathcal{F}| < \mathfrak{c}$ such that $\mathcal{F} \cap \mathcal{I}_h = \emptyset$ and a function $f \in \omega^\omega$ is given. Then there exists $G \subseteq \omega$ such that $f[G] \in \mathcal{I}_g$ and $G \cap F \notin \mathcal{I}_h$ for every $F \in \mathcal{F}$.

$\mathcal{I}_g$-ultrafilters need not be rapid

Proof of Theorem 4. — more details

Lemma 4c.

($MA_{\sigma-\text{centered}}$) Assume $\mathcal{I}_g$ and $\mathcal{I}_h$ are two tall summable ideals such that $\mathcal{I}_g \not\leq_K \mathcal{I}_h$. Assume $\mathcal{F}$ is a filter base with $|\mathcal{F}| < \mathfrak{c}$ such that $\mathcal{F} \cap \mathcal{I}_h = \emptyset$ and a function $f \in \omega^\omega$ is given. Then there exists $G \subseteq \omega$ such that $f[G] \in \mathcal{I}_g$ and $G \cap F \notin \mathcal{I}_h$ for every $F \in \mathcal{F}$.

Lemma 4d.

($MA_{\sigma-\text{centered}}$) Assume $\mathcal{I}_h$ is a tall summable ideal and $\mathcal{F}$ is a filter base with $|\mathcal{F}| < \mathfrak{c}$ such that $\mathcal{F} \cap \mathcal{I}_h = \emptyset$. Then there exists a set $H \subseteq \omega$ such that $H \notin \mathcal{I}_h$ and $H \setminus F$ is finite for every $F \in \mathcal{F}$.

$\mathcal{I}_g$-ultrafilters need not be rapid

Proof of Theorem 4. — more details

Lemma 4e.

Assume $f \in \omega^\omega$, $\mathcal{I}_g$ and $\mathcal{I}_h$ are tall summable ideals with $\mathcal{I}_g \not\leq_K \mathcal{I}_h$. If H is an infinite subset of ω such that $H \notin \mathcal{I}_h$ and $f[H] \notin \mathcal{I}_g$ then there exists $A \subseteq f[H]$ such that $A \in \mathcal{I}_g$ and $f^{-1}[A] \cap H \notin \mathcal{I}_h$.

$\mathcal{I}_g$-ultrafilters need not be rapid

Proof of Theorem 4. — more details

Lemma 4e.

Assume $f \in \omega^\omega$, $\mathcal{I}_g$ and $\mathcal{I}_h$ are tall summable ideals with $\mathcal{I}_g \not\leq_K \mathcal{I}_h$. If H is an infinite subset of ω such that $H \notin \mathcal{I}_h$ and $f[H] \notin \mathcal{I}_g$ then there exists $A \subseteq f[H]$ such that $A \in \mathcal{I}_g$ and $f^{-1}[A] \cap H \notin \mathcal{I}_h$.

Lemma 4f.

Assume $\mathcal{I}_g$ is a tall summable ideal determined by a decreasing function g, A is a subset of ω and $B \subseteq A$. Then

1. $A \in \mathcal{I}_g$ if and only if $A + 1 \in \mathcal{I}_g$
2. $A \in \mathcal{I}_g$ if and only if $B + 1 \cup (A \setminus B) \in \mathcal{I}_g$

References

J. Flašková, Rapid ultrafilters and summable ideals, *preprint*, http://home.zcu.cz/~flaskova/english/index.html

J. Flašková, $\mathcal{I}$-ultrafilters and summable ideals, in: *Proceedings of the 10th Asian Logic Conference*, Kobe 2008.

A. Miller, There are no Q-points in Laver's model for the Borel conjecture, *Proc. Amer. Math. Soc.* **78** (1980), 498 – 502.

P. Vojtáš, On ω^* and absolutely divergent series, *Topology Proceedings* **19** (1994), 335 – 348.