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DOCTORAL THESIS

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Ultrafilters and small sets

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I declare this thesis is all my own work and I used only the quoted literature. The thesis is freely available for lending.

Prague, January 18, 2006 Jana Flašková

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Contents

Introduction

Ultrafilters on natural numbers have been receiving much attention over the years and many results and constructions of special types of ultrafilters became part of mathematical folklore. There have been several attempts to connect ultrafilters with families of “small” sets. For our purposes two of them are important — Gryzlov’s 0-points and $\mathscr{I}$-ultrafilters introduced by Baumgartner. Both notions denote ultrafilters that contain “small” sets where “smallness” is defined by zero asymptotic density in the first case and a prescribed family $\mathscr{I}$ in the second case. Not only the ultrafilter itself contains such a set, but also many other ultrafilters, images under permutations in the first case and under all functions in the second case.

Gryzlov defined 0-points in his talk during the 12th Winter School on Abstract Analysis in Srní and he constructed such ultrafilters in ZFC (see [17], [18]). His investigation was stimulated by a question of van Douwen.

The definition of $\mathscr{I}$-ultrafilter which was given by Baumgartner in [2]: Let $\mathscr{I}$ be a family of subsets of a set X such that $\mathscr{I}$ contains all singletons and is closed under subsets. Given a free ultrafilter $\mathscr{U}$ on ω, we say that $\mathscr{U}$ is an $\mathscr{I}$-ultrafilter if for any $F : \omega \to X$ there is $A \in \mathscr{U}$ such that $F[A] \in \mathscr{I}$.

Baumgartner defined in his article discrete ultrafilters, scattered ultrafilters, measure zero ultrafilters and nowhere dense ultrafilters which he obtained by taking $X = 2^\omega$, the Cantor set, and $\mathscr{I}$ the collection of discrete sets, scattered sets, sets with closure of measure zero, nowhere dense sets respectively. If we let $\mathscr{I}$ be the collection of sets with countable closure then we obtain countably closed ultrafilters which were introduced by Brendle [6]. Yet another class of $\mathscr{I}$-ultrafilters was introduced by Barney in [1] by taking $\mathscr{I}$ to be the sets with σ-compact closure. All these classes of ultrafilters are proved to be pairwise distinct under some additional set-theoretic assumptions (Continuum Hypothesis or some form of Martin’s Axiom). It seems that some additional set-theoretic assumptions cannot be avoided completely when speaking about $\mathscr{I}$-ultrafilters because Shelah [25] proved that it is consistent with ZFC that there are no nowhere dense ultrafilters, which implies that the existence of any of these ultrafilters (being a subclass of nowhere dense ultrafilters) is not provable in ZFC. (These “topologically” defined ultrafilters are of certain importance, e.g., in the forcing theory, as the result of Błaszczyk and Shelah [4] shows).

Another example of $\mathscr{I}$-ultrafilters are ordinal ultrafilters which were defined also in [2] by taking $X = \omega_1$ and $\mathscr{I} = \{A \subseteq \omega_1 : A \text{ has order type } \leq \alpha\}$ for an indecomposable ordinal α. It was also Baumgartner [2] and Brendle [6] who studied $\mathscr{I}$-ultrafilters in this setting and presented several interesting results, but it remains an open question whether these “ordinal” ultrafilters exist in ZFC. The ambition of this thesis, however, is not to solve this intriguing question whose solution probably requires advanced forcing techniques.

We study in this thesis $\mathscr{I}$-ultrafilters in the setting $X = \omega$ and $\mathscr{I}$ is an ideal on ω or another family of “small” subsets of natural numbers that contains finite sets and is closed under subsets. So we consider as $\mathscr{I}$ the family of sets with asymptotic density zero, the summable ideal or the family of thin sets or (SC)-sets. We prove that it is consistent with ZFC that such ultrafilters exist and investigate sums and product of these ultrafilters.

We investigate also relationships of such ultrafilters to other well-known classes of ultrafilters among others to P-points which can be described as $\mathscr{I}$-ultrafilters in two ways: If $X = 2^\omega$ then P-points are precisely the $\mathscr{I}$-ultrafilters for $\mathscr{I}$ consisting of all finite and converging sequences, if $X = \omega_1$ then P-points are precisely the $\mathscr{I}$-ultrafilters for $\mathscr{I} = \{A \subseteq \omega_1 : A \text{ has order type } \leq \omega\}$ (see [2]). It seems that there is no family $\mathscr{I}$ of subsets of natural numbers such that P-points are precisely the $\mathscr{I}$-ultrafilters, but we can relate all the introduced classes of $\mathscr{I}$-ultrafilters to P-points.

Finally, we approach the position of Gryzlov. We weaken the notion of $\mathscr{I}$-ultrafilter so that we restrict the functions considered in definition of an $\mathscr{I}$-ultrafilter to finite-to-one functions at first and then to one-to-one functions and we construct in ZFC such an ultrafilter with the summable ideal chosen for $\mathscr{I}$, which strengthens Gryzlov's result.

The structure of the dissertation is as follows: After reviewing basic notions we introduce in chapter 1 several collections of “small” subsets that we use to define corresponding classes of $\mathscr{I}$-ultrafilters. Chapter 2 is devoted entirely to $\mathscr{I}$-ultrafilters and the relationship of various classes of ultrafilters and it contains, for instance, a construction of a hereditarily rapid ultrafilter that is not a Q-point. Sums and products are studied in chapter 3. The thesis ends with chapter 4 in which we adopt Gryzlov's approach. We focus on weaker forms of $\mathscr{I}$-ultrafilters and construct a summable ultrafilter.

Some parts of this thesis have been already published or accepted for publication. Some results from section 2.3 can be found in [13] or [14] (eventually under different set-theoretic assumptions); section 4.2 is based on [15].

Basic notions

Given a non-empty set X we will denote by $\mathscr{P}(X)$ the power set of X, i.e., the set of all subsets of X. The set of all natural numbers is ω and we denote $\mathbb{N} = \omega \setminus \{0\}$. We will denote by $\mathfrak{c}$ the cardinality of the continuum or the cardinality of $\mathscr{P}(\omega)$.

The set of all finite subsets of ω is denoted by $[\omega]^{<\omega}$, the set of all infinite subsets of ω by $[\omega]^{\omega}$ as usual. We denote the set of all functions from ω to ω by ${}^{\omega}\omega$. Let us recall the quasiorder $\leq^*$ on ${}^{\omega}\omega$: for $f, g \in {}^{\omega}\omega$ we write $f \leq^* g$ if and only if there is $n \in \omega$ such that $f(m) \leq g(m)$ for every $m \geq n$. A family $\mathcal{F} \subseteq {}^{\omega}\omega$ is called a dominating family in $({}^{\omega}\omega, \leq^*)$ if for every $g \in {}^{\omega}\omega$ there exists $f \in \mathcal{F}$ with $g \leq^* f$.

Continuum Hypothesis and Martin's Axiom

It was already stated in the introduction that some additional set-theoretic assumptions seem to be necessary when speaking about $\mathscr{I}$-ultrafilters. We mention here two of them: the Continuum Hypothesis and Martin's Axiom.

The Continuum Hypothesis (CH in abbreviation), $2^{\omega} = \omega_1$, enables us for example enumerate all functions from ω to ω by countable ordinals. MA stands for Martin's Axiom, which is implied by CH, but not equivalent to it (see [20]).

We deal mostly with Martin's Axiom for countable posets (in abbreviation $\mathrm{MA}_{\mathrm{ctble}}$), which is a weaker form of Martin's Axiom. However, before we say what $\mathrm{MA}_{\mathrm{ctble}}$ is, let us recall some definitions concerning posets.

Let $(P, \leq_P)$ be a poset. A set $D \subseteq P$ is *dense in* P if $(\forall p \in P)\ (\exists q \leq_P p)$ $q \in D$. A set $\mathscr{G} \subseteq P$ is a *filter in* P if $(\forall p, q \in \mathscr{G})\ (\exists r \in \mathscr{G})\ r \leq_P p\ \&\ r \leq_P q$ and $(\forall p \in \mathscr{G})\ (\forall q \in P)\ p \leq_P q$ implies $q \in \mathscr{G}$.

$\mathrm{MA}_{\mathrm{ctble}}$ is the statement: Whenever $(P, \leq_P)$ is a non-empty countable poset, and $\mathscr{D}$ is a family of $< \mathfrak{c}$ dense subsets of P, then there is a filter $\mathscr{G}$ (called a *$\mathscr{D}$-generic filter*) in P such that $(\forall D \in \mathscr{D})\ \mathscr{G} \cap D \neq \emptyset$.

Filters and ideals

Let X be a nonempty set and $\mathscr{F} \subseteq \mathscr{P}(X)$, $\mathscr{F} \neq \emptyset$. We say that $\mathscr{F}$ is

a *k-linked family* if $F_0 \cap F_1 \cap \cdots \cap F_k$ is infinite whenever $F_i \in \mathscr{F}$, $i \leq k$.

a *centered system* if $\mathscr{F}$ is k-linked for every k.

a *filter base* if $F \neq \emptyset$ for every $F \in \mathscr{F}$ and if $F_1, F_2 \in \mathscr{F}$ then there is $F \in \mathscr{F}$ such that $F \subseteq F_1 \cap F_2$.

a *filter (on X)* if $\mathscr{F}$ is a filter base and whenever $F \in \mathscr{F}$ and $F \subseteq G \subseteq X$ then $G \in \mathscr{F}$.

an *ultrafilter (on X)* if $\mathscr{F}$ is a filter and for every $M \subseteq X$ either M or $X \setminus M$ belongs to $\mathscr{F}$, i.e., $\mathscr{F}$ is a maximal filter on X.

Observe that a filter on X is precisely a filter in the poset $(\mathscr{P}(X), \subseteq)$.

If $\mathscr{F}$ is a centered system then the smallest filter (base) that contains $\mathscr{F}$ is called the *filter (base) generated by* $\mathscr{F}$ and we denote it by $\langle\langle \mathscr{F} \rangle\rangle$ ($\langle \mathscr{F} \rangle$). To obtain a filter base we have to add all finite intersections of sets from $\mathscr{F}$ and we have to add all supersets of sets in the filter base to get a filter. An example: if $\mathscr{F} \subseteq \mathscr{P}(X)$ is a filter base that is closed under finite intersections and $A \subseteq X$ such that $\mathscr{F} \cup \{A\}$ is a centered system (we say that A is *compatible with* F) then $\langle \mathscr{F} \cup A \rangle = \mathscr{F} \cup \{A\} \cup \{F \cap A : F \in \mathscr{F}\}$ and $\langle\langle \mathscr{F} \rangle\rangle = \{M \subseteq X : (\exists F \in \mathscr{F}) F \cap A \subseteq M\}$.

An (ultra)filter $\mathscr{F}$ is called *free* if $\bigcap\{U : U \in \mathscr{F}\} = \emptyset$ and it is called *fixed* (or principal) if $\bigcap\{U : U \in \mathscr{F}\} \neq \emptyset$.

The *character of* $\mathscr{F}$ is the minimal cardinality of a subfamily of $\mathscr{F}$ that generates $\mathscr{F}$, we write $\chi(\mathscr{F}) = \min\{|\mathscr{B}| : \mathscr{B} \subseteq \mathscr{F}, \langle\langle \mathscr{B} \rangle\rangle = \mathscr{F}\}$.

An ideal is a dual notion to filter. Hence $\mathscr{I} \subseteq \mathscr{P}(X)$ is an *ideal on* X if it is a non-empty proper subset of $\mathscr{P}(X)$ and it is closed under subsets and finite unions.

If $\mathscr{F} \subseteq \mathscr{P}(X)$ is a filter then $\mathscr{F}^* = \{X \setminus F : F \in \mathscr{F}\}$ is the *dual ideal to* $\mathscr{F}$ and if $\mathscr{I} \subseteq \mathscr{P}(X)$ is an ideal then $\mathscr{I}^* = \{X \setminus A : A \in \mathscr{I}\}$ is the *dual filter to* $\mathscr{I}$.

A basic example of an ideal is the principal ideal $\mathscr{I}_A = \{B \subseteq X : B \subseteq A\}$ for a given $A \subseteq X$ or the Fréchet ideal, the family of all finite subsets of the given set. The dual filter is called the Fréchet filter and consists of cofinite subsets. Dual ideals to ultrafilters are called maximal ideals.

The smallest ideal that contains a family $\mathcal{A} \subseteq \mathscr{P}(\omega)$ is the *ideal generated by* $\mathcal{A}$, denoted $\langle A \rangle$. A family $\mathcal{A}$ that generates an ideal $\mathscr{I}$ is the *base of* $\mathscr{I}$ and the *character of* $\mathscr{I}$ is the minimal cardinality of a base of the ideal, i.e., $\chi(\mathscr{I}) = \min\{|\mathcal{A}| : (\forall I \in \mathscr{I})(\exists k \in \omega)(\exists A_1, \ldots, A_k \in \mathcal{A})\, I \subseteq A_1 \cup \cdots \cup A_k\}$.

The following definition is crucial for our future considerations:

An ideal $\mathscr{I} \subseteq \mathscr{P}(\omega)$ is called *tall* if every $A \notin \mathscr{I}$ contains an infinite subset that belongs to the ideal $\mathscr{I}$.

For every $A, B \subseteq \omega$ we say that A is *almost contained* in B and we write $A \subseteq^* B$ if $A \setminus B$ is finite. Using this notation we recall the definition of the *pseudointersection number*:

$\mathfrak{p} = \min\{|\mathscr{F}| : \mathscr{F} \subseteq \mathscr{P}(\omega) \text{ is centered}, \neg((\exists A \in [\omega]^\omega)(\forall F \in \mathscr{F}) A \subseteq^* F)\}$

It is not difficult to prove that $\chi(\mathscr{I}) \geq \mathfrak{p}$ for every tall ideal $\mathscr{I}$.

We say that an ideal $\mathscr{I}$ is a *P-ideal* if whenever $A_n \in \mathscr{I}$, $n \in \omega$, then there is $A \in \mathscr{I}$ such that $A_n \subseteq^* A$ for every n.

Rudin-Keisler order and Katětov order

The Čech-Stone compactification of ω, denoted by $\beta\omega$, is the unique (up to homeomorphism) compact space that contains ω as a dense subset and such that for every compact space K and every continuous function $f : \omega \to K$ there is a continuous extension $\beta f : \beta\omega \to K$ called the Stone extension. It implies that every function $f : \omega \to \omega$ has its Stone extension $\beta f : \beta\omega \to \beta\omega$.

We identify points of $\beta\omega$ with ultrafilters on ω. The points of the remainder $\omega^* = \beta\omega \setminus \omega$ correspond to the free ultrafilters on ω, the fixed ultrafilters are identified with points of ω.

Let $\mathscr{U}, \mathscr{V} \in \beta\omega$. Observe that $\mathscr{B} = \{f[U] : U \in \mathscr{U}\}$ is a filter base. We denote by $\beta f(\mathscr{U})$ the filter generated by $\mathscr{B}$ and it is easy to check that $\beta f(\mathscr{U})$ is indeed an ultrafilter. It is easily verified that $\beta f(\mathscr{U}) = \mathscr{V}$ iff $(\forall U \in \mathscr{U})\ f[U] \in \mathscr{V}$ iff $(\forall V \in \mathscr{V})\ f^{-1}[V] \in \mathscr{U}$.

We write $\mathscr{U} \approx \mathscr{V}$ if there exists a permutation π of ω such that $\beta\pi(\mathscr{V}) = \mathscr{U}$. It is clear that the relation $\approx$ is an equivalence relation on $\beta\omega$.

For $\mathscr{U}, \mathscr{V} \in \beta\omega$ we write $\mathscr{U} \leq_{RK} \mathscr{V}$ iff there is $f \in {}^{\omega}\omega$ such that $\beta f(\mathscr{V}) = \mathscr{U}$. The relation $\leq_{RK}$ is a quasiorder since the relation is not antisymmetric, but we get the *Rudin-Keisler order* if we consider the quotient relation defined by $\leq_{RK}$ on $\beta\omega/\approx$.

Katětov order $\leq_K$ is an extension of the Rudin-Keisler order to arbitrary filters or ideals. We write $\mathscr{F} \leq_K \mathscr{G}$ if there exists a function $f : \omega \to \omega$ such that $f^{-1}[U] \in \mathscr{G}$ for every $U \in \mathscr{F}$. It is easy to check that $\mathscr{F} \leq_K \mathscr{G}$ if and only if $\mathscr{F}^* \leq_K \mathscr{G}^*$.

We say that $\mathcal{C} \subseteq \beta\omega$ is *closed downward under* $\leq_{RK}$ if $\mathscr{U} \in \mathcal{C}$ and $\mathscr{V} \leq_{RK} \mathscr{U}$ implies $\mathscr{V} \in \mathcal{C}$.

Some well-known ultrafilters

We will investigate relations between some classes of $\mathscr{I}$-ultrafilters and several well-known classes of ultrafilters in chapter 2. We summarize in this section the definitions and equivalent descriptions of ultrafilters on ω that we will consider. Two types of ultrafilters, hereditarily Q-points and hereditarily rapid ultrafilters namely, are newly introduced.

A free ultrafilter $\mathscr{U}$ is called a *P-point* if for all partitions of ω, $\{R_i : i \in \omega\}$, either for some i, $R_i \in \mathscr{U}$, or $(\exists U \in \mathscr{U})\ (\forall i \in \omega)\ |U \cap R_i| < \omega$. An equivalent combinatorial description is: a free ultrafilter $\mathscr{U}$ is a P-point if and only if whenever $U_n \in \mathscr{U}$, $n \in \omega$, there is $U \in \mathscr{U}$ such that $U \subseteq^* U_n$ for each n (i.e. P-points are dual filters to maximal P-ideals). The class of P-points is downward closed under Rudin-Keisler order (see e.g. [11]).

A free ultrafilter $\mathscr{U}$ is called a *selective ultrafilter* (or a Ramsey ultrafilter) if for all partitions of ω, $\{R_i : i \in \omega\}$, either for some i, $R_i \in \mathscr{U}$, or $(\exists U \in \mathscr{U})$ $(\forall i \in \omega)$ $|U \cap R_i| \leq 1$. We will profit also from the following equivalent characterization of selective ultrafilters: if $\mathscr{U}$ is a selective ultrafilter on ω then for every $f \in {}^{\omega}\omega$ there is $U \in \mathscr{U}$ such that $f \restriction U$ is either one-to-one or constant (see [11]). It is also proved in [11] that selective ultrafilters are minimal in Rudin-Keisler order on ultrafilters.

Every selective ultrafilter is a P-point, but the converse is not true.

A free ultrafilter $\mathscr{U}$ is called a *Q-point* if for every partition $\{Q_n : n \in \omega\}$ of ω into finite sets there exists $U \in \mathscr{U}$ such that $|U \cap Q_n| \leq 1$ for every $n \in \omega$. The notion of Q-point was introduced by Choquet [10]. An equivalent description of Q-points, known also as rare ultrafilters, was given by Mathias in [21]: an ultrafilter $\mathscr{U}$ is a Q-point if every finite-to-one function is one-to-one on a set in $\mathscr{U}$.

It folows from the definition that every selective ultrafilter is a Q-point. A Q-point need not be a selective ultrafilter, but it is selective if the ultrafilter is also a P-point.

A free ultrafilter $\mathscr{U}$ is called a *rapid ultrafilter* if the enumeration functions of its sets form a dominating family in $({}^{\omega}\omega, \leq^*)$ where enumeration function of a set A is the unique strictly increasing function e_A from ω onto A. Rapid ultrafilters (called also semi-Q-points by some authors) are due to Choquet [10] resp. Mokobodzki [23].

We say that a free ultrafilter $\mathscr{U}$ is a *hereditarily Q-point (rapid ultrafilter)* if it is a Q-point (rapid ultrafilter) such that for every $\mathscr{V} \leq_{RK} \mathscr{U}$ the ultrafilter $\mathscr{V}$ is again a Q-point (rapid ultrafilter).

It is known that every Q-point is a rapid ultrafilter and, obviously, every hereditarily Q-point is then a hereditarily rapid ultrafilter. Bukovský, Copláková showed in [7] under additional set-theoretic assumptions that rapid ultrafilter need not be a Q-point. This result is strengthened in Propositon 2.4.6 in chapter 2 where we construct a hereditarily rapid ultrafilter which is not a Q-point assuming Martin's Axiom for countable posets.

It is consistent that all the above mentioned types of ultrafilters exist under various set-theoretic assumptions (for instance, Booth [5] proved that selective ultrafilters exist if Martin's Axiom holds). However, there exist various models of set theory showing that it is consistent with ZFC that no such ultrafilters exist. A model with no P-points constructed Shelah [24]. Miller [22] showed that there are no Q-points (or even rapid ultrafilters) in the Laver model.

1 Small subsets of natural numbers

Several collections of "small" subsets of ω are presented in this chapter. We summarize definitions, give some equivalent descriptions in some cases and show the relationships among the various types of small sets.

The following diagram summarizes inclusions between the classes of thin sets, (SC)-sets, (S)-sets and (H)-sets which are defined in the following sections.

$$\begin{tikzcd}
 & \text{thin sets} \arrow[dl] \arrow[dr] & \\
(SC)\text{-sets} \arrow[dr] & & (S)\text{-sets} \arrow[dl] \\
 & (H)\text{-sets} &
\end{tikzcd}$$

Some more ideals on ω are described in the last section of this chapter. They do not appear in the diagram because their relation to the other classes is not clear enough.

1.1 Thin and almost thin sets

Definition 1.1.1. An infinite set $A \subseteq \omega$ with enumeration $A = \{a_n : n \in \mathbb{N}\}$ is called *thin* (see [3]) if $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 0$.

An example of a thin set is the set $\{n! : n \in \omega\}$. The family of thin sets is the smallest subset of $\mathscr{P}(\omega)$ we will consider. A slightly larger collection of subsets of ω represent the almost thin sets.

Definition 1.1.2. An infinite set $A \subseteq \omega$ with enumeration $A = \{a_n : n \in \mathbb{N}\}$ is called *almost thin* if $\limsup_{n\to\infty} \frac{a_n}{a_{n+1}} < 1$.

It is obvious that every thin set is almost thin. The converse is not true, see for example the set $\{2^n : n \in \omega\}$.

Neither the family of thin sets nor the family of almost thin sets is an ideal. To see this consider sets $A = \{n! : n \in \omega\}$ and $B = \{n! + 1 : n \in \omega\}$, which are thin but the union $A \cup B$ is not even almost thin.

We will denote the ideal generated by thin sets by $\mathscr{T}$ and the ideal generated by almost thin sets by $\mathscr{A}$. Both ideals extend the Fréchet ideal. Obviously, $\mathscr{A} \supseteq \mathscr{T}$ and the following lemma shows that the ideals do not coincide.

Lemma 1.1.3. *$A = \{2^n : n \in \omega\} \in \mathscr{A} \setminus \mathscr{T}$.*

Proof. Since A is almost thin it belongs to the ideal $\mathscr{A}$. Assume for the contrary that there are thin sets $A_1, \dots, A_k$ such that $A \subseteq A_1 \cup \dots \cup A_k$. For every $i = 1, \dots, k$ there is $n_i \in \omega$ such that whenever a, b are two elements of A_i with $n_i < a < b$ then $\frac{a}{b} < \frac{1}{2^k}$. Let $n_0 = \max\{n_i : i = 1, \dots, k\}$ and consider the set $\{2^{n_0}, 2^{n_0+1}, \dots, 2^{n_0+k}\}$. Each of its $k+1$ elements belongs to A_i for some i, so there is i_0 such that A_{i_0} contains two of them. For these elements we have $n_{i_0} < a < b$ and $\frac{a}{b} \geq \frac{2^{n_0}}{2^{n_0+k}} = \frac{1}{2^k}$ — a contradiction. □

Lemma 1.1.4. *Let A be a subset of ω. If $A \in \mathscr{T}$ then $(\exists k \in \omega)$ $(\forall n \in \omega)$ $|A \cap [2^n, 2^n + n]| < k$.*

Proof. Assume for the contrary that $A \in \mathscr{T}$ and $(\forall k \in \omega)$ $(\exists n \in \omega)$ $|A \cap [2^n, 2^n + n]| \geq k$. It follows from $A \in \mathscr{T}$ that there are thin sets $A_1, \dots, A_m$ such that $A = \bigcup_{i=1}^m A_i$. For every $i = 1, \dots, m$ there exists n_i such that the ratio of any two successive elements in A_i which are greater than n_i is less than $\frac{1}{2}$. Let $n_0 = \max\{m+1, n_i : i = 1, \dots, m\}$. According to the assumption there exists $n \in \omega$ such that $|A \cap [2^n, 2^n + n]| \geq n_0$. Now from the Pigeon Hole Principle we have $|A_i \cap [2^n, 2^n + n]| \geq 2$ for some i. Hence there are two successive elements in A_i greater than n_i whose ratio is greater (or equal to) $\frac{2^n}{2^n+n} > \frac{1}{2}$ — a contradiction. □

Lemma 1.1.5. *Neither $\mathscr{A}$ nor $\mathscr{T}$ are P-ideals.*

Proof. Consider thin sets $A_k = \{n! + k : n \in \omega\}$, $k \in \omega$. We want to prove that whenever $A \subseteq \omega$ contains all but finitely many elements of each A_k then A cannot be written as a finite union of almost thin sets, i.e. $A \notin \mathscr{A}$.

Let $A = \bigcup_{j \leq l} B_j$. There exists $n_0 \geq l$ such that the interval $[n!, n! + l]$ is contained in A for every $n \geq n_0$. Therefore one of the sets B_j contains two of its elements. Since there are only finitely many sets B_j, but infinitely many $n \geq n_0$ there exists B_j such that $|B_j \cap [n!, n! + l]| \geq 2$ for infinitely many n. It follows that $B_j = \{b_n^j : n \in \omega\}$ is not almost thin because

$$\limsup_{n \to \infty} \frac{b_n^j}{b_{n+1}^j} \geq \limsup_{n \to \infty} \frac{n!}{n! + l} = 1.$$

□

1.2 Sets with property (SC) and (C)

Given $A \subseteq \omega$ and $k \in \omega$ we define $A + k = \{n : n - k \in A\}$.

Definition 1.2.1. We say that set $A \subseteq \omega$ *has property* (SC), in short, A is an (SC)*-set*, if $(A + k) \cap A$ is finite for all $k \in \mathbb{N}$.

Lemma 1.2.2. *Every almost thin set has property* (SC).

Proof. Let $A = \{a_n : n \in \mathbb{N}\}$. If there is $k \in \mathbb{N}$ such that $\{n : a_n + k \in A\}$ is infinite then

$$\limsup_{n\to\infty} \frac{a_n}{a_{n+1}} \geq \limsup_{n\to\infty} \frac{a_n}{a_n + k} = 1$$

and A is not almost thin. $\square$

The set of all squares of natural numbers $\{n^2 : n \in \omega\}$ has property (SC) and it is not almost thin.

Lemma 1.2.3. *Set* $A = \{a_n : n \in \mathbb{N}\} \subseteq \omega$ *has property* (SC) *if and only if for every* $k \in \mathbb{N}$ *there is* n_k *such that* $a_{n+1} - a_n > k$ *for every* $n \geq n_k$.

Proof. Assume first that A has property (SC). Then $M_i = \{n : a_n + i \in A\}$ is finite for every $i \leq k$. Let $n_k = 1 + \max \bigcup_{i \leq k} M_i$. It is easy to see that $a_{n+1} - a_n > k$ whenever $n \geq n_k$.

On the other hand, if for every $k \in \mathbb{N}$ there is n_k such that $a_{n+1} - a_n > k$ for every $n \geq n_k$ then $(A + k) \cap A \subseteq \{a_1, a_2, \ldots, a_{n_k}\}$ is finite. $\square$

The family of sets with property (SC) is not an ideal, consider, e.g., the union of sets $\{n^2 : n \in \omega\}$ and $\{n^2 + 1 : n \in \omega\}$ that does not satisfy condition (SC). But it still satisfies a weaker condition (C).

Definition 1.2.4. We say a set $A \subseteq \omega$ has *property* (C) if $(A + k) \cap A$ is finite for all but finitely many $k \in \mathbb{N}$.

However, even the larger family of sets with property (C) is not an ideal. There exist two thin sets whose union does not fulfil condition (C).

Example 1.2.5. Let us enumerate prime numbers as $\{p_k : k \in \omega\}$. Put $A = \{n! : n \in \omega\}$, $B = \{(p_k{}^n)! + k : k, n \in \omega\}$. We know that A is thin. If $b = (p_k{}^n)! + k \in B$ and $b' \in B$ is the immediate succesor of b in B then $\frac{b}{b'} \leq \frac{(p_k{}^n)!+k}{(p_k{}^n+1)!} \leq \frac{2(p_k{}^n)!}{(p_k{}^n+1)!} = \frac{2}{p_k{}^n+1}$. Hence B is thin. Obviously, $A \cup B$ does not satisfy condition (C) since $((A \cup B) + k) \cap (A \cup B) \supseteq \{(p_k{}^n)! + k : n \in \omega\}$ is infinite for each $k \in \mathbb{N}$.

It follows from the definition that every set with property (SC) has property (C). Although the converse implication is not true it turns out that the ideals generated by families (SC) and (C) coincide.

Lemma 1.2.6. *Families* (C) *and* (SC) *generate the same ideal on* ω.

Proof. It suffices to prove that every set with property (C) belongs to the ideal generated by sets with property (SC). Assign to every set A with property (C) finite set $K_A = \{k \in \mathbb{N} : (A+k) \cap A \text{ is infinite}\}$. We will proceed by induction on $n = |K_A|$.

If $K_A = \emptyset$ then A has property (SC) and it trivially belongs to the ideal.

Now, suppose that every set B with $|K_B| \leq n$ is a finite union of sets with property (SC) and consider A with $|K_A| = n+1$. Define $k = \max K_A$ and set $A_0 = (A+k) \cap A$ and $A_1 = A \setminus A_0$. We get $(A_0+k) \cap A_0 \subseteq (A+2k) \cap A$ which is a finite set, and $(A_1+k) \cap A_1 \subseteq (A+k) \cap A \cap A_1 = \emptyset$ hence $|K_{A_0}| \leq |K_A \setminus \{k\}| = n$ and $|K_{A_1}| \leq |K_A \setminus \{k\}| = n$. According to the induction assumption A_0 and A_1 can be written as a finite union of sets with property (SC), thus the set $A = A_0 \cup A_1$ belongs to the ideal generated by (SC)-sets. $\square$

Lemma 1.2.7. *Let A be a subset of ω. Set A belongs to the ideal generated by (SC)-sets if and only if* $(\exists k)$ $(\forall d)$ $(\exists n_d)$ $(\forall n \geq n_d)$ $|[n, n+d] \cap A| \leq k$.

Proof. If $A = A_1 \cup \cdots \cup A_k$ where $A_i \in (SC)$ for $i = 1, \ldots, k$ then according to Lemma 1.2.3 there is n_d^i, $i = 1, \ldots, k$, such that $|[n, n+d] \cap A_i| \leq 1$ for every $n \geq n_d^i$. Let $n_d = \max\{n_d^i : i = 1, \ldots, k\}$. It is obvious that $(\forall n \geq n_d)$ $|[n, n+d] \cap A| \leq k$.

On the other hand, if $(\exists k)$ $(\forall d)$ $(\exists n_d)$ $(\forall n \geq n_d)$ $|[n, n+d] \cap A| \leq k$ then put $A_i = \{a_{mk+i} : m \in \omega\}$ where $\{a_m : m \in \omega\}$ is an increasing enumeration of A. Obviously, $A = A_1 \cup \cdots \cup A_k$ and it is easy to see that $A_i \in (SC)$ for every $i = 1, \ldots, k$ because $|A_i \cap [n, n+d]| \leq 1$ whenever $n \geq n_d$. $\square$

Lemma 1.2.8. *The ideal generated by (SC)-sets is not a P-ideal.*

Proof. Consider thin sets $A_k = \{n! + k : n \in \omega\}$, $k \in \omega$, as in the proof of Lemma 1.1.5. They have property (SC) and we prove that whenever $A \subseteq \omega$ contains all but finitely many elements of each A_k then A cannot be written as finite union of sets with property (SC).

Let $A = \bigcup_{j \leq l} B_j$. There exists $n_0 \geq l$ such that the interval $[n!, n!+l]$ is contained in A for every $n \geq n_0$. Therefore one of the sets B_j contains two of its elements. Since there are only finitely many sets B_j, but infinitely many $n > n_0$ there exists B_j such that $|B_j \cap [n!, n!+l]| \geq 2$ for infinitely many n. It follows that B_j does not satisfy condition (SC) because there are infinitely many elements in B_j with difference i for some $i \leq l$. $\square$

1.3 Summable ideal

Definition 1.3.1. *Summable ideal* is the family $\{A \subseteq \mathbb{N} : \sum_{a \in A} \frac{1}{a} < +\infty\}$. We call the sets from the summable ideal *(S)-sets.*

Although the summable ideal is defined as an ideal on $\mathbb{N}$ we will often regard it as an ideal on ω (which is generated by the summable ideal on $\mathbb{N}$ and $\{0\}$).

It follows from the definition that every almost thin set belongs to the summable ideal. However, there is no inclusion between the summable ideal and the ideal generated by (SC)-sets.

Example 1.3.2. Consider the set $\bigcup\{[2^n, 2^n + n) : n \in \omega\}$. It belongs to the summable ideal because

$$\sum_{n=0}^{\infty}\sum_{i=0}^{n-1}\frac{1}{2^n+i} \leq \sum_{n=0}^{\infty}\frac{n}{2^n} < +\infty,$$

but it is obviously not in the ideal generated by (SC)-sets.

Example 1.3.3. Consider a sequence $\langle a_n : n \in \mathbb{N}\rangle$ of natural numbers defined by recursion: $a_1 = 1$ and $a_{n+1} = a_n + k$ if $n \in [2^k, 2^{k+1})$. It is easy to see that the set $A = \{a_n : n \in \mathbb{N}\}$ has property (SC). To check that it does not belong to the summable ideal observe first that for elements of A we have $a_{2^k} < k \cdot (2^k - 1)$ for every $k \geq 2$. So we get for $k \geq 2$

$$\sum_{n=2^k}^{2^{k+1}-1}\frac{1}{a_n} \geq \frac{1}{a_{2^k}} + \frac{1}{a_{2^k}+k} + \cdots + \frac{1}{a_{2^k}+k(2^k-1)} > \frac{2^k}{a_{2^k}+k(2^k-1)} > \frac{1}{2k}$$

It follows that

$$\sum_{n=1}^{\infty}\frac{1}{a_n} \geq \sum_{k=2}^{\infty}\sum_{n=2^k}^{2^{k+1}-1}\frac{1}{a_n} \geq \sum_{k=2}^{\infty}\frac{1}{2k} = +\infty$$

and set A is not in the summable ideal.

It is a known fact that summable ideal is a P-ideal, but we give the proof for the sake of completeness.

Lemma 1.3.4. *Summable ideal is a P-ideal.*

Proof. Let A_k, $k \in \omega$, be (S)-sets. For every k there is n_k such that

$$\sum_{a \in A_k \cap [n_k, +\infty)} \frac{1}{a} < \frac{1}{2^k}.$$

Set $A = \bigcup\{A_k \cap [n_k, +\infty) : k \in \omega\}$. It is easy to check that A belongs to the summable ideal and $A_k \subseteq^* A$ for every $k \in \omega$. $\square$

Lemma 1.3.5. *If $A = \{a_n : n \in \mathbb{N}\}$ is an (S)-set then $\lim_{n\to\infty} \frac{n}{a_n} = 0$.*

Proof. We will show that if $\limsup_{n\to\infty} \frac{n}{a_n} = c > 0$ then A is not in the summable ideal. Take $n_0 \in \mathbb{N}$ such that $\frac{n_0}{a_{n_0}} > \frac{c}{2}$. Then $\sum_{n=1}^{n_0} \frac{1}{a_n} \geq \frac{n_0}{a_{n_0}} > \frac{c}{2}$. By induction construct a sequence $\langle n_k \rangle_{k\in\omega}$ such that $\sum_{n=1}^{n_k} \frac{1}{a_n} \geq (k+2)\frac{c}{4}$ for every k. Assume we know already $n_0, \ldots, n_k$. Since $\limsup_{n\to\infty} \frac{n}{a_n} = c > 0$ we can choose $n_{k+1} > 2n_k$ such that $\frac{n_{k+1}}{a_{n_{k+1}}} > \frac{c}{2}$. We get $\frac{n_{k+1}-n_k}{a_{n_{k+1}}} > \frac{1}{2} \cdot \frac{n_{k+1}}{a_{n_{k+1}}}$ and $\sum_{n=1}^{n_{k+1}} \frac{1}{a_n} = \sum_{n=1}^{n_k} \frac{1}{a_n} + \sum_{n=n_k+1}^{n_{k+1}} \frac{1}{a_n} \geq (k+2)\frac{c}{4} + \frac{1}{2} \cdot \frac{c}{2} = (k+3)\frac{c}{4}$.

Finally, $\sum_{n\in\mathbb{N}} \frac{1}{a_n}$ is minorized by a divergent series, hence it diverges. $\square$

1.4 Density ideal

Definition 1.4.1. We say that *upper asymptotic density* of set $A \subseteq \omega$ is $d^*(A) = \limsup_{n\to\infty} \frac{|A\cap n|}{n}$. If $d^*(A) = 0$ then A *has asymptotic density zero*, in short, A is an (H)*-set*.

Notice that $B \subseteq^* A$ implies $d^*(B) \leq d^*(A)$.

Lemma 1.4.2. *For $A = \{a_n : n \in \mathbb{N}\} \subseteq \omega$ we have*

$$d^*(A) = \limsup_{n\to\infty} \frac{n}{a_n + 1}$$

Proof. Set $\alpha_k = \sup\{\frac{|A\cap n|}{n} : n \geq k\}$ and $\beta_k = \sup\{\frac{n}{a_n+1} : n \geq k\}$. Obviously, $\beta_k \leq \alpha_{a_k+1}$. If $n \geq a_k+1$ then there exists a unique $m \geq k$ such that $a_m+1 \leq n < a_{m+1} + 1$ and we have $\frac{|A\cap n|}{n} = \frac{m}{n} \leq \frac{m}{a_m+1} \leq \beta_k$. Hence $\beta_k \geq \alpha_{a_k+1}$. So $\langle \beta_k \rangle_{k\in\omega}$ is a subsequence of $\langle \alpha_k \rangle_{k\in\omega}$ and $\lim_{k\to\infty} \beta_k = \lim_{k\to\infty} \alpha_k$. $\square$

It follows from the definition that the collection of sets with asymptotic density zero is closed under subsets and under finite unions. Hence it is an ideal and we call the ideal density ideal.

Lemma 1.4.3. *Every (S)-set has asymptotic density zero.*

Proof. If $A = \{a_n : n \in \mathbb{N}\}$ belongs to the summable ideal then according to Lemma 1.3.5 $\lim_{n\to\infty} \frac{n}{a_n} = 0$. Then also $\lim_{n\to\infty} \frac{n}{a_n+1} = 0$ and from Lemma 1.4.2 we conclude that A has asymptotic density zero. $\square$

Lemma 1.4.4. *Every (SC)-set has asymptotic density zero.*

Proof. Assume $A = \{a_n : n \in \mathbb{N}\}$ has property (SC) and that $a_{n+1} - a_n > k$ whenever $n \geq n_k$. Define $A_k = \{a_n : n > n_k\}$. For every k we have $d^*(A_k) = \limsup_{m\to\infty} \frac{m}{a_{n_k+m+1}} \leq \limsup_{m\to\infty} \frac{m}{a_{n_k}+km+1} \leq \frac{1}{k}$. Since $A \subseteq^* A_k$ for every k, it follows that $d^*(A) = 0$. $\square$

Lemma 1.4.5. *Density ideal is a P-ideal.*

Proof. Assume A_k, $k \in \mathbb{N}$, are sets with asymptotic density zero. We may assume that $A_k \subseteq A_{k+1}$ for every $k \in \omega$ (otherwise we can switch to sets $B_k = \bigcup_{i \leq k} A_i$). Fix n_k such that $\frac{|A_k \cap n|}{n} < \frac{1}{k}$ whenever $n \geq n_k$ and let $A = \bigcup_{k \in \mathbb{N}} (A_k \cap [n_k, n_{k+1}))$. For $n \in [n_k, n_{k+1})$ we have $\frac{|A \cap n|}{n} \leq \frac{|A_k \cap n|}{n} < \frac{1}{k}$ and we get

$$\limsup_{n \to \infty} \frac{|A \cap n|}{n} \leq \lim_{k \to \infty} \frac{1}{k} = 0.$$

Hence A belongs to the density ideal and obviously $A_k \subseteq^* A$ for every k. $\square$

1.5 More ideals on ω

We say that $A \subseteq \omega$ contains an arithmetic progression of length n if there exist $a \in \omega$ and $d > 0$ such that all the members of arithmetic progression $a + j \cdot d$ for $j = 0, \ldots, n-1$ belong to the set A.

Definition 1.5.1. *Van der Waerden ideal* is the family $\mathscr{W} = \{A \subseteq \omega : A$ does not contain arithmetic progressions of arbitrary length$\}$.

It is obvious that the family $\mathscr{W}$ is closed under subsets and that $\omega \notin \mathscr{W}$. It follows from the van der Waerden Theorem that $\mathscr{W}$ is closed under finite unions and hence an ideal. We mention here two different formulations of this well-known theorem. The proof of the van der Waerden Theorem can be found for example in [16] or [9].

Theorem 1.5.2 (van der Waerden).

1. *If $A = A_1 \cup \cdots \cup A_r$ is a subset of natural numbers that contains arithmetic progressions of arbitrary length then at least one of the sets $A_1, \ldots, A_r$ has the same property.*

2. *For every $k, l \in \omega$ there exists $N(k, l) \in \omega$ such that for every colouring of the set $\{1, 2, \ldots, N(k, l)\}$ with k colours there is a homogeneous set that contains an arithmetic progression of length l.*

It is easy to see that $\mathscr{W}$ contains all finite sets. The set $\{2^n : n \in \omega\}$ is an infinite set that belongs to $\mathscr{W}$ because it contains no arithmetic progression of length 3. Another example of an infinite set in $\mathscr{W}$ is the set $\{n^2 : n \in \omega\}$ which contains no arithmetic progression of length 4 (an observation made already by L. Euler). These two examples show also that sets in $\mathscr{W}$ need not be thin or almost thin. However, every almost thin set belongs to the van der Waerden ideal.

Lemma 1.5.3. *If $A = \{a_n : n \in \mathbb{N}\} \subseteq \omega$ is an almost thin set then $A \in \mathscr{W}$.*

Proof. We want to prove that $\limsup_{n\to\infty} \frac{a_n}{a_{n+1}} = 1$ if $A = \{a_n : n \in \mathbb{N}\} \subseteq \omega$ contains arithmetic progressions of arbitrary length. In such a case for every $k \in \omega$ there are $b_k \in \omega$ and $d_k > 0$ such that $b_k + j \cdot d_k \in A$ for $j = 0, 1, \ldots, k-1$. Actually, there are infinitely many pairs b_k and d_k for every k. So there are infinitely many $n \in \mathbb{N}$ such that $a_n = b_k + (k-2)d_k$ and $b_k + (k-1)d_k \in A$. For such indices we get:

$$\frac{a_n}{a_{n+1}} \geq \frac{b_k + (k-2)d_k}{b_k + (k-1)d_k} \geq \frac{(k-2)d_k}{(k-1)d_k} = \frac{k-2}{k-1}$$

It follows from $\lim_{k\to\infty} \frac{k-2}{k-1} = 1$ that $\limsup_{n\to\infty} \frac{a_n}{a_{n+1}} = 1$ and the set A is not almost thin. □

Now, we can conclude that the set from Example 1.2.5 belongs to the ideal $\mathscr{W}$ while it does not belong to the ideal generated by (SC)-sets. In fact, there is no inclusion between the latter ideal and $\mathscr{W}$. Remember the (SC)-set from Example 1.3.3 that obviously does not belong to the ideal $\mathscr{W}$.

Surprisingly, there is an inclusion between the van der Waerden ideal and the density ideal. Szemeredi [26] proved that every set from the van der Waerden ideal has asymptotic density zero. To see that the density ideal is strictly greater consider the set $\{[n^3, n^3+n) : n \in \omega\}$ or $\{[2^n, 2^n+n) : n \in \omega\}$.

The latter set belongs not only to the density ideal, but also to the summable ideal. Hence the van der Waerden ideal and the summable ideal differ, but it is still not known whether there is an inclusion between these two ideals, which is a famous conjecture of Paul Erdös.

Conjecture 1.5.4 (Erdös). *If A is a subset of natural numbers such that $\sum_{a\in A} \frac{1}{a} = +\infty$ then A contains arithmetic progressions of arbitrary length.*

The last collection of small subsets of natural numbers that we introduce is inspired by the summable ideal and it can be found for example in [12].

Definition 1.5.5. For any function $g : \mathbb{N} \to (0, +\infty)$ we define a *generalized summable ideal* $\mathscr{I}_g$ as the family $\{A \subseteq \mathbb{N} : \sum_{a\in A} g(a) < +\infty\}$.

It is obvious that every generalized summable ideal extends the Fréchet ideal. If $\sum_{n\in\mathbb{N}} g(n) < +\infty$ then $\mathscr{I}_g = \mathscr{P}(\mathbb{N})$. If $\lim_{n\to\infty} g(n) > 0$ then $\mathscr{I}_g$ consists precisely of all finite sets. Therefore we assume in the following that $\sum_{n\in\mathbb{N}} g(n) = +\infty$ and $\lim_{n\to\infty} g(n) = 0$ to obtain a proper ideal that is strictly greater than the Fréchet filter.

Lemma 1.5.6. *Ideal $\mathscr{I}_g$ is a P-ideal for any function g.*

Proof. The proof is analogous to the proof of Lemma 1.3.4. □

2 $\mathscr{I}$-ultrafilters

In the first section of this chapter we present several general results about $\mathscr{I}$-ultrafilters. We recall the definition of $\mathscr{I}$-ultrafilter and give some necessary conditions on the existence of $\mathscr{I}$-ultrafilters. We show also that $\mathscr{I}$-ultrafilters exist in ZFC for every maximal ideal $\mathscr{I}$ with $\chi(\mathscr{I}) = \mathfrak{c}$ and it is consistent with ZFC that $\mathscr{I}$-ultrafilters exist for any tall ideal $\mathscr{I}$.

Since all the ideals defined in chapter 1 are tall it is consistent with ZFC that $\mathscr{I}$-ultrafilters for these families exist. We speak about (almost) thin ultrafilters, (SC)-ultrafilters, (S)-ultrafilters, (H)-ultrafilters, etc. and we focus in the rest of the chapter on these particular classes of ultrafilters. So we prove in the second section that thin ultrafilters and almost thin ultrafilters coincide; in the third and fourth section we study the relationships between the above mentioned classes of $\mathscr{I}$-ultrafilters and some well-known classes of ultrafilters; the fifth section contains three results on $\mathscr{W}$-ultrafilters and $\mathscr{I}_g$-ultrafilters that are not included in the previous sections.

2.1 General results

Definition 2.1.1. Let $\mathscr{I}$ be a family of subsets of a set X such that $\mathscr{I}$ contains all singletons and is closed under subsets. Given an ultrafilter $\mathscr{U}$ on ω, we say that $\mathscr{U}$ is an *$\mathscr{I}$-ultrafilter* if for any $F : \omega \to X$ there is $A \in \mathscr{U}$ such that $F[A] \in \mathscr{I}$.

In the following we will always consider $X = \omega$ although some results are true for arbitrary X.

The family $\mathscr{I}$ need not be an ideal in general, but it is enough to consider ideals on ω if we want to study the classes of $\mathscr{I}$-ultrafilters because replacing $f[U] \in \mathscr{I}$ by $f[U] \in \langle\mathscr{I}\rangle$ in the definition of $\mathscr{I}$-ultrafilter, where $\langle\mathscr{I}\rangle$ is the ideal generated by $\mathscr{I}$, we get the same concept (noticed in [2]). The following lemma shows that $\mathscr{I}$-ultrafilters and $\langle\mathscr{I}\rangle$-ultrafilters coincide.

Lemma 2.1.2. *For an ultrafilter $\mathscr{U}$ the following are equivalent:*

(i) $\mathscr{U}$ is an $\mathscr{I}$-ultrafilter

(ii) $\mathscr{U}$ is an $\langle\mathscr{I}\rangle$-ultrafilter

Proof. It suffices to prove that (ii) implies (i) since (i) implies (ii) trivially. Therefore assume that $\mathscr{U}$ is an $\langle\mathscr{I}\rangle$-ultrafilter and let $f \in {}^\omega\omega$. There exists $V \in \mathscr{U}$ such that $f[V] \in \langle\mathscr{I}\rangle$ so there are for some $k \in \omega$ sets $A_1, \ldots, A_k \in \mathscr{I}$ such that $f[V] \subseteq A_1 \cup \cdots \cup A_k$. Now $f^{-1}[A_1] \cup \cdots \cup f^{-1}[A_k] = f^{-1}[A_1 \cup \cdots \cup$

$A_k] \supseteq V \in \mathscr{U}$. So $f^{-1}[A_i] \in \mathscr{U}$ for some $i \leq k$. Put $U = f^{-1}[A_i]$. Then $U \in \mathscr{U}$ and $f[U] = A_i \in \mathscr{I}$. It follows that $\mathscr{U}$ is an $\mathscr{I}$-ultrafilter. $\square$

Let $\mathscr{I}$ be an ideal on ω. If an ultrafilter $\mathscr{U}$ extends the dual filter of $\mathscr{I}$, i.e., $\mathscr{U} \cap \mathscr{I} = \emptyset$, then $\mathscr{U}$ is not an $\mathscr{I}$-ultrafilter. However, the converse does not hold.

Example 2.1.3. Take $\mathscr{V} \in \omega^*$, $\mathscr{V} \cap \mathscr{I} = \emptyset$. Let $A = \{a_n : n \in \omega\}$ be an infinite set from the ideal $\mathscr{I}$. Define $f : \omega \to \omega$ so that $f(a_n) = n + 1$ and $f(k) = 0$ for any $k \notin A$. Now let $\mathscr{U}$ be the ultrafilter generated by $\{f^{-1}[V] : V \in \mathscr{V}\}$. Then $\mathscr{U} \cap \mathscr{I} \neq \emptyset$ since $A = f^{-1}[\omega \setminus \{0\}] \in \mathscr{U} \cap \mathscr{I}$ but $\mathscr{U}$ is not an $\mathscr{I}$-ultrafilter since $(\forall U \in \mathscr{U})\ f[U] \in \mathscr{V}$, i.e., $f[U] \notin \mathscr{I}$.

Baumgartner noticed in [1] that the class of $\mathscr{I}$-ultrafilters is closed downward under the Rudin-Keisler order $\leq_{RK}$. Recall that $\mathscr{U} \leq_{RK} \mathscr{V}$ if there is a function $f : \omega \to \omega$ whose Stone extension $\beta f : \beta\omega \to \beta\omega$ maps $\mathscr{V}$ on $\mathscr{U}$ (see [5]).

Lemma 2.1.4. *If $\mathcal{C}$ is a class of ultrafilters closed downward under $\leq_{RK}$ and $\mathscr{I}$ an ideal on ω then the following are equivalent:*

(i) There exists $\mathscr{U} \in \mathcal{C}$ which is not an $\mathscr{I}$-ultrafilter

(ii) There exists $\mathscr{V} \in \mathcal{C}$ which extends $\mathscr{I}^*$, the dual filter to $\mathscr{I}$

Proof. No ultrafilter extending $\mathscr{I}^*$ is an $\mathscr{I}$-ultrafilter, so (ii) implies (i) trivially. To prove (i) implies (ii) assume that $\mathscr{U} \in \mathcal{C}$ is not an $\mathscr{I}$-ultrafilter. Hence there is a function $f \in {}^\omega\omega$ such that $(\forall A \in \mathscr{I})\ f^{-1}[A] \notin \mathscr{U}$. Let $\mathscr{V} = \{V \subseteq \omega : f^{-1}[V] \in \mathscr{U}\}$. Obviously $\mathscr{V}$ extends $\mathscr{I}^*$ and $\mathscr{V} \leq_{RK} \mathscr{U}$. Since $\mathcal{C}$ is closed downward under $\leq_{RK}$ and $\mathscr{U} \in \mathcal{C}$ we get $\mathscr{V} \in \mathcal{C}$. $\square$

If we consider two ideals $\mathscr{I}, \mathscr{J}$ we may ask whether the classes of $\mathscr{I}$-ultrafilters and $\mathscr{J}$-ultrafilters coincide or not. The following corollary of the lemma above suggests in what form we can find ultrafilters demonstrating that the two classes are distinct.

Corollary 2.1.5. *Let $\mathscr{I}$, $\mathscr{J}$ be ideals on ω. If there is an $\mathscr{J}$-ultrafilter that is not an $\mathscr{I}$-ultrafilter then there is an $\mathscr{J}$-ultrafilter that extends $\mathscr{I}^*$.* $\square$

Let us recall the definition of Katětov order $\leq_K$ on ideals on ω. We say that $\mathscr{I} \leq_K \mathscr{J}$ if there is a function $f : \omega \to \omega$ such that $f^{-1}[A] \in \mathscr{J}$ for every $A \in \mathscr{I}$. For filters $\mathscr{F}, \mathscr{G}$ is the Katětov order defined analogously: we write $\mathscr{F} \leq_K \mathscr{G}$ if there is a function $f : \omega \to \omega$ such that $f^{-1}[F] \in \mathscr{G}$ for every $F \in \mathscr{F}$.

Lemma 2.1.6. *An ultrafilter $\mathscr{U}$ is an $\mathscr{I}$-ultrafilter if and only if $\mathscr{I}^* \not\leq_K \mathscr{U}$.*

Proof. If $\mathscr{I}^* \leq_K \mathscr{U}$ then there exists $f : \omega \to \omega$ such that $f^{-1}[A] \in \mathscr{U}$ for every $A \in \mathscr{I}^*$. Since $U \subseteq f^{-1}[f[U]] \in \mathscr{U}$ for every $U \in \mathscr{U}$ and $\mathscr{U}$ is a filter we get $f[U] \notin \mathscr{I}$ for every $U \in \mathscr{U}$ and $\mathscr{U}$ is not an $\mathscr{I}$-ultrafilter.

If $\mathscr{I}^* \not\leq_K \mathscr{U}$ then for every $f : \omega \to \omega$ there is $A \in \mathscr{I}^*$ such that $f^{-1}[A] \notin \mathscr{U}$. Since $\mathscr{U}$ is an ultrafilter we get $\omega \setminus f^{-1}[A] \in \mathscr{U}$ and we have also $f[\omega \setminus f^{-1}[A]] \subseteq \omega \setminus A \in \mathscr{I}$. Hence $\mathscr{U}$ is an $\mathscr{I}$-ultrafilter. □

Lemma 2.1.7. *Let $\mathscr{F}$ be a filter on ω and $\mathscr{U}$ a (free) ultrafilter on ω. Then $\mathscr{F} \leq_K \mathscr{U}$ if and only there is an ultrafilter $\mathscr{V}$ such that $\mathscr{V} \leq_{RK} \mathscr{U}$ and $\mathscr{V} \supseteq \mathscr{F}$.*

Proof. If $\mathscr{F} \leq_K \mathscr{U}$ then there is a function $f : \omega \to \omega$ such that $f^{-1}[F] \in \mathscr{U}$ for every $F \in \mathscr{F}$. Put $\mathscr{V} = \{A \subseteq \omega : (\exists U \in \mathscr{U}) f[U] \subseteq A\}$. It is easy to see that $\mathscr{F} \subseteq \mathscr{V}$, $\mathscr{V}$ is an ultrafilter and $\beta f(\mathscr{U}) = \mathscr{V}$. Hence $\mathscr{V} \leq_{RK} \mathscr{U}$.

If $\mathscr{V} \leq_{RK} \mathscr{U}$ then there exists $f : \omega \to \omega$ such that $f^{-1}[V] \in \mathscr{U}$ for every $V \in \mathscr{V}$. In particular, $f^{-1}[F] \in \mathscr{U}$ for every $F \in \mathscr{F} \subseteq \mathscr{V}$ and we have $\mathscr{F} \leq_K \mathscr{U}$. □

Putting together Lemma 2.1.6 and Lemma 2.1.7 we have proved the following proposition characterizing $\mathscr{I}$-ultrafilters for an ideal $\mathscr{I}$.

Proposition 2.1.8. *Let $\mathscr{I}$ be an ideal on ω. For an ultrafilter $\mathscr{U} \in \omega^*$ the following are equivalent:*

(i) $\mathscr{U}$ is an $\mathscr{I}$-ultrafilter

(ii) $\mathscr{I}^* \not\leq_K \mathscr{U}$

(iii) $\mathscr{V} \not\leq_{RK} \mathscr{U}$ for every ultrafilter $\mathscr{V} \supseteq \mathscr{I}^*$ □

As an immediate consequence of the previous proposition we get a result that generalizes the obvious fact that if $\mathscr{I} \subseteq \mathscr{J}$ then every $\mathscr{I}$-ultrafilter is a $\mathscr{J}$-ultrafilter.

Corollary 2.1.9. *If $\mathscr{I} \leq_K \mathscr{J}$ then each $\mathscr{I}$-ultrafilter is a $\mathscr{J}$-ultrafilter.* □

There are many ultrafilters that are not $\mathscr{I}$-ultrafilters for a given ideal $\mathscr{I}$ because any ultrafilter extending the dual filter of $\mathscr{I}^*$ is not an $\mathscr{I}$-ultrafilter. So there are, for instance, no $\mathscr{I}$-ultrafilters where $\mathscr{I}$ is the Fréchet ideal. However, the Fréchet ideal is not the only one ideal for which $\mathscr{I}$-ultrafilters do not exist. The following proposition provides a necessary condition on $\mathscr{I}$ for the existence of $\mathscr{I}$-ultrafilters.

Proposition 2.1.10. *There are no $\mathscr{I}$-ultrafilters for an ideal $\mathscr{I}$ which is not tall.*

Proof. Suppose that for $A \in [\omega]^\omega \setminus \mathscr{I}$ we have $\mathscr{I} \cap \mathscr{P}(A) = [A]^{<\omega}$ and let $e_A : \omega \to A$ be an increasing enumeration of the set A.

Now assume for the contrary that there exists an $\mathscr{I}$-ultrafilter $\mathscr{U} \in \omega^*$. According to the definition of an $\mathscr{I}$-ultrafilter there exists $U \in \mathscr{U}$ such that $e_A[U] \in \mathscr{I}$. Since $e_A[U] \subseteq A$ the set $e_A[U]$ is finite. It follows that U is finite because e_A is one-to-one — a contradiction to the assumption that no set in $\mathscr{U}$ is finite. $\square$

The next proposition provides a sufficient condition for the existence of $\mathscr{I}$-ultrafilters.

Proposition 2.1.11. *If $\mathscr{I}$ is a maximal ideal on ω such that $\chi(\mathscr{I}) = \mathfrak{c}$ then $\mathscr{I}$-ultrafilters exist.*

Proof. Enumerate all functions from ω to ω as $\{f_\alpha : \alpha < \mathfrak{c}\}$. By transfinite induction on $\alpha < \mathfrak{c}$ we will construct filter bases $\mathscr{F}_\alpha$ satisfying

(i) $\mathscr{F}_0$ is the Fréchet filter
(ii) $\mathscr{F}_\alpha \subseteq \mathscr{F}_\beta$ whenever $\alpha \leq \beta$
(iii) $\mathscr{F}_\gamma = \bigcup_{\alpha<\gamma} \mathscr{F}_\alpha$ for γ limit
(iv) $(\forall\alpha)\ |\mathscr{F}_\alpha| \leq |\alpha| \cdot \omega$
(v) $(\forall\alpha)\ (\exists F \in \mathscr{F}_{\alpha+1})\ f_\alpha[F] \in \mathscr{I}$

Suppose we know already $\mathscr{F}_\alpha$. If there is a set $F \in \mathscr{F}_\alpha$ such that $f_\alpha[F] \in \mathscr{I}$ then put $\mathscr{F}_{\alpha+1} = \mathscr{F}_\alpha$. Hence we may assume that $f_\alpha[F] \notin \mathscr{I}$. Then $\omega \setminus f_\alpha[F] \in \mathscr{I}$ for every $F \in \mathscr{F}_\alpha$ and since $\chi(\mathscr{I}) = \mathfrak{c} > |\mathscr{F}_\alpha|$ we can find $M \in \mathscr{I}$ such that $M \cap f_\alpha[F]$ is infinite for every $F \in \mathscr{F}_\alpha$. To complete the induction step let $\mathscr{F}_{\alpha+1}$ be the filter base generated by $\mathscr{F}_\alpha$ and $f_\alpha^{-1}[M]$.

It is obvious that any ultrafilter that extends the filter base $\mathscr{F} = \bigcup_{\alpha<\mathfrak{c}} \mathscr{F}_\alpha$ is an $\mathscr{I}$-ultrafilter. $\square$

Proposition 2.1.11 may be considered as a special case of a result proved by Butkovičová in [8] not using the terminology of $\mathscr{I}$-ultrafilters. We present here the theorem reformulated in terms of $\mathscr{I}$-ultrafilters.

Theorem 2.1.12 (Butkovičová). *Let $\mathscr{I}$ be a maximal ideal on ω such that $\chi(\mathscr{I}) = \mathfrak{c}$ and assume κ is a cardinal, $\kappa < \mathfrak{c}$. There exist 2^κ (distinct) $\mathscr{I}$-ultrafilters.*

The last two results in this section are consistency results. Proposition 2.1.13 states that $\mathscr{I}$-ultrafilters exist for every tall ideal $\mathscr{I}$ under the assumption $\mathfrak{p} = \mathfrak{c}$ (this is a slightly stronger assumption than $\mathrm{MA}_{\mathrm{ctble}}$). Finally, it turns out in Proposition 2.1.14 that $\mathscr{I}$-ultrafilters need not be P-points if we assume Continuum Hypothesis and $\mathscr{I}$ is a tall P-ideal.

Proposition 2.1.13. *($\mathfrak{p} = \mathfrak{c}$) If $\mathscr{I}$ is a tall ideal then $\mathscr{I}$-ultrafilters exist.*

Proof. Enumerate all functions from ω to ω as $\{f_\alpha : \alpha < \mathfrak{c}\}$. By transfinite induction on $\alpha < \mathfrak{c}$ we will construct filter bases $\mathscr{F}_\alpha$ satisfying

(i) $\mathscr{F}_0$ is the Fréchet filter

(ii) $\mathscr{F}_\alpha \subseteq \mathscr{F}_\beta$ whenever $\alpha \leq \beta$

(iii) $\mathscr{F}_\gamma = \bigcup_{\alpha<\gamma} \mathscr{F}_\alpha$ for γ limit

(iv) $(\forall \alpha)$ $|\mathscr{F}_\alpha| \leq |\alpha| \cdot \omega$

(v) $(\forall \alpha)$ $(\exists F \in \mathscr{F}_{\alpha+1})$ $f_\alpha[F] \in \mathscr{I}$

Suppose we know already $\mathscr{F}_\alpha$. If there is a set $F \in \mathscr{F}_\alpha$ such that $f_\alpha[F] \in \mathscr{I}$ then put $\mathscr{F}_{\alpha+1} = \mathscr{F}_\alpha$. Hence we may assume that $f_\alpha[F] \notin \mathscr{I}$, in particular $f_\alpha[F]$ is infinite, for every $F \in \mathscr{F}_\alpha$.

Since $|\mathscr{F}_\alpha| < \mathfrak{c} = \mathfrak{p}$ there exists $M \in [\omega]^\omega$ such that $M \subseteq^* f_\alpha[F]$ for every $F \in \mathscr{F}_\alpha$. The ideal $\mathscr{I}$ is tall, so there is $A \in \mathscr{I}$ which is an infinite subset of M and we have $A \subseteq^* f_\alpha[F]$ and $f_\alpha^{-1}[A] \cap F$ is infinite for every $F \in \mathscr{F}_\alpha$. It follows that $f_\alpha^{-1}[A]$ is compatible with $\mathscr{F}_\alpha$. To complete the induction step let $\mathscr{F}_{\alpha+1}$ be the filter base generated by $\mathscr{F}_\alpha$ and $f_\alpha^{-1}[A]$.

It is easy to see that every ultrafilter that extends $\mathscr{F} = \bigcup_{\alpha<\mathfrak{c}} \mathscr{F}_\alpha$ is an $\mathscr{I}$-ultrafilter. $\square$

Proposition 2.1.14. *(CH) If $\mathscr{I}$ is a tall P-ideal on ω then there is an $\mathscr{I}$-ultrafilter which is not a P-point.*

Proof. Fix a partition $\{R_n : n \in \omega\}$ of ω into infinite sets and enumerate ${}^\omega\omega = \{f_\alpha : \alpha < \omega_1\}$. By transfinite induction on $\alpha < \omega_1$ we will construct countable filter bases $\mathscr{F}_\alpha$ satisfying

(i) $\mathscr{F}_0$ is generated by the Fréchet filter and $\{\omega \setminus R_n : n \in \omega\}$

(ii) $\mathscr{F}_\alpha \subseteq \mathscr{F}_\beta$ whenever $\alpha \leq \beta$

(iii) $\mathscr{F}_\gamma = \bigcup_{\alpha<\gamma} \mathscr{F}_\alpha$ for γ limit

(iv) $(\forall \alpha)$ $(\forall F \in \mathscr{F}_\alpha)$ $\{n : |F \cap R_n| = \omega\}$ is infinite

(v) $(\forall \alpha)$ $(\exists F \in \mathscr{F}_{\alpha+1})$ $f_\alpha[F] \in \mathscr{I}$

Suppose we know already $\mathscr{F}_\alpha$. If there is a set $F \in \mathscr{F}_\alpha$ such that $f_\alpha[F] \in \mathscr{I}$ then put $\mathscr{F}_{\alpha+1} = \mathscr{F}_\alpha$. If $(\forall F \in \mathscr{F}_\alpha)$ $f_\alpha[F] \notin \mathscr{I}$ then one of the following cases occurs.

Case A. $(\forall F \in \mathscr{F}_\alpha)$ $\{n : |f_\alpha[F \cap R_n]| = \omega\}$ is infinite

Fix an enumeration $\{F_k : k \in \omega\}$ of $\mathscr{F}_\alpha$. According to the assumption the set $M_k = \{n : |f_\alpha[F_k \cap R_n]| = \omega\}$ is infinite for all $k \in \omega$. For every $k \in \omega$, $n \in M_k$ we can find an infinite set $I_{k,n} \subseteq f_\alpha[F_k \cap R_n]$ with $I_{k,n} \in \mathscr{I}$ because $\mathscr{I}$ is tall. Since $\mathscr{I}$ is a P-ideal there exist $I \in \mathscr{I}$ such that $I_{k,n} \subseteq^* I$ for every $k \in \omega$ and $n \in M_k$. It is easy to see that for every $F_k \in \mathscr{F}_\alpha$ the set $\{n : |f_\alpha^{-1}[I] \cap F_k \cap R_n| = \omega\} \supseteq M_k$ is infinite. To complete the induction step let $\mathscr{F}_{\alpha+1}$ be the countable filter base generated by $\mathscr{F}_\alpha$ and $f_\alpha^{-1}[I]$.

Case B. $(\exists F_0 \in \mathscr{F}_\alpha)$ $\{n : |f_\alpha[F_0 \cap R_n]| = \omega\}$ is finite

Enumerate $\mathscr{F}_\alpha \setminus \{F_0\} = \{F_k : k > 0\}$. For every $k > 0$ the set $M_k = \{n : |F_0 \cap F_k \cap R_n| = \omega$ and $|f_\alpha[F_k \cap F_0 \cap R_n]| < \omega\}$ is infinite. For every $n \in M_k$ define $u_n = \max\{u \in f_\alpha[F_k \cap F_0 \cap R_n] : |f_\alpha^{-1}\{u\} \cap F_k \cap F_0 \cap R_n| = \omega\}$.

Let $A_k = \{u_n : n \in M_k\}$. If A_k is finite for some k then let $\mathscr{F}_{\alpha+1}$ be the filter base generated by $\mathscr{F}_\alpha$ and $f_\alpha^{-1}[A_k]$. Otherwise, we can choose for every k an infinite set $I_k \in \mathscr{I}$ such that $I_k \subseteq A_k$. Note that for every $u \in I_k$ we have $|f_\alpha^{-1}\{u\} \cap F_k \cap F_0 \cap R_n| = \omega\}$ for some $n \in M_k$. Since $\mathscr{I}$ is a P-ideal there exists $I \in \mathscr{I}$ such that $I_k \subseteq^* I$ for every k. It is easy to see that $\{n : |f_\alpha^{-1}[I] \cap F_k \cap F_0 \cap R_n| = \omega\}$ is infinite for every k. To complete the induction step let $\mathscr{F}_{\alpha+1}$ be the countable filter base generated by $\mathscr{F}_\alpha$ and $f_\alpha^{-1}[I]$.

Finally, let $\mathscr{F} = \bigcup_{\alpha<\omega_1} \mathscr{F}_\alpha$. It is clear that every ultrafilter which extends $\mathscr{F}$ is an $\mathscr{I}$-ultrafilter because of condition (v). The filter base $\mathscr{F}$ satisfies also condition (iv) and the following claim shows that such a filter base can be extended to an ultrafilter that is not a P-point.

Claim: If $\mathscr{F}$ is a filter base satisfying (iv) and $A \subseteq \omega$ then either $\langle \mathscr{F} \cup \{A\}\rangle$ or $\langle \mathscr{F} \cup \{\omega \setminus A\}\rangle$ satisfies (iv).

Whenever $\mathscr{F}$ is a filter base satisfying (iv) and $A \subseteq \omega$ then either for every $F \in \mathscr{F}$ exist infinitely many $n \in \omega$ such that $|A \cap F \cap R_n| = \omega$, so the filter base generated by $\mathscr{F}$ and A satisfies (iv) or there is $F_0 \in \mathscr{F}$ such that for all but finitely many $n \in \omega$ we have $|A \cap F_0 \cap R_n| < \omega$. Then since for every $F \in \mathscr{F}$ exist infinitely many $n \in \omega$ for which $|F \cap F_0 \cap R_n| = \omega$ the filter base generated by $\mathscr{F}$ and $\omega \setminus A$ satisfies (iv). Hence for every subset of ω we may extend $\mathscr{F}$ either by the set itself or its complement. Consequently, $\mathscr{F}$ may be extended to an ultrafilter satisfying (iv). $\square$

2.2 Thin and almost thin ultrafilters

Let us recall that an ultrafilter $\mathscr{U} \in \omega^*$ is an *(almost) thin ultrafilter* if for every $f \in {}^\omega\omega$ there exists $U \in \mathscr{U}$ such that $f[U]$ is (almost) thin.

Every thin ultrafilter is an almost thin ultrafilter because the corresponding families of subsets of ω are in inclusion. The following proposition states that thin ultrafilters and almost thin ultrafilters actually coincide.

Proposition 2.2.1. *Every almost thin ultrafilter is a thin ultrafilter.*

Proof. Because of the Corollary 2.1.5 it suffices to prove that every almost thin ultrafilter contains a thin set. So assume that $\mathscr{U}$ is an almost thin ultrafilter and $U_0 \in \mathscr{U}$ is an almost thin set with an increasing enumeration $U_0 = \{u_n : n \in \omega\}$. If U_0 is not thin then we have $\limsup_{n\to\infty} \frac{u_n}{u_{n+1}} = q_0 < 1$.

We may assume that the set of even numbers belongs to $\mathscr{U}$ (otherwise the roles of even and odd numbers exchange).

Define $g : \omega \to \omega$ so that $g(u_n) = 2n$, $g[\omega \setminus U_0] = \{2n+1 : n \in \omega\}$.

Since $\mathscr{U}$ is an almost thin ultrafilter there exists $U_1 \in \mathscr{U}$ such that $g[U_1]$ is almost thin. Let $U = U_0 \cap U_1 = \{u_{n_k} : k \in \omega\}$. Almost thin sets are closed under subsets, therefore $g[U] = \{g(u_{n_k}) : k \in \omega\} \subseteq g[U_1]$ is almost thin and $1 > \limsup_{k\to\infty} \frac{g(u_{n_k})}{g(u_{n_{k+1}})} = \limsup_{k\to\infty} \frac{2n_k}{2n_{k+1}}$.

We know that there is n_0 such that $(\forall n \geq n_0)\ \frac{u_n}{u_{n+1}} \leq \frac{q_0+1}{2}$ and that there is k_0 such that $(\forall k\ \geq k_0)\ n_k \geq n_0$. Hence for $k \geq k_0$ we have

$$\frac{u_{n_k}}{u_{n_{k+1}}} = \frac{u_{n_k}}{u_{n_k+1}} \cdot \ \cdots\ \cdot \frac{u_{n_{k+1}-1}}{u_{n_{k+1}}} \leq \left(\frac{q_0+1}{2}\right)^{n_{k+1}-n_k}$$

It follows from $\limsup_{k\to\infty} \frac{n_k}{n_{k+1}} < 1$ that $\lim_{k\to\infty}(n_{k+1} - n_k) = +\infty$. Hence

$$\lim_{k\to\infty} \frac{u_{n_k}}{u_{n_{k+1}}} \leq \lim_{k\to\infty} \left(\frac{q_0+1}{2}\right)^{n_{k+1}-n_k} = 0$$

and the set $U \in \mathscr{U}$ is thin. $\square$

2.3 Connections to selective ultrafilters and P-points

We know from the definition that every selective ultrafilter is a P-point. From the inclusion of coresponding ideals we obtain inclusions for the classes of thin ultrafilters, (SC)-ultrafilters, (S)-ultrafilters and (H)-ultrafilters.

The following diagram shows all inclusions between these classes of ultrafilters (an arrow stands for inclusion).

$$\begin{tikzcd}
 & \text{selective ultfs} \arrow[dl] \arrow[dr] & & \\
P\text{-points} \arrow[dr] & & \text{thin ultfs} \arrow[dl] \arrow[dr] & \\
 & (SC)\text{-ultfs} \arrow[dr] & & (S)\text{-ultfs} \arrow[dl] \\
 & & (H)\text{-ultfs} &
\end{tikzcd}$$

We will show that assuming Martin's Axiom for countable posets none of the arrows reverses and no arrow can be added.

Proposition 2.3.1. *Every selective ultrafilter is a thin ultrafilter.*

Proof. Consider the partition of ω, $\{R_n : n \in \omega\}$, where $R_0 = \{0\}$ and $R_n = [n!, (n+1)!)$ for $n > 0$. If $\mathscr{U}$ is a selective ultrafilter then there exists $U_0 \in \mathscr{U}$ such that $|U_0 \cap R_n| \leq 1$ for every $n \in \omega$. Since $\mathscr{U}$ is an ultrafilter either $A_0 = \bigcup\{R_n : n \text{ is even}\}$ or $A_1 = \bigcup\{R_n : n \text{ is odd}\}$ belongs to $\mathscr{U}$. Without loss of generality, assume $A_0 \in \mathscr{U}$. Enumerate $U = U_0 \cap A_0 \in \mathscr{U}$ as $\{u_k : k \in \omega\}$. If $u_k \in [(2m_k)!, (2m_k+1)!)$ then $u_{k+1} \geq (2m_k+2)!$ and we have $\frac{u_k}{u_{k+1}} \leq \frac{(2m_k+1)!}{(2m_k+2)!} = \frac{1}{2m_k+2} \leq \frac{1}{2k+2}$. Hence U is thin and we have proved that every selective ultrafilter contains a thin set. Selective ultrafilters are minimal points in Rudin-Keisler ordering, hence the class is downward closed under $\leq_{RK}$ and we may apply Lemma 2.1.4 to conclude that every selective ultrafilter is thin. $\square$

Corollary 2.3.2. *A free ultrafilter is selective if and only if it is a P-point and thin ultrafilter.*

Proof. Every selective ultrafilter is according to the previous proposition a thin ultrafilter and it is also known to be a P-point. Every thin ultrafilter is a Q-point (see Proposition 2.4.1), so every ultrafilter that is thin and P-point is selective. $\square$

Proposition 2.3.3. *Every P-point is an (SC)-ultrafilter.*

Proof. Let $\mathscr{U}$ be a P-point. Consider an arbitrary function $f : \omega \to \omega$. Our aim is to find $U \in \mathscr{U}$ such that $f[U] \in (SC)$.

Take arbitrary $U_0 \in \mathscr{U}$. If $f[U_0] \in (SC)$ then set $U = U_0$. Otherwise, we will proceed by induction. Suppose we know already $U_i \in \mathscr{U}$, $i = 0, 1, \ldots, k-1$, such that $U_i \subseteq U_{i-1}$ for $i > 0$ and the difference of two successive elements of $f[U_i]$ is greater or equal to 2^i for every $i < k$. Enumerate $f[U_{k-1}] = \{u_n : n \in \omega\}$. Since $\mathscr{U}$ is an ultrafilter either $f^{-1}[\{u_{2n} : n \in \omega\}] \cap U_{k-1}$ or $f^{-1}[\{u_{2n+1} : n \in \omega\}] \cap U_{k-1}$ belongs to $\mathscr{U}$. Denote this set by U_k. If $f[U_k] \in (SC)$ then let $U = U_k$. If $f[U_k] \notin (SC)$ then we may continue the induction because the difference of two successive elements of $f[U_k]$ is greater or equal to $2 \cdot 2^{k-1} = 2^k$.

If we obtain an infinite sequence of sets $U_n \in \mathscr{U}$ such that $U_n \supseteq U_{n+1}$ and the difference of two succesive elements of $f[U_n]$ is greater or equal to 2^n for every n then since $\mathscr{U}$ is a P-point there is $U \in \mathscr{U}$ such that $U \subseteq^* U_n$ for every $n \in \omega$. For this U we have $f[U] \subseteq^* f[U_n]$ for every $n \in \omega$. Thus for every $k \in \omega$ all but finitely many pairs of succesive elements in $f[U]$ have difference greater or equal to 2^k and it follows that $f[U] \in (SC)$. $\square$

Notice the following interesting consequence of the previous proposition: Since every (SC)-ultrafilter is an (H)-ultrafilter we obtain as a corollary of this proposition that every P-point is an (H)-ultrafilter. In particular, it means that every P-point contains a set with asymptotic density zero. Hence the dual filter to the density ideal is an example of a P-filter (filter dual to a P-ideal) that cannot be extended to a P-point.

In Proposition 2.1.14 we constructed under CH for a given tall P-ideal on ω an $\mathscr{I}$-ultrafilter which is not a P-point. We cannot apply the proposition to obtain a thin ultrafilter that is not a P-point because the ideal $\mathscr{T}$ is not a P-ideal (see Proposition 1.1.5. Nevertheless, we construct a thin ultrafilter which is not a P-point even under a strictly weaker assumption $\mathrm{MA}_{\mathrm{ctble}}$.

Proposition 2.3.4. *(MA_{ctble}) There exists a thin ultrafilter which is not a P-point.*

Proof. Enumerate ${}^{\omega}\omega = \{f_\alpha : \alpha < \mathfrak{c}\}$ and fix a partition $\{R_n : n \in \omega\}$ of ω into infinite sets. Our aim is to construct a filter base $\mathscr{F}$ such that for every $F \in \mathscr{F}$ there are infinitely many n such that $|F \cap R_n| = \omega$.

By transfinite induction on $\alpha < \mathfrak{c}$ we will construct filter bases $\mathscr{F}_\alpha$, $\alpha < \mathfrak{c}$, so that the following conditions are satisfied:
(i) $\mathscr{F}_0$ is generated by the Fréchet filter and $\{\omega \setminus R_n : n \in \omega\}$
(ii) $\mathscr{F}_\alpha \subseteq \mathscr{F}_\beta$ whenever $\alpha \leq \beta$
(iii) $\mathscr{F}_\gamma = \bigcup_{\alpha<\gamma} \mathscr{F}_\alpha$ for γ limit
(iv) $(\forall \alpha)$ $|\mathscr{F}_\alpha| \leq |\alpha| \cdot \omega$
(v) $(\forall \alpha)$ $(\forall F \in \mathscr{F}_\alpha)$ $\{n : |F \cap R_n| = \omega\}$ is infinite
(vi) $(\forall \alpha)$ $(\exists F \in \mathscr{F}_{\alpha+1})$ $f_\alpha[F] \in \mathscr{T}$

Suppose we know already $\mathscr{F}_\alpha$. If there is a set $F \in \mathscr{F}_\alpha$ such that $f_\alpha[F] \in \mathscr{T}$ then put $\mathscr{F}_{\alpha+1} = \mathscr{F}_\alpha$. So we may assume that $(\forall F \in \mathscr{F}_\alpha)$ $f_\alpha[F] \notin \mathscr{T}$, in particular, $f_\alpha[F]$ is infinite.

If there exists $K \in [\omega]^{<\omega}$ such that $f_\alpha^{-1}[K] \cap F \cap R_n$ is infinite for infinitely many n for every $F \in \mathscr{F}_\alpha$ then we let $\mathscr{F}_{\alpha+1}$ be the filter base generated by $\mathscr{F}_\alpha$ and $f_\alpha^{-1}[K]$. In the following we will assume that no such set exists, i.e., (♣) for every $K \in [\omega]^{<\omega}$ there is $F_K \in \mathscr{F}_\alpha$ such that $f_\alpha^{-1}[K] \cap F_K \cap R_n$ is infinite for only finitely many n.

Case I. $(\forall F \in \mathscr{F}_\alpha)$ $\{n \in \omega : |f_\alpha[F \cap R_n]| = \omega\}$ *is infinite*

Set $I_F = \{n \in \omega : f_\alpha[F \cap R_n] \text{ is infinite}\}$. Consider poset $P = \{K \in [\omega]^{<\omega} : v > u^2 \text{ whenever } [u, v] \cap f_\alpha[K] = \{u, v\}\}$ with partial order given by $K \leq_P L$ if $K = L$ or $K \supset L$ and $\min(K \setminus L) > \max L$. For every $F \in \mathscr{F}_\alpha$, $n \in I_F$ and $k \in \omega$ define $D_{F,n,k} = \{K \in P : |K \cap F \cap R_n| \geq k\}$.

Claim 1: $D_{F,n,k}$ is dense in P for every $F \in \mathscr{F}_\alpha$, $n \in I_F$, $k \in \omega$.

Take $L \in P$ arbitrary. Since $F \cap R_n$ and $f_\alpha[F \cap R_n]$ are infinite sets we may choose $L' \subseteq F \cap R_n$ such that $|L'| = k$, $\min L' > \max L$, $f_\alpha[\min L'] >$

$(\max f_\alpha[L])^2$ and $f_\alpha(u) > (f_\alpha(v))^2$ whenever $u, v \in L'$, $u > v$. Let $K = L \cup L'$. It is obvious that $K \in D_{F,n,k}$ and $K \leq_P L$.

The family $\mathscr{D} = \{D_{F,n,k} : F \in \mathscr{F}_\alpha, n \in I_F, k \in \omega\}$ consists of less than $\mathfrak{c}$ many dense sets in P. By Martin's Axiom for countable posets there exists a $\mathscr{D}$-generic filter $\mathscr{G}$. Let $U = \bigcup\{K : K \in \mathscr{G}\}$.

Now it is easy to check that U satisfies the following

• $(\forall F \in \mathscr{F}_\alpha)$ $\{n \in \omega : |U \cap F \cap R_n| = \omega\}$ is infinite

Given $F \in \mathscr{F}_\alpha$ for every $n \in I_F$ and every $k \in \omega$ there is $K \in \mathscr{G} \cap D_{F,n,k}$. So $|U \cap F \cap R_n| \geq |K \cap F \cap R_n| \geq k$ and it implies that $|U \cap F \cap R_n| = \omega$ for every $n \in I_F$.

• $f_\alpha[U] \in \mathscr{T}$

Enumerate $f_\alpha[U] = \{u_n : n \in \omega\}$. For every $n \in \omega$ there is $K_n \in \mathscr{G}$ such that $u_n, u_{n+1} \in f_\alpha[K_n]$. Since $K_n \in P$ we have $u_{n+1} > (u_n)^2$ and $\frac{u_n}{u_{n+1}} < \frac{1}{u_n}$. Thus $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} \leq \lim_{n\to\infty} \frac{1}{u_n} = 0$.

To complete the induction step let $\mathscr{F}_{\alpha+1}$ be the filter base generated by $\mathscr{F}_\alpha$ and U.

Case II. $(\exists F_0 \in \mathscr{F}_\alpha)$ $\{n \in \omega : |f_\alpha[F_0 \cap R_n]| = \omega\}$ *is finite*

For every $F \in \mathscr{F}_\alpha$ let $I_F = \{n \in \omega : F \cap F_0 \cap R_n$ is infinite and $f_\alpha[F \cap F_0 \cap R_n]$ is finite$\}$. Observe that I_F is infinite for every $F \in \mathscr{F}_\alpha$ according to the assumption. For every $n \in I_F$ define $h(n) = \max\{m \in f_\alpha[F \cap F_0 \cap R_n] : |f_\alpha^{-1}\{m\} \cap F \cap F_0 \cap R_n| = \omega\}$. The latter set is non-empty and finite, whence the definition is correct.

Claim 2: $\{h(n) : n \in I_F\}$ is infinite.

Assume for the contrary that there is $h \in \omega$ such that $h(n) \leq h$ for every $n \in I_F$. We know from (♣) that there is $F_h \in \mathscr{F}_\alpha$ such that $\{n : |f_\alpha^{-1}[0,h] \cap F_h \cap R_n| = \omega\}$ is finite. Hence $\{n : |f_\alpha^{-1}[0,h] \cap F \cap F_0 \cap F_h \cap R_n| = \omega\}$ is finite. Since $I_{F\cap F_h}$ is infinite and $I_{F\cap F_h} \subseteq^* I_F$ there are infinitely many $n \in I_F$ such that $|(F \cap F_0 \cap F_h \cap R_n) \setminus f_\alpha^{-1}[0,h]| = \omega$ while $f_\alpha[F \cap F_0 \cap F_h \cap R_n]$ is finite. It follows that we can find $m \in f_\alpha[F \cap F_0 \cap F_h \cap R_n] \setminus [0,h]$ such that $|f_\alpha^{-1}\{m\} \cap F \cap F_0 \cap F_h \cap R_n| = \omega$. We have $m > h$ — a contradiction to the definition of $h(n)$.

Choose a sequence $H_F = \langle h_i : i \in \omega\rangle \subseteq \{h(n) : n \in I_F\}$ such that $h_{i+1} > (h_i)^2$ for every i. It is obvious that H_F is thin and infinite. Remember that for every $h_i \in H_F$ there is $n_i \in I_F$ such that $f_\alpha^{-1}\{h_i\} \cap F \cap F_0 \cap R_{n_i}$ is infinite. Note that $n_i \neq n_j$ for $i \neq j$, so $|f_\alpha^{-1}[H_F] \cap F \cap F_0 \cap R_n| = \omega$ for infinitely many n.

Consider poset $P = \{K \in [\omega]^{<\omega} : v > u^2$ whenever $[u,v] \cap K = \{u,v\}\}$ with partial order given by $K \leq_P L$ if $K = L$ or $K \supset L$ and $\min(K \setminus L) > \max L$. For $F \in \mathscr{F}_\alpha$ and $k \in \omega$ define $D_{F,k} = \{K \in P : |K \cap H_F| \geq k\}$.

Claim 3: $D_{F,k}$ is dense in P for every $F \in \mathscr{F}_\alpha$ and $k \in \omega$.

Take $L \in P$ arbitrary. There is $L' \subseteq H_F \setminus [0, (\max L)^2]$ such that $|L'| = k$. Let $K = L \cup L'$. Obviously, $K \in D_{F,k}$ and $K \leq_P L$.

The family $\mathscr{D} = \{D_{F,k} : F \in \mathscr{F}_\alpha, k \in \omega\}$ consists of less than $\mathfrak{c}$ many dense sets in P. By Martin's Axiom for countable posets there exists a $\mathscr{D}$-generic filter $\mathscr{G}$. Let $H = \bigcup\{K : K \in \mathscr{G}\}$.

Now it remains to check that the filter base generated by $\mathscr{F}_\alpha$ and $f_\alpha^{-1}[H]$ satisfies conditions (v) and (vi).

• $(\forall F \in \mathscr{F}_\alpha)$ $\{n \in \omega : |f_\alpha^{-1}[H] \cap F \cap R_n| = \omega\}$ is infinite

For every $F \in \mathscr{F}_\alpha$ and for every $k \in \omega$ there exists $K \in \mathscr{G} \cap D_{F,k}$. It follows that $|\{n : |f_\alpha^{-1}[H] \cap F \cap R_n| = \omega\}| \geq |\{n : |f_\alpha^{-1}[K] \cap F \cap R_n| = \omega\}| \geq k$.

• $f_\alpha[f_\alpha^{-1}[H]] = H \in \mathscr{T}$

Enumerate $H = \{u_n : n \in \omega\}$. For every $n \in \omega$ there is $K \in \mathscr{G}$ such that $u_n, u_{n+1} \in K$. Since $K \in P$ we have $u_{n+1} > (u_n)^2$ and $\frac{u_n}{u_{n+1}} < \frac{1}{u_n}$. Thus $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} \leq \lim_{n\to\infty} \frac{1}{u_n} = 0$.

To complete the induction step let $\mathscr{F}_{\alpha+1}$ be the filter base generated by $\mathscr{F}_\alpha$ and $f_\alpha^{-1}[H]$.

Finally, let $\mathscr{F} = \bigcup_{\alpha<\mathfrak{c}} \mathscr{F}_\alpha$. The filter base $\mathscr{F}$ has the property that for every $F \in \mathscr{F}$ there are infinitely many n such that $F \cap R_n$ is infinite therefore it can be extended to an ultrafilter which is not a P-point (see the proof of Proposition 2.1.14). It is obvious that every ultrafilter extending $\mathscr{F}$ is a thin ultrafilter because of condition (vi). □

The following proposition implies that under Martin's Axiom for countable posets there are: P-points which are not thin, (SC)-ultrafilters which are not thin, (SC)-ultrafilters which are not (S)-ultrafilters and (H)-ultrafilters which are not (S)-ultrafilters.

Proposition 2.3.5. *(MA_{ctble}) There exists a P-point which is not an (S)-ultrafilter.*

Proof. Enumerate all infinite partitions of ω (into infinite sets) as $\{\mathscr{R}^\alpha : \alpha < \mathfrak{c}\}$. By transfinite induction on $\alpha < \mathfrak{c}$ we will construct filter bases $\mathscr{F}_\alpha$, $\alpha < \mathfrak{c}$, so that the following conditions are satisfied:

(i) $\mathscr{F}_0$ is the Fréchet filter

(ii) $\mathscr{F}_\alpha \subseteq \mathscr{F}_\beta$ whenever $\alpha \leq \beta$

(iii) $\mathscr{F}_\gamma = \bigcup_{\alpha<\gamma} \mathscr{F}_\alpha$ for γ limit

(iv) $(\forall\alpha)$ $|\mathscr{F}_\alpha| \leq |\alpha| \cdot \omega$

(v) $(\forall\alpha)$ $(\forall F \in \mathscr{F}_\alpha)$ $\sum_{a\in F} \frac{1}{a} = +\infty$

(vi) $(\forall\alpha)$ $(\exists F \in \mathscr{F}_{\alpha+1})$ either $(\exists R_n^\alpha \in \mathscr{R}^\alpha)$ $F \subseteq R_n^\alpha$ or $(\forall R_n^\alpha \in \mathscr{R}^\alpha)$ $|F \cap R_n^\alpha| < \omega$

Induction step: Suppose we know already $\mathscr{F}_\alpha$ and we construct $\mathscr{F}_{\alpha+1}$.

Case A. $(\exists K \in [\omega]^{<\omega})$ $(\forall F \in \mathscr{F}_\alpha)$ $F \cap \bigcup_{n\in K} R^\alpha_n \notin (S)$

For some $n_0 \in K$ the filter base generated by $R^\alpha_{n_0}$ and $\mathscr{F}_\alpha$ satisfies condition (v). Otherwise, there would be for every $n \in K$ a set $F_n \in \mathscr{F}_\alpha$ such that $F_n \cap R^\alpha_n \in (S)$. We would have $\bigcap_{n\in K} F_n \cap \bigcup_{n\in K} R^\alpha_n \in (S)$ — a contradiction to the assumption of *Case A*. So we let $\mathscr{F}_{\alpha+1}$ be the filter base generated by $\mathscr{F}_\alpha$ and the set $R^\alpha_{n_0}$.

Case B. $(\forall K \in [\omega]^{<\omega})$ $(\exists F_K \in \mathscr{F}_\alpha)$ $F_K \cap \bigcup_{n\in K} R^\alpha_n \in (S)$

Consider $P = \{\langle K, n\rangle \in [\omega]^{<\omega} \times \omega : K \subseteq \bigcup_{i\leq n} R^\alpha_i,\ K \cap R^\alpha_n \neq \emptyset\}$ with ordering given by $\langle K, n\rangle \leq_P \langle L, m\rangle$ if $\langle K, n\rangle = \langle L, m\rangle$ or $K \supset L$, $\min(K \setminus L) > \max L$, $n > m$ and $(K \setminus L) \cap \bigcup_{i\leq m} R^\alpha_i = \emptyset$. For every $F \in \mathscr{F}_\alpha$ and $k \in \omega$ define $D_{F,k} = \{\langle K, n\rangle \in P : \sum_{a\in K\cap F} \frac{1}{a} \geq k\}$ and $D_j = \{\langle K, n\rangle \in P : n \geq j\}$.

Claim: $D_{F,k}$ is dense in P for every $F \in \mathscr{F}_\alpha$ and $k \in \omega$; D_j is dense in P for every $j \in \omega$.

Take $\langle L, m\rangle \in P$ arbitrary. According to the assumption there is $F_m \in \mathscr{F}_\alpha$ such that $F_m \cap \bigcup_{i\leq m} R^\alpha_i \in (S)$. It follows that $(F_m \cap F) \setminus \bigcup_{i\leq m} R^\alpha_i \notin (S)$. Hence we can choose a finite set $L' \subseteq (F_m \cap F) \setminus \bigcup_{i\leq m} R^\alpha_i$ such that $\sum_{a\in L'} \frac{1}{a} \geq k$. Let $n = \max\{i : L' \cap R^\alpha_i \neq \emptyset\}$ and $K = L \cup L'$. It is easy to see that $\langle K, n\rangle \leq_P \langle L, m\rangle$ and $\langle K, n\rangle \in D_{F,k}$. So $D_{F,k}$ is dense. For $j \leq m$ we have $\langle L, m\rangle \in D_j$ and for any $j > m$ we can choose arbitrary $r \in R^\alpha_j$ such that $r > \max L$. Let $K' = L \cup \{r\}$. Of course, $\langle K', j\rangle \leq_P \langle L, m\rangle$ and $\langle K', j\rangle \in D_j$. So D_j is dense.

The family $\mathscr{D} = \{D_{F,k} : F \in \mathscr{F}_\alpha, k \in \omega\} \cup \{D_j : j \in \omega\}$ consists of dense subsets in P and $|\mathscr{D}| < \mathfrak{c}$. Therefore there is a $\mathscr{D}$-generic filter $\mathscr{G}$.

Let $U = \bigcup\{K : \langle K, n\rangle \in \mathscr{G}\}$. It remains to check that:

- $(\forall F \in \mathscr{F}_\alpha)$ $\sum_{a\in U\cap F} \frac{1}{a} = +\infty$

We have $U \cap F \notin (S)$ for every $F \in \mathscr{F}_\alpha$ because for every $k \in \omega$ there exists $\langle K, n\rangle \in \mathscr{G} \cap D_{F,k}$ and we get $\sum_{a\in U\cap F} \frac{1}{a} \geq \sum_{a\in K\cap F} \frac{1}{a} \geq k$.

- $(\forall R^\alpha_n \in \mathscr{R}_\alpha)$ $|U \cap R^\alpha_n| < \omega$

Take $\langle K_n, j_n\rangle \in \mathscr{G} \cap D_n$ where $j_n = \min\{j : (\exists K \in [\omega]^{<\omega})\langle K, j\rangle \in \mathscr{G} \cap D_n\}$. Now observe that for $\langle K, m\rangle \in \mathscr{G}$ we have $K \cap R^\alpha_n = \emptyset$ if $m < n$ and that $K \cap R^\alpha_n = K_n \cap R^\alpha_n$ if $m \geq n$. To see the latter consider $\langle L, m'\rangle \in \mathscr{G}$ such that $\langle L, m'\rangle \leq_P \langle K, m\rangle$ and $\langle L, m'\rangle \leq_P \langle K_n, j_n\rangle$ (such a condition exists because $\mathscr{G}$ is a filter) for which we get $L \cap R^\alpha_n = K \cap R^\alpha_n$ and $L \cap R^\alpha_n = K_n \cap R^\alpha_n$. It follows that $U \cap R^\alpha_n = K_n \cap R^\alpha_n$ is finite.

To complete the induction step let $\mathscr{F}_{\alpha+1}$ be the filter base generated by $\mathscr{F}_\alpha$ and U.

It is clear from condition (vi) that every ultrafilter which extends $\mathscr{F} = \bigcup_{\alpha<\mathfrak{c}} \mathscr{F}_\alpha$ is a P-point. Because of condition (v) there exists ultrafilter extending $\mathscr{F}$ which extends the dual filter of (S). So there exists a P-point which is not an (S)-ultrafilter. $\square$

Proposition 2.3.6. *If an (S)-ultrafilter exists then there is an (S)-ultrafilter which is not an (SC)-ultrafilter.*

Proof. We shall prove in Proposition 3.2.1 that the class of (S)-ultrafilters is closed under products (for the definition of products of ultrafilters see the first paragraph of chapter 3). Hence if $\mathscr{U}$ is an (S)-ultrafilter then $\mathscr{U} \cdot \mathscr{U}$ is an (S)-ultrafilter too. But the ultrafilter $\mathscr{U} \cdot \mathscr{U}$ cannot be an (SC)-ultrafilter according to Proposition 3.2.2. $\square$

Corollary 2.3.7. *(MA_{ctble}) There is an (S)-ultrafilter which is not an (SC)-ultrafilter.*

Proof. If Martin's Axiom holds then selective ultrafilters exist. Every selective ultrafilter is a thin ultrafilter (see Proposition 2.3.1) and hence an (S)-ultrafilter. So (S)-ultrafilters exist under Martin's Axiom and from the previous proposition we get an (S)-ultrafilter which is not an (SC)-ultrafilter. $\square$

2.4 Connections to Q-points and rapid ultrafilters

The following diagram shows all inclusions between the classes of (hereditarily) Q-points, (hereditarily) rapid ultrafilters, thin ultrafilters and (S)-ultrafilters (an arrow stands for inclusion). No arrow can be reversed or added if we assume Martin's Axiom for countable posets.

$$\begin{tikzcd}
 & \boxed{\begin{array}{c}\text{her. } Q\text{-points}\\ = \text{thin ultfs}\end{array}} \arrow[dl] \arrow[dr] & & \\
\boxed{Q\text{-points}} \arrow[dr] & & \boxed{\text{her. rapid ultfs}} \arrow[dl] \arrow[dr] & \\
 & \boxed{\text{rapid ultfs}} & & \boxed{(S)\text{-ultfs}}
\end{tikzcd}$$

Proposition 2.4.1. *Every thin ultrafilter is a Q-point.*

Proof. Let $\mathscr{U}$ be a thin ultrafilter and $\mathscr{Q} = \{Q_n : n \in \omega\}$ a partition of ω into finite sets. Enumerate $Q_n = \{q_i^n : i = 0, \dots, k_n\}$ (where $k_n = |Q_n| - 1$). We want to find $U \in \mathscr{U}$ such that $|U \cap Q_n| \le 1$ for every $n \in \omega$.

Define a strictly increasing function $f : \omega \to \omega$ in the following way:

$f(q_0^0) = 0$, $f(q_0^{n+1}) = (n+2) \cdot \max\{f(q_{k_n}^n), k_{n+1}\}$ for $n \in \omega$ and $f(q_i^n) = f(q_0^n) + i$ for $i \leq k_n, n \in \omega$.

Since $\mathscr{U}$ is a thin ultrafilter there exists $U_0 \in \mathscr{U}$ such that $f[U_0] = \{v_m : m \in \omega\}$ is a thin set. Hence there is $m_0 \in \omega$ such that $\frac{v_m}{v_{m+1}} < \frac{1}{2}$ for every $m \geq m_0$.

Function f is one-to-one and $\mathscr{Q}$ a partition of ω into finite sets so we can find $K \subseteq \omega$ of size at most m_0 such that $\{f^{-1}(v_i) : i < m_0\} \subseteq \bigcup_{n \in K} Q_n$. The latter set is finite, which implies $U = U_0 \setminus \bigcup_{n \in K} Q_n \in \mathscr{U}$.

From the definition we have $U \cap Q_n = \emptyset$ for $n \in K$ and it remains to check that

• $(\forall n \notin K)\ |U \cap Q_n| \leq 1$

Assume for the contrary that for some $n \notin K$ there are two distinct elements $u_1, u_2 \in U \cap Q_n$, $u_1 < u_2$. Then $f(u_1) = v_m$ for some $m \geq m_0$ and $f(u_2) = v_n$ for some $n \geq m+1$. We get $\frac{v_m}{v_{m+1}} \geq \frac{v_m}{v_n} = \frac{f(u_1)}{f(u_2)} \geq \frac{f(q_0^n)}{f(q_0^n)+k_n} \geq \frac{(n+1) \cdot M}{(n+1) \cdot M + M} = \frac{n+1}{n+2}$ where $M = \max\{f(q_{k_{n-1}}^{n-1}), k_n\}$. But $\frac{n+1}{n+2} \geq \frac{1}{2}$ — a contradiction. □

Lemma 2.4.2. *Every Q-point contains a thin set.*

Proof. It follows from the proof of Proposition 2.3.1 that every Q-point contains a thin set because the partition considered in the proof consists of finite sets. □

Corollary 2.4.3. *A free ultrafilter on ω is thin if and only if it is a hereditarily Q-point.*

Proof. If $\mathscr{U}$ is a thin ultrafilter then every $\mathscr{V} \leq_{RK} \mathscr{U}$ is also a thin ultrafilter because thin ultrafilters are downward closed under Rudin-Keisler order and Proposition 2.4.1 implies that $\mathscr{U}$ is a hereditarily Q-point.

If $\mathscr{U}$ is a hereditarily Q-point then for every function f the ultrafilter $\beta f(\mathscr{U})$ is a Q-point and hence contains according to Lemma 2.4.2 a thin set. It follows from the definition of $\beta f(\mathscr{U})$ that there exists $U \in \mathscr{U}$ such that $f[U]$ is thin and $\mathscr{U}$ is a thin ultrafilter. □

Lemma 2.4.4. *Every rapid ultrafilter contains an (S)-set.*

Proof. If $\mathscr{U}$ is a rapid ultrafilter then there is $U = \{u_n : n \in \omega\} \in \mathscr{U}$ such that $2^n \leq e_U(n) = u_n$ for all but finitely many n. So

$$\sum_{n \in \omega} \frac{1}{u_n} \leq \sum_{n \leq n_0} \frac{1}{u_n} + \sum_{n > n_0} \frac{1}{2^n} < +\infty.$$

We see that the set U is an (S)-set. □

Corollary 2.4.5. *Every hereditarily rapid ultrafilter is an (S)-ultrafilter.*

Proof. If $\mathscr{U}$ is a hereditarily rapid ultrafilter then for every function f the ultrafilter $\beta f(\mathscr{U})$ is a rapid ultrafilter and hence contains according to Lemma 2.4.4 an (S)-set. It follows from the definition of $\beta f(\mathscr{U})$ that there exists $U \in \mathscr{U}$ such that $f[U]$ belongs to the summable ideal and $\mathscr{U}$ is an (S)-ultrafilter. $\square$

Proposition 2.4.6. *(MA_{ctble}) There is a hereditarily rapid ultrafilter which is not a Q-point.*

Proof. Enumerate ${}^{\omega}\omega \times {}^{\omega}\omega = \{\langle f_\alpha, g_\alpha\rangle : \alpha < \mathfrak{c}\}$ and fix a partition of ω into finite sets $\{Q_n : n \in \omega\}$ (such that $\limsup_{n\to\infty} |Q_n| = +\infty$). By transfinite induction on $\alpha < \mathfrak{c}$ we will construct filter bases $\mathscr{F}_\alpha$, $\alpha < \mathfrak{c}$, so that the following conditions are satisfied:

(i) $\mathscr{F}_0$ is the Fréchet filter
(ii) $\mathscr{F}_\alpha \subseteq \mathscr{F}_\beta$ whenever $\alpha \leq \beta$
(iii) $\mathscr{F}_\gamma = \bigcup_{\alpha<\gamma} \mathscr{F}_\alpha$ for γ limit
(iv) $(\forall\alpha)$ $|\mathscr{F}_\alpha| \leq |\alpha| \cdot \omega$
(v) $(\forall\alpha)$ $(\forall F \in \mathscr{F}_\alpha)$ $(\forall k \in \omega)$ $(\exists n \in \omega)$ $|F \cap Q_n| \geq k$
(vi) $(\forall\alpha)$ $(\exists U \in \mathscr{F}_{\alpha+1})$ such that $g_\alpha \leq^* e_{f_\alpha[U]}$ or $f_\alpha[U]$ is finite

Let us first prove that any filter base satisfying condition (v) can be extended to an ultrafilter which is not a Q-point.

Claim 1. Let $\mathscr{F}$ be a filter base on ω such that $(\forall F \in \mathscr{F})$ $(\forall k \in \omega)$ $(\exists n \in \omega)$ $|F \cap Q_n| \geq k$. For every $A \subseteq \omega$ either $\langle \mathscr{F} \cup \{A\}\rangle$ or $\langle \mathscr{F} \cup \{\omega \setminus A\}\rangle$ has the same property.

If $\langle \mathscr{F} \cup \{A\}\rangle$ does not have the required property then there is $F_0 \in \mathscr{F}$ and $k_0 \in \omega$ such that $|F_0 \cap A \cap Q_n| < k_0$ for every $n \in \omega$. Since $F \cap F_0 \in \mathscr{F}$ we know that for every $k \in \omega$ there is n_k such that $|F \cap F_0 \cap Q_{n_k}| \geq k + k_0$. It follows that $|F \cap (\omega \setminus A) \cap Q_{n_k}| \geq k$ for every $F \in \mathscr{F}$ and $\langle \mathscr{F} \cup \{\omega \setminus A\}\rangle$ has the required property.

Induction step: Suppose we know already $\mathscr{F}_\alpha$. If there is $U \in \mathscr{F}_\alpha$ such that $g_\alpha \leq^* e_{f_\alpha[U]}$ then simply put $\mathscr{F}_{\alpha+1} = \mathscr{F}_\alpha$. If there is not such a set U we will construct a suitable set eventually making use of Martin's Axiom.

Case A. $(\exists K \in [\omega]^{<\omega})$ $(\forall F \in \mathscr{F}_\alpha)$ $(\forall k)$ $(\exists n)$ $|f_\alpha^{-1}[K] \cap F \cap Q_n| \geq k$

Let $\mathscr{F}_{\alpha+1}$ be the filter base generated by $\mathscr{F}_\alpha$ and $U = f_\alpha^{-1}[K]$. Then for every ultrafilter $\mathscr{U}$ which extends the filter base $\mathscr{F}_{\alpha+1}$ the ultrafilter $\beta f_\alpha(\mathscr{U})$ is principal. It is easy to see that there is $V \in \mathscr{U}$ such that $g_\alpha \leq^* e_{f_\alpha[V]}$.

Case B. $(\forall K \in [\omega]^{<\omega})$ $(\exists F_K \in \mathscr{F}_\alpha)$ $(\exists k_K)$ $(\forall n)$ $|f_\alpha^{-1}[K] \cap F_K \cap Q_n| < k_K$

Consider $P = \{\langle L, m\rangle \in [\omega]^{<\omega} \times \omega : L \subseteq \bigcup_{i\leq m} Q_i, L \cap Q_m \neq \emptyset, e_{f_\alpha[L]} > g_\alpha \restriction |L|\}$ with partial ordering given by $\langle K, n\rangle \leq_P \langle L, m\rangle$ if $\langle K, n\rangle = \langle L, m\rangle$

or $n > m$, $K \supset L$, $\min(K \setminus L) > \max L$ and $(K \setminus L) \cap \bigcup_{i \le m} Q_i = \emptyset$. For every $F \in \mathscr{F}_\alpha$ and $k \in \omega$ let $D_{F,k} = \{\langle L, m\rangle \in P : (\exists n) |L \cap F \cap Q_n| \ge k\}$.

Claim 2. $D_{F,k}$ is a dense subset of $(P, \le_P)$ for every $F \in \mathscr{F}_\alpha$ and $k \in \omega$.

Let $\langle L, m\rangle \in P$ be arbitrary. Set $M_0 = \max\{g_\alpha(n) : n \le |L| + k\}$, $M_1 = \max f_\alpha[L]$ and $M_2 = 1 + \max \bigcup_{i \le m} Q_i$. Finally, let $M = \max\{M_0, M_1, M_2\}$. Interval $[0, M]$ is finite so there exist $F_M \in \mathscr{F}_\alpha$ and $k_M \in \omega$ such that for every n we have $|f_\alpha^{-1}[0, M] \cap F_M \cap Q_n| < k_M$. According to condition (v) there exists n (we may assume $n > m$) such that $|F \cap F_M \cap Q_n| \ge M + k_M + k$. Since $|F \cap F_M \cap Q_n \cap f_\alpha^{-1}[0, M]| < k_M$ we can choose $L' \subseteq F \cap F_M \cap Q_n$ such that $|L'| = k$ and $f_\alpha(a) > M$. Let $K = L \cup L'$. It is not difficult to check that $\langle K, n\rangle \in P$ and then it is obvious that $\langle K, n\rangle \in D_{F,k}$ and $\langle K, n\rangle \le_P \langle L, m\rangle$. So $D_{F,k}$ is dense in P.

The family $\mathscr{D} = \{D_{F,k} : F \in \mathscr{F}_\alpha, k \in \omega\}$ consists of dense subsets of P and has cardinality less than $\mathfrak{c}$. Therefore there exists a $\mathscr{D}$-generic filter $\mathscr{G}$ on P according to Martin's Axiom for countable posets.

Let $U = \bigcup\{L : (\exists m)\langle L, m\rangle \in \mathscr{G}\}$ and verify that the set U satisfies the following conditions:

• $(\forall F \in \mathscr{F}_\alpha)$ $(\forall k \in \omega)$ $(\exists n \in \omega)$ $|U \cap F \cap Q_n| \ge k$

For every $\langle K, m\rangle \in \mathscr{G} \cap D_{F,k}$ we have $U \supseteq K$ and there is n such that $|K \cap F \cap Q_n| \ge k$.

• $g_\alpha \le^* e_{f_\alpha[U]}$

There exist $\langle K_j, m_j\rangle \in \mathscr{G}$, $j \in \omega$, such that $\langle K_{j+1}, m_{j+1}\rangle \le_P \langle K_j, m_j\rangle$ for every $j \in \omega$ and $U = \bigcup_{j \in \omega} K_j$. Since $e_{f_\alpha[K_j]} > g_\alpha \restriction |K_j|$ for every j we have $g_\alpha \le^* e_{f_\alpha[U]}$.

To complete the induction step let $\mathscr{F}_{\alpha+1}$ be the filter base generated by $\mathscr{F}_\alpha$ and U.

It is obvious that every ultrafilter which extends $\mathscr{F} = \bigcup_{\alpha < \mathfrak{c}} \mathscr{F}_\alpha$ is a hereditarily rapid ultrafilter because of condition (vi) and it can be extended to a non-Q-point because of condition (v). □

Proposition 2.4.7. *(MA_{ctble}) For any (tall) ideal $\mathscr{I}$ on ω, there is a Q-point which is not an $\mathscr{I}$-ultrafilter.*

Proof. Enumerate all partitions of ω into finite sets as $\{\mathscr{Q}_\alpha : \alpha < \mathfrak{c}\}$ and fix a partition $\{R_n : n \in \omega\}$ of ω into infinite sets. By transfinite induction on $\alpha < \mathfrak{c}$ we will construct filter bases $\mathscr{F}_\alpha$, $\alpha < \mathfrak{c}$, so that the following conditions are satisfied:

(i) $\mathscr{F}_0$ is the Fréchet filter
(ii) $\mathscr{F}_\alpha \subseteq \mathscr{F}_\beta$ whenever $\alpha \le \beta$
(iii) $\mathscr{F}_\gamma = \bigcup_{\alpha < \gamma} \mathscr{F}_\alpha$ for γ limit
(iv) $(\forall \alpha)$ $|\mathscr{F}_\alpha| \le |\alpha| \cdot \omega$

(v) $(\forall\alpha)$ $(\forall F \in \mathscr{F}_\alpha)$ $(\forall n \in \omega)$ $|F \cap R_n| = \omega$
(vi) $(\forall\alpha)$ $(\exists F \in \mathscr{F}_{\alpha+1})$ $(\forall Q \in \mathscr{Q}_\alpha)$ $|F \cap Q| \leq 1$

Induction step: Suppose we know already $\mathscr{F}_\alpha$. If there is a set $F \in \mathscr{F}_\alpha$ such that $|F \cap Q| \leq 1$ for every $Q \in \mathscr{Q}_\alpha$ then put $\mathscr{F}_{\alpha+1} = \mathscr{F}_\alpha$. If this is not the case, we can construct such a set using Martin's Axiom.

Consider $P = \{K \in [\omega]^{<\omega} : (\forall Q \in \mathscr{Q}_\alpha)|K \cap Q| \leq 1\}$ with partial order defined by $K_2 \leq_P K_1$ if $K_2 = K_1$ or $K_2 \supset K_1$ and $\min(K_2 \setminus K_1) > \max K_1$. For every $F \in \mathscr{F}_\alpha$ and $n, k \in \omega$ let $D_{F,n,k} = \{K \in P : |K \cap F \cap R_n| \geq k\}$.

Claim: $D_{F,n,k}$ is a dense subset of $(P, \leq_P)$ for every $F \in \mathscr{F}_\alpha$, $n, k \in \omega$.

Whenever we take $L \in P$ there is a finite set $S \subseteq \omega$ such that $L \subseteq \bigcup_{i \in S} Q_i$. Since $(F \cap R_n) \setminus [0, \max \bigcup_{i \in S} Q_i]$ is infinite we can choose for $j = 1, 2, \dots, k$ distinct $n_j \in \omega \setminus S$ and elements $q_j \in F \cap R_n \cap Q_{n_j}$. Let $K = L \cup \{q_1, q_2, \dots q_k\}$. Obviously, $K \leq_P L$ and $K \in D_{F,n,k}$.

The family $\mathscr{D} = \{D_{F,n,k} : F \in \mathscr{F}_\alpha, n, k \in \omega\}$ consists of dense subsets of P and has cardinality less than $\mathfrak{c}$. So there exists a $\mathscr{D}$-generic filter $\mathscr{G}$ on P.

Let $U = \bigcup\{K : K \in \mathscr{G}\}$. The set U satisfies the following conditions:

• $(\forall F \in \mathscr{F}_\alpha)$ $(\forall n \in \omega)$ $U \cap F \cap R_n$ is infinite

For every $k \in \omega$ and every $K \in \mathscr{G} \cap D_{F,n,k}$ we have $U \supset K$ and $|K \cap F \cap R_n| \geq k$. Thus $U \cap F \cap R_n$ is infinite.

• $(\forall Q \in \mathscr{Q}_\alpha)$ $|U \cap Q| \leq 1$

If $u, v \in U$ then there is $K \in \mathscr{G}$ such that $u, v \in K$ and according to the definiton of P elements u, v belong to distinct sets from partition $\mathscr{Q}_\alpha$.

To complete the induction step let $\mathscr{F}_{\alpha+1}$ be the filter base generated by $\mathscr{F}_\alpha$ and U.

Finally, let $\mathscr{F} = \bigcup_{\alpha<\mathfrak{c}} \mathscr{F}_\alpha$. It is obvious that each ultrafilter extending $\mathscr{F}$ is a Q-point and $F \cap R_n$ is infinite for every $F \in \mathscr{F}$, $n \in \omega$.

Hence the set $R_A = \bigcup_{n \in A} R_n$ is compatible with $\mathscr{F}$ for every $A \subseteq \omega$. Let $\mathscr{G} = \{R_A : A \in \mathscr{I}^*\}$ and observe that any ultrafilter extending $\mathscr{F} \cup \mathscr{G}$ is a Q-point because it extends $\mathscr{F}$ and it is not an $\mathscr{I}$-ultrafilter because of the function f defined by $f[R_n] = \{n\}$. $\square$

Proposition 2.4.8. *(MA_{ctble}) There is an (S)-ultrafilter which is not a rapid ultrafilter.*

Proof. Enumerate ${}^\omega\omega = \{f_\alpha : \alpha < \mathfrak{c}\}$. By transfinite induction on $\alpha < \mathfrak{c}$ we will construct filter bases $\mathscr{F}_\alpha$, $\alpha < \mathfrak{c}$, so that the following conditions are satisfied:

(i) $\mathscr{F}_0$ is the Fréchet filter
(ii) $\mathscr{F}_\alpha \subseteq \mathscr{F}_\beta$ whenever $\alpha \leq \beta$
(iii) $\mathscr{F}_\gamma = \bigcup_{\alpha<\gamma} \mathscr{F}_\alpha$ for γ limit
(iv) $(\forall\alpha)$ $|\mathscr{F}_\alpha| \leq |\alpha| \cdot \omega$

(v) $(\forall\alpha)$ $(\forall F \in \mathscr{F}_\alpha)$ $(\forall k \in \omega)$ $(\exists n \in \omega)$ $|F \cap [2^n, 2^{n+1})| \geq n^k$

(vi) $(\forall\alpha)$ $(\exists F \in \mathscr{F}_{\alpha+1})$ $f_\alpha[F] \in (S)$, i.e., $\sum_{a \in f_\alpha[F]} \frac{1}{a} < +\infty$

Claim 1: Let $A \subseteq \omega$. If $(\forall k \in \omega)$ $(\exists n \in \omega)$ $|A \cap [2^n, 2^{n+1})| \geq n^k$ then $f \not\leq^* e_A$ where $f(n) = 2^{n+1}$.

It follows from the assumption that there are infinitely many n such that $A \cap [2^n, 2^{n+1}) \geq n$. For such indices n we get $a_n < 2^{n+1}$ where a_n is the nth element of A. So we have $e_A(n) < f(n)$ for infinitely many n and $f \not\leq^* e_A$.

Claim 2: If A is a subset of ω and $\mathscr{F}$ is a filter base on ω such that $(\forall F \in \mathscr{F})$ $(\forall k)$ $(\exists n)$ $|F \cap [2^n, 2^{n+1})| \geq n^k$ then either the filter base generated by $\mathscr{F}$ and A or the filter base generated by $\mathscr{F}$ and $\omega \setminus A$ has the same property.

If the filter generated by $\mathscr{F}$ and A does not have the required property then it means that there exists $F_0 \in \mathscr{F}$ and $k_0 \in \omega$ such that $|F_0 \cap A \cap [2^n, 2^{n+1})| < n^{k_0}$ for every $n \in \omega$. Since $F \cap F_0 \in \mathscr{F}$ we know that for every $k \in \omega$ there is some $\bar{n} \in \omega$ such that $|F \cap F_0 \cap [2^{\bar{n}}, 2^{\bar{n}+1})| \geq \bar{n}^{k+k_0}$. It follows that $|F \cap (\omega \setminus A) \cap [2^{\bar{n}}, 2^{\bar{n}+1})| \geq \bar{n}^{k+k_0} - \bar{n}^{k_0} > \bar{n}^k$ for every $F \in \mathscr{F}$. So the filter generated by $\mathscr{F}$ and $\omega \setminus A$ has the required property.

Induction step: Suppose we know already $\mathscr{F}_\alpha$. If there is $F \in \mathscr{F}_\alpha$ such that $f_\alpha[F] \in (S)$ then simply put $\mathscr{F}_{\alpha+1} = \mathscr{F}_\alpha$. If $f_\alpha[F] \notin (S)$ (in particular, $f_\alpha[F]$ is infinite) for every $F \in \mathscr{F}_\alpha$ we will construct a suitable set eventually making use of Martin's Axiom.

If there exists $K \in [\omega]^{<\omega}$ such that for every $F \in \mathscr{F}_\alpha$ and every $k \in \omega$ there is $n \in \omega$ such that $|F \cap f_\alpha^{-1}[K] \cap [2^n, 2^{n+1})| \geq n^k$ then we let $\mathscr{F}_{\alpha+1}$ be the filter base generated by $\mathscr{F}_\alpha$ and $f_\alpha^{-1}[K]$. In the following we will assume that no such set exists, i.e., (♣) for every $K \in [\omega]^{<\omega}$ there is $F_K \in \mathscr{F}_\alpha$ and $k_K \in \omega$ such that for every $n \in \omega$ we have $|F_K \cap f_\alpha^{-1}[K] \cap [2^n, 2^{n+1})| < n^{k_K}$.

Case I. $(\forall F \in \mathscr{F}_\alpha)$ $(\forall k \in \omega)$ $(\exists n \in \omega)$ $|f_\alpha[F \cap [2^n, 2^{n+1})]| \geq n^k$

Let $P = \{K \in [\omega]^{<\omega} : \sum_{a \in f_\alpha[K]} \frac{1}{a} \leq (2 - \frac{1}{2^{|K|}}) \frac{1}{\min f_\alpha[K]}\}$ and define a partial order $\leq_P$ on P in the following way: $K \leq_P L$ if and only if $K = L$ or $K \supset L$ and $\min K \setminus L > \max L$. For every $F \in \mathscr{F}_\alpha$ and $k \in \omega$ define $D_{F,k} = \{K \in P : (\exists n \in \omega) |K \cap F \cap [2^n, 2^{n+1})| \geq n^k\}$.

Claim 1: $D_{F,k}$ is a dense subset of $(P, \leq_P)$ for every $F \in \mathscr{F}_\alpha$, $k \in \omega$.

Let $L \in P$ be arbitrary. According to the assumption of *Case I.* there exists $n \in \omega$ such that $|f_\alpha[F \cap [2^n, 2^{n+1})]| \geq n^{k+(|L|+k+1)\cdot\max f_\alpha[L]}$ (we may assume that n is large enough so that we have $2^n > \max L$ and $n^{(|L|+k+1)\cdot\max f_\alpha[L]} > \max f_\alpha[L] \cdot n^{|L|+k+1}$).

Since $n^{k+(|L|+k+1)\cdot\max f_\alpha[L]} > n^k + n^{(|L|+k+1)\cdot\max f_\alpha[L]}$ there exists $L' \subseteq F \cap [2^n, 2^{n+1})$ of size n^k such that $a > \max L$ and $f_\alpha(a) > n^{(|L|+k+1)\cdot\max f_\alpha[L]} > \max f_\alpha[L] \cdot n^{|L|+k+1}$ for every $a \in L'$. Let $K = L \cup L'$.

To see that $K \in P$ observe that $\sum_{a \in f_\alpha[K]} \frac{1}{a} = \sum_{a \in f_\alpha[L]} \frac{1}{a} + \sum_{a \in f_\alpha[L']} \frac{1}{a} \leq$

$(2-\frac{1}{2^{|L|}})\frac{1}{\min f_\alpha[L]}+\frac{n^k}{n^{|L|+k+1}\cdot\max f_\alpha[L]}\leq(2-\frac{1}{2^{|L|}})\frac{1}{\min f_\alpha[L]}+\frac{1}{n^{|L|+1}\cdot\max f_\alpha[L]}\leq(2-\frac{1}{2^{|L|}}+\frac{1}{2^{|L|+1}})\frac{1}{\min f_\alpha[L]}=(2-\frac{1}{2^{|L|+1}})\frac{1}{\min f_\alpha[L]}$. Since $\min f_\alpha[L]=\min f_\alpha[K]$ and $|K|\geq|L|+1$ we get $\sum_{a\in f_\alpha[K]}\frac{1}{a}\leq(2-\frac{1}{2^{|K|}})\frac{1}{\min f_\alpha[K]}$. It is obvious that $K\leq_P L$ and $K\in D_{F,k}$. Therefore $D_{F,k}$ is dense in P.

Since $\mathscr{D}=\{D_{F,k}:F\in\mathscr{F}_\alpha,k\in\omega\}$ is a family of dense subsets of a countable poset and $|\mathscr{D}|<\mathfrak{c}$ there is a $\mathscr{D}$-generic filter $\mathscr{G}$ on P.

Let $U=\bigcup\{K:K\in\mathscr{G}\}$. It remains to check that:

- $(\forall F\in\mathscr{F}_\alpha)$ $(\forall k\in\omega)$ $(\exists n\in\omega)$ $|U\cap F\cap[2^n,2^{n+1})|\geq n^k$

For every $K\in\mathscr{G}\cap D_{F,k}$ we have $U\supseteq K$ and there is some n such that $|K\cap F\cap[2^n,2^{n+1})|\geq n^k$.

- $\sum_{a\in f_\alpha[U]}\frac{1}{a}<+\infty$, i.e., $f_\alpha[U]\in(S)$

Enumerate $f_\alpha[U]=\{u_n:n\in\omega\}$. For every n there exists $K_n\in\mathscr{G}$ such that $u_n\in K_n$. Since $\mathscr{G}$ is a filter we may assume $K_{n+1}\leq_P K_n$ for every $n\in\omega$. Obviously, $U=\bigcup_{n\in\omega}K_n$ and we get $\sum_{a\in f_\alpha[U]}\frac{1}{a}\leq\frac{2}{\min f_\alpha[U]}$ because $\sum_{a\in f_\alpha[K_n]}\frac{1}{a}\leq(2-\frac{1}{2^{|K_n|}})\frac{1}{\min f_\alpha[K_n]}$ for every n.

To complete the induction step let $\mathscr{F}_{\alpha+1}$ be the filter base generated by $\mathscr{F}_\alpha$ and U.

Case II. $(\exists F_0\in\mathscr{F}_\alpha)$ $(\exists k_0\in\omega)$ $(\forall n\in\omega)$ $|f_\alpha[F_0\cap[2^n,2^{n+1})|<n^{k_0}$

Let $P=\{K\in[\omega]^{<\omega}:(\forall u,v\in K)\,u<v$ implies $2u<v\}$ and define a partial order $\leq_P$ on P in the following way: $K\leq_P L$ if and only if $K=L$ or $K\supset L$ and $\min K\setminus L>\max L$. For every $F\in\mathscr{F}_\alpha$ and $k\in\omega$ define $D_{F,k}=\{K\in P:(\exists n\in\omega)|F\cap f_\alpha^{-1}[K]\cap[2^n,2^{n+1})|\geq n^k\}$.

Claim 2: $D_{F,k}$ is a dense subset of $(P,\leq_P)$ for every $F\in\mathscr{F}_\alpha$, $k\in\omega$.

Let $L\in P$ be arbitrary. According to the assumption (♣) there is $F_L\in\mathscr{F}_\alpha$ and $k_L\in\omega$ such that for every n we have $|f_\alpha^{-1}[0,2\max f_\alpha[L]]\cap F_L\cap[2^n,2^{n+1})|<n^{k_L}$. From condition (v) we know that there is $n\in\omega$ such that $|F\cap F_L\cap F_0\cap[2^n,2^{n+1})|\geq n^{k+k_L+k_0}>n^{k+k_0}+n^{k_L}$. Hence there exists $M\subseteq F\cap F_L\cap F_0\cap[2^n,2^{n+1})$ of size n^{k+k_0} such that $f_\alpha(a)>2\max L$ for every $a\in M$. It follows from the assumption of *Case II.* that there is $h\in f_\alpha[M]$ such that $|f_\alpha^{-1}(h)\cap M|\geq n^k$. Let $K=L\cup\{h\}$. Since $h\in f_\alpha[M]$ we have $h>2\max L$ and $K\in P$. It is obvious that $K\leq_P L$ and $K\in D_{F,k}$. Hence $D_{F,k}$ is dense.

Since $\mathscr{D}=\{D_{F,k}:F\in\mathscr{F}_\alpha,k\in\omega\}$ is a family of dense subsets of a countable poset and $|\mathscr{D}|<\mathfrak{c}$ there is a $\mathscr{D}$-generic filter $\mathscr{G}$ on P.

Let $H=\bigcup\{K:K\in\mathscr{G}\}$. It remains to check that:

- $(\forall F\in\mathscr{F}_\alpha)$ $(\forall k\in\omega)$ $(\exists n\in\omega)$ $|F\cap f_\alpha^{-1}[H]\cap[2^n,2^{n+1})|\geq n^k$

For every $K\in\mathscr{G}\cap D_{F,k}$ we have $H\supseteq K$ and there is some n such that $|F\cap f_\alpha^{-1}[K]\cap[2^n,2^{n+1})|\geq n^k$.

- $f_\alpha[f_\alpha^{-1}[H]]=H\in(S)$

Enumerate $H = \{h_n : n \in \omega\}$. Since $h_{n+1} > 2h_n$ for every n we get $\sum_{n\in\omega} \frac{1}{h_n} \le \frac{1}{h_0} \sum_{n\in\omega} \frac{1}{2^n} = \frac{2}{h_0}$.

To complete the induction step let $\mathscr{F}_{\alpha+1}$ be the filter base generated by $\mathscr{F}_\alpha$ and $f_\alpha^{-1}[H]$.

It is obvious that every ultrafilter which extends $\mathscr{F} = \bigcup_{\alpha<\mathfrak{c}} \mathscr{F}_\alpha$ is an (S)-ultrafilter and it can be extended to an ultrafilter that is not a rapid ultrafilter because of condition (v). $\square$

2.5 Some other classes of $\mathscr{I}$-ultrafilters

We know that van der Waerden ideal contains all thin sets therefore every thin ultrafilters is a $\mathscr{W}$-ultrafilter and it is consistent that $\mathscr{W}$-ultrafilters exist. Every $\mathscr{W}$-ultrafilter is an (H)-ultrafilter because van der Waerden ideal is a subideal of the density ideal. We present in this chapter two more results concerning $\mathscr{W}$-ultrafilters and we show also that it is consistent that $\mathscr{I}_g$-ultrafilters exist for every generalized summable ideal $\mathscr{I}_g$.

Proposition 2.5.1. *(MA_{ctble}) There is a P-point which is not a $\mathscr{W}$-ultrafilter.*

Proof. Enumerate all partitions of ω (into infinite sets) as $\{\mathscr{R}_\alpha : \alpha < \mathfrak{c}\}$. By transfinite induction on $\alpha < \mathfrak{c}$ we will construct filter bases $\mathscr{F}_\alpha$, $\alpha < \mathfrak{c}$, so that the following conditions are satisfied:

(i) $\mathscr{F}_0$ is the Fréchet filter

(ii) $\mathscr{F}_\alpha \subseteq \mathscr{F}_\beta$ whenever $\alpha \le \beta$

(iii) $\mathscr{F}_\gamma = \bigcup_{\alpha<\gamma} \mathscr{F}_\alpha$ for γ limit

(iv) $(\forall\alpha)$ $|\mathscr{F}_\alpha| \le |\alpha| \cdot \omega$

(v) $(\forall\alpha)$ $(\forall F \in \mathscr{F}_\alpha)$ $F \notin \mathscr{W}$

(vi) $(\forall\alpha)$ $(\exists F \in \mathscr{F}_{\alpha+1})$ either $(\exists R_n^\alpha \in \mathscr{R}_\alpha)$ $F \subseteq R_n^\alpha$ or $(\forall R_n^\alpha \in \mathscr{R}_\alpha)$ $|F \cap R_n^\alpha| < \omega$

Induction step: Suppose we already know $\mathscr{F}_\alpha$ and we construct $\mathscr{F}_{\alpha+1}$.

Case A. $(\exists K \in [\omega]^{<\omega})$ $(\forall F \in \mathscr{F}_\alpha)$ $F \cap \bigcup_{n\in K} R_n^\alpha \notin \mathscr{W}$

For some $n_0 \in K$ the filter base generated by $R_{n_0}^\alpha$ and $\mathscr{F}_\alpha$ satisfies condition (v). Otherwise, there would be for every $n \in K$ a set $F_n \in \mathscr{F}_\alpha$ such that $F_n \cap R_n^\alpha \in \mathscr{W}$. We would have $\bigcap_{n\in K} F_n \cap \bigcup_{n\in K} R_n^\alpha \in \mathscr{W}$ — a contradiction to the assumption of *Case A*. Now we let $\mathscr{F}_{\alpha+1}$ be the filter base generated by $\mathscr{F}_\alpha$ and the set $R_{n_0}^\alpha$.

Case B. $(\forall K \in [\omega]^{<\omega})$ $(\exists F_K \in \mathscr{F}_\alpha)$ $F_K \cap \bigcup_{n\in K} R_n^\alpha \in \mathscr{W}$

Consider $P = \{\langle K, n\rangle \in [\omega]^{<\omega} \times \omega : K \subseteq \bigcup_{i\le n} R_i^\alpha,\ K \cap R_n^\alpha \ne \emptyset\}$ and define $\langle K, n\rangle \le_P \langle L, m\rangle$ if $\langle K, n\rangle = \langle L, m\rangle$ or $K \supset L$, $\min(K \setminus L) > \max L$, $n > m$ and $(K \setminus L) \cap \bigcup_{i\le m} R_i^\alpha = \emptyset$. For every $F \in \mathscr{F}_\alpha$ and $k \in \omega$ let

$D_{F,k} = \{\langle K, n\rangle \in P : K \cap F$ contains an arithmetic progression of length $k\}$ and $D_j = \{\langle K, n\rangle \in P : n \geq j\}$.

Claim: $D_{F,k}$ is dense in $(P, \leq_P)$ for every $F \in \mathscr{F}_\alpha$ and $k \in \omega$; D_j is dense in $(P, \leq_P)$ for every $j \in \omega$.

Take $\langle L, m\rangle \in P$ arbitrary. According to the assumption there is $F_m \in \mathscr{F}_\alpha$ such that $F_m \cap \bigcup_{i\leq m} R_i^\alpha \in \mathscr{W}$. It follows that $(F_m \cap F) \setminus \bigcup_{i\leq m} R_i^\alpha \notin \mathscr{W}$. Hence we can choose an arithmetic progression $L' \subseteq (F_m \cap F) \setminus \bigcup_{i\leq m} R_i^\alpha$ such that $\min L' > \max L$ and the length of L' is k. Let $n = \max\{i : L' \cap R_i^\alpha \neq \emptyset\}$ and $K = L \cup L'$. It is easy to see that $\langle K, n\rangle \leq_P \langle L, m\rangle$ and $\langle K, n\rangle \in D_{F,k}$. So $D_{F,k}$ is dense. For $j \leq m$ we have $\langle L, m\rangle \in D_j$ and for any $j > m$ we can choose arbitrary $r \in R_j^\alpha$ such that $r > \max L$. Let $K' = L \cup \{r\}$. Of course, $\langle K', j\rangle \leq_P \langle L, m\rangle$ and $\langle K', j\rangle \in D_j$. So D_j is dense.

The family $\mathscr{D} = \{D_{F,k} : F \in \mathscr{F}_\alpha, k \in \omega\} \cup \{D_j : j \in \omega\}$ consists of dense subsets in P and $|\mathscr{D}| < \mathfrak{c}$. Therefore there is a $\mathscr{D}$-generic filter $\mathscr{G}$.

Let $U = \bigcup\{K : \langle K, n\rangle \in \mathscr{G}\}$. It remains to check that:

• $(\forall F \in \mathscr{F}_\alpha)$ $U \cap F$ contains arithmetic progressions of arbitrary length

Take $k \in \omega$ arbitrary. For every $K \in \mathscr{G} \cap D_{F,k}$ we have $U \supseteq K$ and $K \cap F$ contains an arithmetic progression of length k. Hence $U \cap F$ contains arithmetic progressions of arbitrary length.

• $(\forall R_n^\alpha \in \mathscr{R}_\alpha)$ $|U \cap R_n^\alpha| < \omega$

Take $\langle K_n, j_n\rangle \in \mathscr{G} \cap D_n$ where $j_n = \min\{j : (\exists K \in [\omega]^{<\omega})\langle K, j\rangle \in \mathscr{G} \cap D_n\}$. Now observe that for $\langle K, m\rangle \in \mathscr{G}$ we have $K \cap R_n^\alpha = \emptyset$ if $m < n$ and that $K \cap R_n^\alpha = K_n \cap R_n^\alpha$ if $m \geq n$. To see the latter consider $\langle L, m'\rangle \in \mathscr{G}$ such that $\langle L, m'\rangle \leq_P \langle K, m\rangle$ and $\langle L, m'\rangle \leq_P \langle K_n, j_n\rangle$ (such a condition exists because $\mathscr{G}$ is a filter) for which we get $L \cap R_n^\alpha = K \cap R_n^\alpha$ and $L \cap R_n^\alpha = K_n \cap R_n^\alpha$. It follows that $U \cap R_n^\alpha = K_n \cap R_n^\alpha$ is finite.

To complete the induction step let $\mathscr{F}_{\alpha+1}$ be the filter base generated by $\mathscr{F}_\alpha$ and U.

It is obvious that every ultrafilter which extends $\mathscr{F} = \bigcup_{\alpha<\mathfrak{c}} \mathscr{F}_\alpha$ is a P-point. Because of condition (v) there exists an ultrafilter extending $\mathscr{F}$ which extends the dual filter of $\mathscr{W}$, i.e. it is not a $\mathscr{W}$-ultrafilter. □

Proposition 2.5.2. *(MA_{ctble}) There exists an (S)-ultrafilter which is not a $\mathscr{W}$-ultrafilter.*

Proof. Enumerate ${}^\omega\omega = \{f_\alpha : \alpha < \mathfrak{c}\}$. By transfinite induction on $\alpha < \mathfrak{c}$ we will construct filter bases $\mathscr{F}_\alpha$, $\alpha < \mathfrak{c}$, so that the following conditions are satisfied:

(i) $\mathscr{F}_0$ is the Fréchet filter

(ii) $\mathscr{F}_\alpha \subseteq \mathscr{F}_\beta$ whenever $\alpha \leq \beta$

(iii) $\mathscr{F}_\gamma = \bigcup_{\alpha<\gamma} \mathscr{F}_\alpha$ for γ limit

(iv) $(\forall\alpha)\ |\mathscr{F}_\alpha| \le |\alpha|\cdot\omega$

(v) $(\forall\alpha)\ (\forall F\in\mathscr{F}_\alpha)\ F\notin\mathscr{W}$

(vi) $(\forall\alpha)\ (\exists F\in\mathscr{F}_{\alpha+1})\ f_\alpha[F]\in(S)$, i.e., $\sum_{a\in f_\alpha[F]}\frac{1}{a}<+\infty$

Induction step: Suppose we know already $\mathscr{F}_\alpha$. If there is $F\in\mathscr{F}_\alpha$ such that $f_\alpha[F]\in(S)$ then simply put $\mathscr{F}_{\alpha+1}=\mathscr{F}_\alpha$. If $f_\alpha[F]\notin(S)$ for every $F\in\mathscr{F}_\alpha$ we will construct a suitable set to add.

Case A. $(\exists M\in[\omega]^{<\omega})\ (\forall F\in\mathscr{F}_\alpha)\ f_\alpha^{-1}[M]\cap F\notin\mathscr{W}$

Let $\mathscr{F}_{\alpha+1}$ be the filter base generated by $\mathscr{F}_\alpha$ and $f_\alpha^{-1}[M]$.

Case B. $(\forall M\in[\omega]^{<\omega})\ (\exists F_M\in\mathscr{F}_\alpha)\ f_\alpha^{-1}[M]\cap F_M\in\mathscr{W}$ (hence $F_M\setminus f_\alpha^{-1}[M]\notin\mathscr{W}$).

Consider $P=\{K\in[\omega]^{<\omega}:\sum_{a\in f_\alpha[K]}\frac{1}{a}\le(2-\frac{1}{2^{|K|}})\frac{1}{\min f_\alpha[K]}\}$ and define a partial order $\le_P$ on P in the following way: $K\le_P L$ if and only if $K=L$ or $K\supset L$ and $\min K\setminus L>\max L$. For every $F\in\mathscr{F}_\alpha$ and $k\in\omega$ let $D_{F,k}=\{K\in P: K\cap F$ contains an arithmetic progression of length $k\}$.

Claim: $D_{F,k}$ is a dense subset of $(P,\le_P)$ for every $F\in\mathscr{F}_\alpha$, $k\in\omega$.

Take $L\in P$ arbitrary. For $r_L=2^{|L|+1}\cdot k\cdot\max f_\alpha[L]$ we have $F_{r_L}\in\mathscr{F}_\alpha$ such that $F_{r_L}\setminus f_\alpha^{-1}[0,r_L]\notin\mathscr{W}$ and we denote $A_{r_L}=(F\cap F_{r_L})\setminus f_\alpha^{-1}[0,r_L]$. Since $A_{r_L}\notin\mathscr{W}$ we can choose an arithmetic progression $L'\subseteq A_{r_L}$ such that $\min L'>\max L$ and $|L'|=k$. Let $K=L\cup L'$. To see that $K\in P$ notice that $\sum_{a\in f_\alpha[K]}\frac{1}{a}=\sum_{a\in f_\alpha[L]}\frac{1}{a}+\sum_{a\in f_\alpha[L']}\frac{1}{a}\le(2-\frac{1}{2^{|L|}})\frac{1}{\min f_\alpha[L]}+\frac{k}{2^{|L|+1}k\max f_\alpha[L]}\le(2-\frac{1}{2^{|L|}}+\frac{1}{2^{|L|+1}})\frac{1}{\min f_\alpha[L]}=(2-\frac{1}{2^{|L|+1}})\frac{1}{\min f_\alpha[L]}$. Since $\min f_\alpha[L]=\min f_\alpha[K]$ and $|K|\ge|L|+1$ we get $\sum_{a\in f_\alpha[K]}\frac{1}{a}\le(2-\frac{1}{2^{|K|}})\frac{1}{\min f_\alpha[K]}$. It is obvious that $K\le_P L$ and $K\in D_{F,k}$ once we have checked that $K\in P$. Therefore $D_{F,k}$ is dense in P.

Since $\mathscr{D}=\{D_{F,k}:F\in\mathscr{F}_\alpha,k\in\omega\}$ is a family of size less than $\mathfrak{c}$ consisting of dense subsets of a countable poset there is a $\mathscr{D}$-generic filter $\mathscr{G}$.

Let $U=\bigcup\{K:K\in\mathscr{G}\}$. It remains to check that:

- $(\forall F\in\mathscr{F}_\alpha)\ U\cap F$ contains arithmetic progressions of arbitrary length

Take $k\in\omega$ arbitrary. For every $K\in\mathscr{G}\cap D_{F,k}$ we have $U\supseteq K$ and $K\cap F$ contains arithmetic progression of length K. Hence $U\cap F$ contains arithmetic progressions of arbitrary length.

- $\sum_{a\in f_\alpha[U]}\frac{1}{a}<+\infty$, i.e., $f_\alpha[U]\in(S)$

Enumerate $f_\alpha[U]=\{u_n:n\in\omega\}$. For every n there exists $K_n\in\mathscr{G}$ such that $u_n\in K_n$. Since $\mathscr{G}$ is a filter we may assume $K_{n+1}\le_P K_n$ for every $n\in\omega$. Obviously, $U=\bigcup_{n\in\omega}K_n$ and we get $\sum_{a\in f_\alpha[U]}\frac{1}{a}\le\frac{2}{\min f_\alpha[U]}$ because $\sum_{a\in f_\alpha[K_n]}\frac{1}{a}\le(2-\frac{1}{2^{|K_n|}})\frac{1}{\min f_\alpha[K_n]}$ for every n.

To complete the induction step let $\mathscr{F}_{\alpha+1}$ be the filter base generated by $\mathscr{F}_\alpha$ and U.

It is obvious that every ultrafilter which extends $\mathscr{F} = \bigcup_{\alpha<\mathfrak{c}} \mathscr{F}_\alpha$ is an (S)-ultrafilter. Because of condition (v) there exists ultrafilter extending $\mathscr{F}$ which extends the dual filter of $\mathscr{W}$. $\square$

The generalized summable ideal $\mathscr{I}_g$ is a tall P-ideal on natural numbers so we can apply Proposition 2.1.13 to show that the existence of $\mathscr{I}_g$-ultrafilters is consistent with ZFC. From Proposition 2.1.14 we get even an $\mathscr{I}_g$-ultrafilter that is not a P-point under the assumption that Continuum Hypothesis holds. The following proposition states that it is sufficient to assume Martin's Axiom for countable posets to construct an $\mathscr{I}_g$-ultrafilter (we regard here $\mathscr{I}_g$ as an ideal on ω, which is possible since ω and $\mathbb{N}$ are isomorphic).

Proposition 2.5.3. *(MA_{ctble}) For every function $g : \omega \to (0, +\infty)$ with $\lim_{n\to\infty} g(n) = 0$ there is an $\mathscr{I}_g$-ultrafilter.*

Proof. Enumerate ${}^\omega\omega = \{f_\alpha : \alpha < \mathfrak{c}\}$. By transfinite induction on $\alpha < \mathfrak{c}$ we will construct filter bases $\mathscr{F}_\alpha$, $\alpha < \mathfrak{c}$, so that the following conditions are satisfied:

(i) $\mathscr{F}_0$ is the Fréchet filter
(ii) $\mathscr{F}_\alpha \subseteq \mathscr{F}_\beta$ whenever $\alpha \le \beta$
(iii) $\mathscr{F}_\gamma = \bigcup_{\alpha<\gamma} \mathscr{F}_\alpha$ for γ limit
(iv) $(\forall\alpha)\ |\mathscr{F}_\alpha| \le |\alpha| \cdot \omega$
(v) $(\forall\alpha)\ (\exists F \in \mathscr{F}_{\alpha+1})\ f_\alpha[F] \in \mathscr{I}_g$, i.e., $\sum_{a\in f_\alpha[F]} g(a) < +\infty$

Induction step: Suppose we know already $\mathscr{F}_\alpha$. If there is $F \in \mathscr{F}_\alpha$ such that $f_\alpha[F] \in \mathscr{I}_g$ then simply put $\mathscr{F}_{\alpha+1} = \mathscr{F}_\alpha$. If $f_\alpha[F] \notin \mathscr{I}_g$ for every $F \in \mathscr{F}_\alpha$ we will construct a suitable set to add.

Consider a poset $P = \{K \in [\omega]^{<\omega} : g(v) \le \frac{1}{2} g(u)$ whenever $u < v$, $u, v \in f_\alpha[K]\}$ and define a partial order $\le_P$ on P in the following way: $K \le_P L$ if and only if $K \supseteq L$. For every $F \in \mathscr{F}_\alpha$ and $m \in \omega$ let $D_{F,m} = \{K \in P : |K \cap F| \ge m\}$.

Claim: $D_{F,m}$ is a dense subset of $(P, \le_P)$ for every $F \in \mathscr{F}_\alpha$, $m \in \omega$.

Take arbitrary $L \in (P, \le_P)$. Since $f_\alpha[F] \notin \mathscr{I}_g$ the set $F \setminus f_\alpha^{-1}[0, \max f_\alpha[L]]$ is infinite. So we can choose $x_1 \in F$ such that $x_1 > \max L$, $f_\alpha(x_1) > \max f_\alpha[L]$ and $g(f_\alpha(x_1)) < \frac{1}{2} \max\{g(u) : u \in f_\alpha[L]\}$. Now, we can proceed by induction and choose elements $x_2, \dots, x_m$ such that $x_i \in F$, $x_i > x_{i-1}$, $f_\alpha(x_i) > f_\alpha(x_{i-1})$ and $g(f_\alpha(x_i)) < \frac{1}{2} g(f_\alpha(x_{i-1}))$ for $i = 2, \dots, m$. Finally, put $K = L \cup \{x_1, x_2, \dots, x_m\}$. Obviously, $K \le_P L$ and $K \in D_{F,m}$ so the set $D_{F,m}$ is dense in P.

Since the family $\mathscr{D} = \{D_{F,m} : F \in \mathscr{F}_\alpha, m \in \omega\}$ consists of dense subsets of a countable poset and $|\mathscr{D}| < \mathfrak{c}$ there is a $\mathscr{D}$-generic filter $\mathscr{G}$ on P.

Let $U = \bigcup\{K : K \in \mathscr{G}\}$. It remains to check that:

- $(\forall F \in \mathscr{F}_\alpha)$ $U \cap F$ is infinite

  For every $m \in \omega$ there exists $K \in \mathscr{G} \cap D_{F,m}$. Since $U \supseteq K$ we get $|U \cap F| \geq |K \cap F| \geq m$ and the set $U \cap F$ is infinite.

- $\sum_{a \in f_\alpha[U]} g(a) < +\infty$, i.e., $f_\alpha[U] \in \mathscr{I}_g$

  Let $f_\alpha[U] = \{u_n : n \in \omega\}$ be an increasing enumeration of $f_\alpha[U]$. According to the definition of P we have $g(u_{n+1}) \leq \frac{1}{2} g(u_n)$ for every n. Hence $\sum_{n\in\omega} g(u_n) \leq g(u_0) \cdot \sum_{n\in\omega} \frac{1}{2^n} < +\infty$.

To complete the induction step let $\mathscr{F}_{\alpha+1}$ be the filter base generated by $\mathscr{F}_\alpha$ and U. It is obvious that every ultrafilter which extends $\mathscr{F} = \bigcup_{\alpha<\mathfrak{c}} \mathscr{F}_\alpha$ is an $\mathscr{I}_g$-ultrafilter. $\square$

3 Sums of $\mathscr{I}$-ultrafilters

Baumgartner in [2] studied closure of $\mathscr{I}$-ultrafilters under ultrafilter sums for the setting $X = \mathbb{R}$ and $\mathscr{I}$ a family of subsets of $\mathbb{R}$. We study the closure of $\mathscr{I}$-ultrafilters under ultrafilter sums for the case $X = \omega$ and $\mathscr{I}$ is an ideal on ω. Two general results can be found in the first section and some results concerning ideals from chapter 1 are in the second section.

Let us recall the definition of ultrafilter sums and products at first:

If $\mathscr{U}$ and $\mathscr{V}_n$, $n \in \omega$, are ultrafilters on ω then $\sum_{\mathscr{U}} \langle \mathscr{V}_n : n \in \omega \rangle$ is the ultrafilter on $\omega \times \omega$ defined by $M \in \sum_{\mathscr{U}} \langle \mathscr{V}_n : n \in \omega \rangle$ if and only if $\{n : \{m : \langle n, m \rangle \in A\} \in \mathscr{V}_n\} \in \mathscr{U}$. We often identify isomorphic ultrafilters so we occasionally regard $\sum_{\mathscr{U}} \langle \mathscr{V}_n : n \in \omega \rangle$ as an ultrafilter on ω. Ultrafilter $\sum_{\mathscr{U}} \langle \mathscr{V}_n : n \in \omega \rangle$ is called the *$\mathscr{U}$-sum of ultrafilters $\mathscr{V}_n$, $n \in \omega$*. If $\mathscr{V}_n = \mathscr{V}$ for every $n \in \omega$ then we write $\sum_{\mathscr{U}} \langle \mathscr{V}_n : n \in \omega \rangle = \mathscr{U} \cdot \mathscr{V}$ and ultrafilter $\mathscr{U} \cdot \mathscr{V}$ is called the *product of ultrafilters $\mathscr{U}$ and $\mathscr{V}$*.

3.1 General results

Definition 3.1.1. Let $\mathcal{C}$ and $\mathcal{D}$ be classes of ultrafilters. We say that $\mathcal{C}$ is *closed under $\mathcal{D}$-sums* provided that whenever $\{\mathscr{V}_n : n \in \omega\} \subseteq \mathcal{C}$ and $\mathscr{U} \in \mathcal{D}$ then $\sum_{\mathscr{U}} \langle \mathscr{V}_n : n \in \omega \rangle \in \mathcal{C}$. In practice we can talk about closure under Ramsey sums, P-point sums, $\mathscr{I}$-sums, thin sums, (S)-sums, (SC)-sums, (H)-sums, etc.

Proposition 3.1.2. *Let $\mathscr{I}$ be an ideal on ω and $\mathcal{C}$ a class of ultrafilters on ω. If there exists an ultrafilter in $\mathcal{C}$ which is not an $\mathscr{I}$-ultrafilter then the class of $\mathscr{I}$-ultrafilters is not closed under $\mathcal{C}$-sums (in other words, if the class of $\mathscr{I}$-ultrafilters is closed under $\mathcal{C}$-sums then $\mathcal{C}$ is a subclass of $\mathscr{I}$-ultrafilters).*

Proof. Let $\mathscr{V}_n$, $n \in \omega$, be arbitrary $\mathscr{I}$-ultrafilters and let $\mathscr{U} \in \mathcal{C}$ be an ultrafilter that is not an $\mathscr{I}$-ultrafilter, i.e., there exists $g : \omega \to \omega$ such that $g[V] \notin \mathscr{I}$ for every $V \in \mathscr{U}$. Define $f : \omega \times \omega \to \omega$ so that $f(\langle n, m \rangle) = g(n)$ for every $n, m \in \omega$. For every $U \subseteq \omega \times \omega$ let $U_n = \{m : \langle n, m \rangle \in U\}$ and $\widetilde{U} = \{n : U_n \in \mathscr{V}_n\}$.

For every $U \in \sum_{\mathscr{U}} \langle \mathscr{V}_n : n \in \omega \rangle$ we have $\{n : (\exists m)\, \langle n, m \rangle \in U\} \supseteq \widetilde{U} \in \mathscr{U}$. Hence $f[U] \supseteq g[\widetilde{U}] \notin \mathscr{I}$ and $\sum_{\mathscr{U}} \langle \mathscr{V}_n : n \in \omega \rangle$ is not an $\mathscr{I}$-ultrafilter. □

Proposition 3.1.3. *If $\mathscr{I}$ is a P-ideal on ω then the class of $\mathscr{I}$-ultrafilters is closed under $\mathscr{I}$-sums.*

Proof. Suppose $\mathscr{U}$ and $\mathscr{V}_n$, $n \in \omega$, are $\mathscr{I}$-ultrafilters. Let $f : \omega \times \omega \to \omega$ be an arbitrary function. We want to find $U \in \sum_{\mathscr{U}} \langle \mathscr{V}_n : n \in \omega \rangle$ such that $f[U] \in \mathscr{I}$.

Define function $f_n : \omega \to \omega$ by $f_n(m) = f(\langle n, m\rangle)$ for every $n \in \omega$.

Since $\mathscr{V}_n$ is an $\mathscr{I}$-ultrafilter there exists $U_n \in \mathscr{V}_n$ such that $f_n[U_n] \in \mathscr{I}$ for every n. Now we can find a set $A \in \mathscr{I}$ such that $f_n[U_n] \subseteq^* A$ for every n because we assumed that $\mathscr{I}$ is a P-ideal.

It is obvious that $f_n^{-1}[f_n[U_n]] \in \mathscr{V}_n$. Therefore either $f_n^{-1}[f_n[U_n] \cap A]$ or $f_n^{-1}[f_n[U_n] \setminus A]$ belongs to $\mathscr{V}_n$. Let $I_0 = \{n \in \omega : f_n^{-1}[f_n[U_n] \cap A] \in \mathscr{V}_n\}$ and $I_1 = \{n \in \omega : f_n^{-1}[f_n[U_n] \setminus A] \in \mathscr{V}_n\}$. Since $\mathscr{U}$ is an ultrafilter one of the sets I_0, I_1 belongs to the ultrafilter $\mathscr{U}$.

Case A. $I_0 \in \mathscr{U}$

Put $U = \{\{n\} \times f_n^{-1}[f_n[U_n] \cap A] : n \in I_0\}$. It is easy to see that $U \in \sum_{\mathscr{U}} \langle \mathscr{V}_n : n \in \omega\rangle$ and $f[U] = \bigcup_{n \in I_0} f_n[U_n] \cap A \subseteq A \in \mathscr{I}$.

Case B. $I_1 \in \mathscr{U}$

Since $f_n[U_n] \setminus A$ is finite and $\mathscr{V}_n$ is an ultrafilter, there exists $k_n \in f_n[U_n] \setminus A$ such that $f_n^{-1}\{k_n\} \in \mathscr{V}_n$. Define $g : \omega \to \omega$ by $g(n) = k_n$. Since $\mathscr{U}$ is an $\mathscr{I}$-ultrafilter there exists $V \in \mathscr{U}$ such that $g[V] \in \mathscr{I}$. It remains to put $U = \{\{n\} \times f_n^{-1}\{k_n\} : n \in I_1 \cap V\}$. It is easy to check that $U \in \sum_{\mathscr{U}} \langle \mathscr{V}_n : n \in \omega\rangle$ and $f[U] \subseteq g[V] \in \mathscr{I}$. $\square$

3.2 Special classes

Proposition 3.2.1. *The class of (S)-ultrafilters is closed under (S)-sums, the class of (H)-ultrafilters is closed under (H)-sums and the class of $\mathscr{I}_g$-ultrafilters is closed under $\mathscr{I}_g$-sums.*

Proof. Since (S), (H) and $\mathscr{I}_g$ are P-ideals it is an immediate consequence of Proposition 3.1.3. $\square$

Ideals generated by thin sets and (SC)-sets are not P-ideals (see Proposition 1.1.5 and Proposition 1.2.8) and it turns out that thin ultrafilters and (SC)-ultrafilters are not closed even under products which are special cases of sums.

Proposition 3.2.2. *$\mathscr{U} \cdot \mathscr{U}$ is neither a thin ultrafilter nor an (SC)-ultrafilter for every $\mathscr{U} \in \omega^*$.*

Proof. Assume $\mathscr{U}$ is a free ultrafilter on ω. Let us recall that $\mathscr{U} \cdot \mathscr{U} = \sum_{\mathscr{U}} \langle \mathscr{V}_n : n \in \omega\rangle$ where $\mathscr{V}_n = \mathscr{U}$ for every $n \in \omega$. For every $U \subseteq \omega \times \omega$ let $U_n = \{m : \langle n, m\rangle \in U\}$ and $\widetilde{U} = \{n : U_n \in \mathscr{U}\}$.

Consider $f : \omega \times \omega \to \omega$ defined by $f(\langle n, m\rangle) = n + m$. For every $n \in \omega$ define $f_n : \omega \to \omega$ by $f_n(m) = f(\langle n, m\rangle)$. Notice that that f_n is one-to-one for every n and f is finite-to-one. For every $A \subseteq \omega$ we have $f_n[A] = A + n = \{a + n : a \in A\}$ according to the definition of f_n. We will show in the following that $f[U]$ is not thin and $f[U] \notin (SC)$ for every $U \in \mathscr{U} \cdot \mathscr{U}$.

Fix $U \in \mathscr{U} \cdot \mathscr{U}$ and let $\{a_n : n \in \omega\}$ be an increasing enumeration of $f[U]$. Since $U \in \mathscr{U} \cdot \mathscr{U}$ we have $\widetilde{U} \in \mathscr{U}$, in particular $\widetilde{U}$ is infinite. Choose two distinct elements $n_1, n_2 \in \widetilde{U}$ and denote $V = U_{n_1} \cap U_{n_2}$. The set V is infinite because $\mathscr{U}$ is a free ultrafilter. We get $f[U] = \bigcup_{n \in \omega} f_n[U_n] \supseteq f_{n_1}[U_{n_1}] \cup f_{n_2}[U_{n_2}] \supseteq (V + n_1) \cup (V + n_2)$. It follows that $|f[U] \cap [v, v + \max\{n_1, n_2\}]| \geq 2$ for every $v \in V$.

If $u \geq \max\{n_1, n_2\}$ then for $a_n, a_{n+1} \in [u, u + \max\{n_1, n_2\}]$ we have $\frac{a_n}{a_{n+1}} \geq \frac{u}{u + \max\{n_1, n_2\}} \geq \frac{1}{2}$ and $a_{n+1} - a_n \leq \max\{n_1, n_2\}$.

There are infinitely many $u \in V$ with $u \geq \max\{n_1, n_2\}$. It follows that the set $f[U]$ is not thin because $\limsup_{n \to \infty} \frac{a_n}{a_{n+1}} \geq \frac{1}{2}$ and $f[U]$ does not belong to (SC) either because there is $j \leq \max\{n_1, n_2\}$ such that $(f[U] + j) \cap f[U]$ is infinite. $\square$

4 Weaker forms of $\mathscr{I}$-ultrafilters

In the first section of this chapter we present several results concerning weak $\mathscr{I}$-ultrafilters which we define analogously to $\mathscr{I}$-ultrafilters with the only difference that functions considered in the definition are finite-to-one.

In the second section we restrict further the family of functions considered in the definition of an $\mathscr{I}$-ultrafilter to one-to-one functions and focus on the summable ideal to get an example of such an ultrafilter in ZFC.

4.1 Weak $\mathscr{I}$-ultrafilters

Definition 4.1.1. Let $\mathscr{I}$ be a family of subsets of a set X such that $\mathscr{I}$ contains all singletons and is closed under subsets. Given an ultrafilter $\mathscr{U}$ on ω, we say that $\mathscr{U}$ is a *weak $\mathscr{I}$-ultrafilter* if for every finite-to-one mapping $F : \omega \to X$ there is $U \in \mathscr{U}$ such that $F[U] \in \mathscr{I}$.

Obviously, every $\mathscr{I}$-ultrafilter is a weak $\mathscr{I}$-ultrafilter.

In the following we concentrate on weak $\mathscr{I}$-ultrafilters, where $X = \omega$ and $\mathscr{I}$ is again a collection of small subsets of ω and we are especially interested in the ideals introduced in chapter 1.

Lemma 4.1.2. *If $\mathscr{I}$ is a tall P-ideal and $\mathscr{U}_n$, $n \in \omega$, weak $\mathscr{I}$-ultrafilters then every accumulation point of the set $\{\mathscr{U}_n : n \in \omega\}$ is a weak $\mathscr{I}$-ultrafilter.*

Proof. Assume f is a finite-to-one function. There exists $U_n \in \mathscr{U}_n$ such that $f[U_n] \in \mathscr{I}$ for every $n \in \omega$. Since $\mathscr{I}$ is a P-ideal there exists an infinite set $A \in \mathscr{I}$ such that $f[U_n] \subseteq^* A$ for every n. It implies $U_n \subseteq^* f^{-1}[A]$ because f is finite-to-one. If $\mathscr{U}$ is an accumulation point of $\{\mathscr{U}_n : n \in \omega\}$ then $U = f^{-1}[A] \in \mathscr{U}$ because $\omega \setminus f^{-1}[A] \notin \mathscr{U}_n$ for every n. Of course, $f[U] = A \in \mathscr{I}$ and it follows that $\mathscr{U}$ is a weak $\mathscr{I}$-ultrafilter. □

Weak thin ultrafilters provide a new description of Q-points.

Proposition 4.1.3. *An ultrafilter on ω is a weak thin ultrafilter if and only if it is a Q-point.*

Proof. It follows from the proof of Proposition 2.4.1 that every weak thin ultrafilter is a Q-point.

Now, assume $\mathscr{U}$ is a Q-point and $f : \omega \to \omega$ is a finite-to-one mapping. Define $Q_n = f^{-1}[n!, (n+1)!)$ for every $n \in \omega$. The family $\{Q_n : n \in \omega\}$ is a partition of ω into finite sets. So there exists $V \in \mathscr{U}$ such that $|V \cap Q_n| \leq 1$ for every n. Since $\mathscr{U}$ is an ultrafilter either $V_0 = \bigcup\{Q_{2n} : n \in \omega\}$, or

$V_1 = \bigcup\{Q_{2n+1} : n \in \omega\}$ belongs to the ultrafilter $\mathscr{U}$. We may assume that $V_0 \in \mathscr{U}$. Let $U = V \cap V_0$. It is easy to verify that $f[U]$ is a thin set:

If $a_n, a_{n+1} \in f[U]$ then there is $k_n \in \omega$ such that $a_n \in [(2k_n)!, (2k_n+1)!)$ and $a_{n+1} \geq (2k_n+2)!$. We get $\limsup_{n\to\infty} \frac{a_n}{a_{n+1}} \leq \limsup_{n\to\infty} \frac{(2k_n+1)!}{(2k_n+2)!} = \limsup_{n\to\infty} \frac{1}{2k_n+2} \leq \limsup_{n\to\infty} \frac{1}{2n+2} = 0$. □

Corollary 4.1.4. *It is consistent that there are no weak thin ultrafilters.* □

It follows from Proposition 2.4.7 that there are rapid ultrafilters which are not (S)-ultrafilters, but there are no rapid ultrafilters which are not weak (S)-ultrafilters.

Proposition 4.1.5. *Every rapid ultrafilter is a weak (S)-ultrafilter.*

Proof. Assume $\mathscr{U}$ is a rapid ultrafilter and $f : \omega \to \omega$ a finite-to-one function. Define $g(n) = \max f^{-1}[0, 2^n] + 1$. Since $\mathscr{U}$ is rapid there is $U \in \mathscr{U}$ such that $g \leq^* e_U$. So we have $u_n \geq g(n)$ (where u_n denotes the nth element of U) for every $n \geq n_0$. The definition of function g gives $f(u_n) > 2^n$. It follows that

$$\sum_{a\in f[U]} \frac{1}{a} \leq \sum_{n<n_0} \frac{1}{f(u_n)} + \sum_{n\geq n_0} \frac{1}{f(u_n)} \leq \sum_{n<n_0} \frac{1}{f(u_n)} + \sum_{n\geq n_0} \frac{1}{2^n} < +\infty.$$

Hence $f[U]$ belongs to the summable ideal and $\mathscr{U}$ is a weak $\mathscr{I}$-ultrafilter. □

4.2 0-points and summable ultrafilters

Let us recall that an ultrafilter $\mathscr{U} \in \mathbb{N}^*$ is called a 0-*point* if for every one-to-one function $f : \mathbb{N} \to \mathbb{N}$ there exists a set $U \in \mathscr{U}$ such that $f[U]$ has asymptotic density zero. Gryzlov constructed such ultrafilters in ZFC (see [17], [18]).

We strengthen Gryzlov's result and construct a *summable ultrafilter* that we define as an ultrafilter $\mathscr{U} \in \mathbb{N}^*$ such that for every one-to-one function $f : \mathbb{N} \to \mathbb{N}$ there exists $U \in \mathscr{U}$ with $f[U]$ in the summable ideal. Our proof was motivated by Gryzlov's original construction as it was written down by K. P. Hart [19].

Let us call a family $\mathscr{F} \subseteq \mathscr{P}(\mathbb{N})$ summable if for every one-to-one function $f : \mathbb{N} \to \mathbb{N}$ there is $A \in \mathscr{F}$ such that $f[A]$ belongs to the summable ideal.

During the construction we make use of the following upper bound for partial sums of the harmonic series:

Fact 4.2.1. $1 + \frac{1}{2} + \cdots + \frac{1}{N} \leq 1 + \ln N \leq 1 + \log_2 N$ *for every* $N \in \mathbb{N}$.

Lemma 4.2.2 is fairly general, but it enables us to construct a summable centered system by applying Proposition 4.2.3 to get summable k-linked families for every k. The summable centered system may then be extended to a summable ultrafilter.

Lemma 4.2.2. *If $\mathscr{F}_k$ is a k-linked family of infinite subsets of $\mathbb{N}$ for every $k \in \mathbb{N}$ then $\mathscr{F} = \{F \subseteq \mathbb{N} : (\forall k)(\exists U^k \in \mathscr{F}_k)\, U^k \subseteq^* F\}$ is a centered system.*

If moreover, $\mathscr{I}$ is a P-ideal, $f \in {}^{\mathbb{N}}\mathbb{N}$ a one-to-one function and for every $k \in \mathbb{N}$ there exists $U^k \in \mathscr{F}_k$ such that $f[U^k] \in \mathscr{I}$ then there exists $U \in \mathscr{F}$ such that $f[U] \in \mathscr{I}$. In particular, if $\mathscr{F}_k$ is summable for every k then $\mathscr{F}$ is summable.

Proof. Take $F_1, F_2, \ldots, F_n \in \mathscr{F}$ and for every $j = 1, \ldots, n$ choose $U_j^k \in \mathscr{F}_k$ such that $U_j^k \subseteq^* F_j$ for every k. For every $k \geq n$ family $\mathscr{F}_k$ is n-linked, hence $\bigcap_{j=1}^n U_j^k$ is an infinite set. We have

$$\bigcap_{j=1}^{n} U_j^k \subseteq^* \bigcap_{j=1}^{n} F_j$$

for every $k \geq n$ and it follows that family $\mathscr{F}$ is centered.

For the moreover part, consider $A \in \mathscr{I}$ such that $f[U^k] \subseteq^* A$ for every $k \in \mathbb{N}$. We get $U^k \subseteq^* f^{-1}[A]$ for every $k \in \mathbb{N}$. According to the definition set $U = f^{-1}[A]$ belongs to $\mathscr{F}$ and $f[U] = A \in \mathscr{I}$. $\square$

Proposition 4.2.3. *Let A be an infinite subset of $\mathbb{N}$. For every $k \in \mathbb{N}$ there exists a summable k-linked family $\mathscr{F}_k \subseteq \mathscr{P}(A)$.*

Proof. Fix $k \in \mathbb{N}$. We divide A into disjoint finite blocks, $A = \bigcup_{n\in\mathbb{N}} B_n$, and for every n enumerate B_n, faithfully, as $\{b(\varphi) : \varphi \in \prod_{j=0}^k Q(j,n)\}$ where $Q(j,n)$ is defined by $Q(j,n) = 2^{n \cdot 2^j}$. Notice that for every $i \leq k$ we have $|Q(i,n)| = 2^n \cdot |\prod_{j=0}^{i-1} Q(j,n)|$.

For every $i \leq k$, $x \in Q(i,n)$ and $s \in \prod_{j=i+1}^k Q(j,n)$ define $B_n(i,x,s) = \{b(\varphi^\frown\langle x\rangle^\frown s) : \varphi \in \prod_{j=0}^{i-1} Q(j,n)\}$. For every one-to-one function $f : \mathbb{N} \to \mathbb{N}$ let $m_x^f = \min f[B_n(i,x,s)]$. Finally, let $x(f,s) \in Q(i,n)$ be that x for which m_x^f is maximal, i.e., $m_{x(f,s)}^f = \max\{m_x^f : x \in Q(i,n)\}$. Now, we may define $A^f \subseteq A$ block by block as the union $A^f = \bigcup_{n\in\mathbb{N}} B_n^f$, where $B_n^f \subseteq B_n$ is defined in two stages: first $B_n^f = \bigcup_{i=0}^k B_n^f(i)$ and second $B_n^f(i) = \bigcup\{B_n^f(i,s) : s \in \prod_{j=i+1}^k Q(j,n)\}$, where $B_n^f(i,s) = B_n(i, x(f,s), s)$.

Claim 1. The family $\mathscr{F}_k = \{A^f : f \in {}^{\mathbb{N}}\mathbb{N}$ one-to-one$\}$ is k-linked.

Consider $f_0, f_1, \ldots, f_k$ distinct one-to-one functions from $\mathbb{N}$ to $\mathbb{N}$. Since

$$\bigcap_{j=0}^{k} A^{f_j} \supseteq \bigcup_{n=1}^{\infty} \bigcap_{j=0}^{k} B_n^{f_j}$$

it suffices to show that $\bigcap_{j=0}^{k} B_n^{f_j} \neq \emptyset$ for every $n \in \mathbb{N}$. To see this fix n and define $\varphi \in \prod_{j=0}^{k} Q(j,n)$ recursively: put $s_0 = \emptyset$ and set $\varphi(k) = x(f_0, s_0)$, next $s_1 = \langle \varphi(k) \rangle$ and $\varphi(k-1) = x(f_1, s_1)$, and so on. It follows that $b(\varphi) \in \bigcap_{j=0}^{k} B_n^{f_j}(k-j, s_j) \subseteq \bigcap_{j=0}^{k} B_n^{f_j}(k-j) \subseteq \bigcap_{j=0}^{k} B_n^{f_j}$.

Claim 2. For every one-to-one function f the set $f[A^f]$ belongs to the summable ideal.

Our aim is to bound the sum $\sum_{a \in B_n^f} \frac{1}{f(a)}$ from above by elements of a convergent series because $f[A^f] = \bigcup_{n \in \mathbb{N}} f[B_n^f]$. At first, we estimate the sum of the reciprocals of elements in $f[B_n^f(i,s)]$ for every $i \leq k$ and $s \in \prod_{j=i+1}^{k} Q(j,n)$.

Since $|f[B_n^f(i,s)]| = |\prod_{j=0}^{i-1} Q(j,n)|$ we have

$$\sum_{a \in B_n^f(i,s)} \frac{1}{f(a)} \leq \Big| \prod_{j=0}^{i-1} Q(j,n) \Big| \cdot \frac{1}{\min f[B_n^f(i,s)]} = \frac{2^{n \cdot (2^i - 1)}}{m_{x(f,s)}^f} \tag{1}$$

Put $q_{i,n} = |\prod_{j=i+1}^{k} Q(j,n)|$ and enumerate $\{m_{x(f,s)}^f : s \in \prod_{j=i+1}^{k} Q(j,n)\}$ increasingly as $\{m_l : l = 1, \ldots, q_{i,n}\}$. It is easy to see that $m_l \geq l \cdot Q(i,n)$ for every l and it follows that

$$\sum_{l=1}^{q_{i,n}} \frac{1}{m_l} \leq \frac{1}{Q(i,n)} \cdot \sum_{l=1}^{q_{i,n}} \frac{1}{l} \leq \frac{1 + \log_2 q_{i,n}}{Q(i,n)} = \frac{1 + \sum_{j=i+1}^{k} \log_2 Q(j,n)}{Q(i,n)} \tag{2}$$

where we used Fact 4.2.1.

Now, observe that

$$1 + \sum_{j=i+1}^{k} \log_2 Q(j,n) \leq 1 + n \sum_{j=0}^{k} 2^j = 1 + n(2^{k+1} - 1) \leq n 2^{k+1} \tag{3}$$

and putting together (1), (2) and (3) we obtain

$$\sum_{a \in B_n^f(i)} \frac{1}{f(a)} \leq \Big| \prod_{j=0}^{i-1} Q(j,n) \Big| \cdot \frac{1 + \sum_{j=i+1}^{k} \log_2 Q(j,n)}{Q(i,n)} = \frac{n 2^{k+1}}{2^n}. \tag{4}$$

Thus we get for every n

$$\sum_{a\in B_n^f} \frac{1}{f(a)} \leq \sum_{i=0}^{k} \frac{n2^{k+1}}{2^n} = \frac{n(k+1)2^{k+1}}{2^n} \tag{5}$$

and finally

$$\sum_{a\in A^f} \frac{1}{f(a)} \leq \sum_{n=1}^{\infty} \frac{n(k+1)2^{k+1}}{2^n} \leq 2(k+1)2^{k+1}, \tag{6}$$

i.e., the set $f[A^f]$ belongs to the summable ideal. □

While constructing a 0-point Gryzlov made use of function $Q(j,n) = n^{2^j}$. We cannot use this function for our purpose because it "grows too slowly". Its polynomial growth with respect to n provides in formula (4) (or (5)) a divergent series as an upper bound for $\sum_{a\in B_n^f} \frac{1}{f(a)}$. So it seems to be necessary that $Q(j,n)$ depends exponentially on n. In formula (4) occurs $|\prod_{j=0}^{i-1} Q(j,n)| \cdot Q(i,n)^{-1}$, which excludes functions of type $2^n \cdot p(j)$ or $2^{n\cdot p(j)}$ where $p(j)$ is a polynomial in j. Hence our definition $Q(j,n) = 2^{n\cdot 2^j}$ seems to be the best possible to use while constructing a summable ultrafilter.

Theorem 4.2.4. *There is a summable ultrafilter on* $\mathbb{N}$.

Proof. Consider an arbitrary countable family $\{A_k : k \in \mathbb{N}\}$ of infinite subsets of natural numbers and apply Proposition 4.2.3 to obtain a summable k-linked family $\mathscr{F}_k$ on A_k for every k. From Lemma 4.2.2 we obtain a summable centered system $\mathscr{F}$ on $\mathbb{N}$. It is obvious that any ultrafilter that extends $\mathscr{F}$ is summable. □

Corollary 4.2.5. *There are* $2^{\mathfrak{c}}$ *distinct summable ultrafilters on* $\mathbb{N}$.

Proof. Assume $\{A_k : k \in \mathbb{N}\}$ is a countable family of disjoint infinite subsets of $\mathbb{N}$ and $\mathscr{F}_k$ is a summable k-linked family on A_k for every k. For every free ultrafilter $\mathscr{U}$ on $\mathbb{N}$ let $\mathscr{F}_{\mathscr{U}} \subseteq \mathscr{P}(\mathbb{N})$ consist of sets $F \subseteq \mathbb{N}$ such that $\{k : F \cap A_k \in \mathscr{F}_k\} \in \mathscr{U}$. It is easy to see that $\mathscr{F}_{\mathscr{U}}$ is a summable filter base and $\mathscr{F}_{\mathscr{U}} \neq \mathscr{F}_{\mathscr{V}}$ whenever $\mathscr{U} \neq \mathscr{V}$. It follows that there are $2^{\mathfrak{c}}$ distinct summable ultrafilters. □

The construction of a summable ultrafilter relies strongly on the fact that functions in question are one-to-one and there is no obvious way to transform the construction to obtain (S)-ultrafilters or even weak (S)-ultrafilters although the moreover part of Lemma 4.2.2 is still true for all finite-to-one functions. This is not the case for Proposition 4.2.3, which can be easily modified just for those finite-to-one functions f for which the size of preimages of singletons, i.e., the sequence $|f^{-1}(n)|_{n\in\mathbb{N}}$, is bounded from above by a natural number p (such functions are called p-to-one). It suffices then to enumerate the block B_n in the proof of Proposition 4.2.3 faithfully as $\{b(r,\varphi) : r \in p, \varphi \in \prod_{j=0}^{k} Q(j,n)\}$ and we may repeat the construction step by step. The only difference is that in formula (6) from Claim 2 we get another upper bound for the sum of reciprocals of the elements of $f[A^f]$, namely $\sum_{a\in A^f} \frac{1}{f(a)} \leq 2p(k+1)2^{k+1}$.

Another interesting question arises if we replace the summable ideal in the definition of a summable ultrafilter by the generalized summable ideal $\mathscr{I}_g$ defined in chapter 1. It is not known at the moment whether it is possible to construct in ZFC an ultrafilter $\mathscr{U}$ such that for every one-to-one function there is $U \in \mathscr{U}$ with $f[U] \in \mathscr{I}_g$ for arbitrary function g.

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