Rational Hypersurfaces with Rational Convolutions

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Abstract

The aim of this article is to focus on the investigation of such rationally parametrized hypersurfaces which admit rational convolution (RC) generally, or in some special cases. Examples of such hypersurfaces are presented and their properties are discussed. We also aim to examine links between well-known curves and surfaces (PH/PN or LN) and general objects defined and explored in this article. In addition, the paper brings a proof that the convolution surfaces of quadratic Bézier surfaces and arbitrary rational surface are always rational.

Key words: Rational curves and surfaces; Convolution hypersurfaces; Offsets; GRC hypersurfaces; Quadratic Bézier triangular surfaces

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1 Introduction

During the past years and decades, geometric methods have played an increasingly important role in a variety of areas dealing with computing for industrial applications such as Computer Aided Design and Manufacturing, Geometric Modeling, Computational Geometry, Robotics, Computer Vision, Pattern Recognition, Reverse Engineering, Image Processing, Computer Graphics etc. Despite the fact that these areas originated from different requirements in specific applications, we can find very similar problems solved by different

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communities. Let us illustrate this on the example of computation of so called convolution curves or surfaces that are not only theoretic geometric objects with extremely interesting properties but they are also very applicable in many other fields like for example in machine tooling.

The choice of an appropriate representation of geometric objects (explicit, parametric, or implicit one) is a fundamental issue for the development of efficient algorithms. Whereas for example Computer Graphics seem to use all the above mentioned representations, CAD focuses on a few of them — mainly on the parametric one. And among all parametrizations, the most important ones are those that can be described with the help of polynomials or rational functions since these descriptions can be easily included into standard CAD systems and then used in the technical praxis. Unfortunately, the existence of rational parametrization of some geometric object still does not guarantee that further derived objects (offsets, hodographs, convolutions etc.) also have parametrized rational description. In this paper, we study the class of curves and surfaces which fulfill this condition for convolutions.

However, the situation is not so easy and clear as one might think at the first sight. The process of finding an exact description of convolution is one of the most challenging problems in Computer Aided Geometric Design (CAGD). Convolutions with spheres (which are nothing else than classical offsets) and their properties have been studied for many years and most often because they play a very important role in the milling theory with the help of sphere cutter. In recent years, interest in studying convolutions with different non-spherical surfaces (which yields the theory of general offsets) has risen and thus many other cases are examined and new techniques are now being explored.

The theory of PH curves (i.e. curves with Pythagorean Hodographs) and PN surfaces (surfaces with Pythagorean Normals) was developed by Farouki and Sakkalis (1990) and by Pottmann (1995). These Pythagorean objects are extremely interesting because of the fact that their convolutions with circles (for PH curves) or spheres (for PN surfaces) also have rational description. Recently, Peternell and Manhart (2003) have studied convolutions between paraboloids and arbitrary rational surfaces. Later, Sampoli, Peternell and Jüttler (2006) generalized this approach using a very interesting concept of LN surfaces (surfaces with Linear field of Normals) introduced by Jüttler (1998). From a practical point of view, the main and most important difference between surfaces with PN, or LN parametrizations is as follows — while PN surfaces as workpieces allow computation of the rational path just of the sphere cutter, LN surfaces can be in addition taken as the universal tool with the guarantee of rationality of the path and thus they easily admit the concept of general offsets.

All of this motivates us to study in more detail hypersurfaces which yield
rational convolution generally, or in some special cases. Examples of concrete curves and surfaces will be shown and their properties will be discussed as well. In particular, close connections with above mentioned curve and surface classes (e.g. PH–/PN– or LN– curves and surfaces) are presented and new classes of geometric objects will be established.

The paper is organized as follows. Elementary facts with references are reviewed in the first part. The second section is devoted to fundamental definitions and theorems connected with the theory of rational convolutions. Then, we introduce some concrete examples of rational curves and surfaces which admit generally rational convolution and discuss some important properties of objects of our interest. Finally, we conclude this paper and show some possibilities for further research.

2 Preliminaries

This section is devoted to basic definitions, theorems and relations between defined notions which are necessary for understanding the following sections of the paper.

2.1 Convolution hypersurfaces

The notions of convolution curves or surfaces in two or three dimensions are used in various fields of mathematics, e.g. mathematical morphology, computer graphics, convex geometry and computational geometry, and there are also many further applications in technical praxis. However, rational convolution hypersurfaces of two general hypersurfaces did not receive much attention in the past but nowadays they are studied and applied much more frequently. In this article, the convolutions of curves or surfaces (defined below) will be denoted by $A \ast B$.

Definition 1 Let $A$ and $B$ be smooth hypersurfaces in the affine space $\mathbb{R}^n$. The convolution hypersurface $C = A \ast B$ is defined

$$C = A \ast B = \{a + b \mid a \in A, b \in B \text{ and } \alpha(a) \parallel \beta(b)\}, \tag{1}$$

where $\alpha(a)$ and $\beta(b)$ are the tangent hyperplanes of $A$ and $B$ at points $a \in A$ and $b \in B$ which are called corresponding points.

The construction can be seen in Fig. 1. Let us point out that in case of arbitrary surfaces $A$ and $B$ there is generally no one-to-one correspondence between corresponding points.
In some other papers, e.g. Peternell and Manhart (2003) or Sampoli, Peternell and Jüttler (2006), the convolution hypersurface $A \star B$ of two smooth hypersurfaces $A$ and $B$ is defined in a slightly different way with the help of normal vectors $\mathbf{N}_A(a)$ and $\mathbf{N}_B(b)$ constructed at points $a \in A$ and $b \in B$. Of course, this assumes that the space $\mathbb{R}^n$ is not only affine but Euclidean. The condition for corresponding points then reads

$$\mathbf{N}_A(a) \parallel \mathbf{N}_B(b).$$

(2)

The construction of convolution $A \star B$ also admits a kinematic interpretation which is very useful for visualization. Consider the hypersurface $A$ together with the origin $O$ as a movable system and let the hypersurface $B$ be fixed. We can denote the different positions of $A$ and $O$ in the time $t$ by $A_t$ and $O_t$. The kinematic system is now moved translatory in the way that the point $O_t$ travels on $B$. Hence, the convolution hypersurface $A \star B$ is then obtained as envelope of $A_t$ under this translation.

Our designed approach is a little bit different compared with the above mentioned because we want to emphasise a study of convolutions from the pure affine point of view. As a result, the convolution hypersurfaces are invariant under affine transformations because the parallelism is generally preserved and our classification will be meaningful from the affine point of view. However, all of our results can be also used in both approaches noted above if we change the affine space into Euclidean one.

In general, the convolution hypersurface $A \star B$ of two smooth hypersurfaces $A$ and $B$ can be computed using the following algorithm. Let $A$ be parametrized by $a(u_1, \ldots, u_{n-1})$ and $B$ by $b(t_1, \ldots, t_{n-1})$ (and we assume in further text that both of these parametrizations are rational). To find corresponding points at
A and B, we have to construct a reparametrization
\[
\phi : (t_1, \ldots, t_{n-1}) \rightarrow (u_1(t_1, \ldots, t_{n-1}), \ldots, u_{n-1}(t_1, \ldots, t_{n-1}))
\]
which is defined for certain parts of A and B with the property that tangent
hyperplanes \(\alpha(a)\) and \(\beta(b)\) at \(a(u_1(t_1, \ldots, t_{n-1}), \ldots, u_{n-1}(t_1, \ldots, t_{n-1})) \in A\) and \(b(t_1, \ldots, t_{n-1}) \in B\) are parallel. Then, the parametric representation of
the convolution hypersurface \(C = A \ast B\) is given by
\[
c(t_1, \ldots, t_{n-1}) = a(u_1(t_1, \ldots, t_{n-1}), \ldots, u_{n-1}(t_1, \ldots, t_{n-1})) + 
+ b(t_1, \ldots, t_{n-1}).
\]
Using the coordinates of tangent hyperplanes \(\alpha(a) = (\alpha_0, \alpha_1, \ldots, \alpha_n)(u_i)\) and
\(\beta(b) = (\beta_0, \beta_1, \ldots, \beta_n)(t_i), i = 1, \ldots, n - 1,\) the condition for corresponding
points reads
\[
\alpha_j(u_1, \ldots, u_{n-1}) = \lambda \cdot \beta_j(t_1, \ldots, t_{n-1}), \lambda \neq 0, \quad j = 1, \ldots, n.
\]
Hypersurfaces A and B are rational — thus, without loss of generality, we can
assume that all coordinates of tangent hyperplanes are polynomial. Unfortunately, reparametrization \(\phi\) still does not have to be expressed in explicit form
as we will see in the next sections. Therefore, the paper is devoted to a special
class of parametrized rational hypersurfaces which yield rational convolution
hypersurfaces — generally (GRC), or just in some special cases (SRC). It
is worth studying these hypersurfaces not only because of their interesting
theoretical properties but mainly for their applicability. Moreover, it is also
necessary in case of SRC hypersurfaces to identify affine classes of geometric
objets with which these hypersurfaces yield rational convolutions.

2.2 Gröbner bases

In this subsection, we will briefly review basics of the theory of Gröbner bases
of a polynomial ideal and some necessary related notions and theorems. Reader
who is more interested in this theory can find more details e.g. in Becker and
Weispfenning (1993); Cox et al. (1992); Winkler (1996).

First of all, it is necessary to explain the notions of a polynomial ideal in a
polynomial ring \(k[x_1, \ldots, x_n]\) (ring of all polynomials in variables \(x_1, \ldots, x_n\)
with coefficients in a field \(k\)) and closely related concept of an affine variety.
The affine variety \(\mathbb{V}(f_1, \ldots, f_s)\) defined by polynomials \(f_1, \ldots, f_s\) is a subset
of an affine space \(k^n\) which contains all solutions of a system of polynomial
equations \(f_1(x_1, \ldots, x_n) = \cdots = f_s(x_1, \ldots, x_n) = 0\). The polynomial ideal \(I\) is
a subset of a polynomial ring \(k[x_1, \ldots, x_n]\) which satisfies three conditions:
1. \(0 \in I,\) 2. \(f, g \in I \Rightarrow f + g \in I,\) 3. \(f \in I \land h \in k[x_1, \ldots, x_n] \Rightarrow hf \in I.\) By
proper ideal we mean the ideal not equal to \( k[x_1, \ldots, x_n] \), i.e. \( I \not\subseteq k[x_1, \ldots, x_n] \).

The real importance of ideals is that they provide a language for computing with affine varieties. The following theorem which relates affine varieties and polynomial ideals is crucial for using Gröbner bases for solving systems of polynomial equations.

**Theorem 2** If \( f_1, \ldots, f_s \) and \( g_1, \ldots, g_t \) generate the same ideal in \( k[x_1, \ldots, x_n] \), so that \( \langle f_1, \ldots, f_s \rangle = \langle g_1, \ldots, g_t \rangle \), then also corresponding affine varieties \( \mathbb{V}(f_1, \ldots, f_s) \) and \( \mathbb{V}(g_1, \ldots, g_t) \) equals.

*Proof.* The proof is very straightforward. \( \square \)

Thus, if we change the basis of the polynomial ideal it does not change the corresponding affine variety.

The important role in the theory of Gröbner bases plays ordering of terms in polynomials, for different monomial orderings we usually obtain different Gröbner bases of the given polynomial ideal. A monomial ordering on \( k[x_1, \ldots, x_n] \) is any relation \( > \) on the set of monomials \( x^\alpha, \alpha \in \mathbb{Z}^n_0 \) which is a total ordering and a well ordering on \( \mathbb{Z}^n_0 \) and which fulfills that if \( \alpha > \beta \) and \( \gamma \in \mathbb{Z}^n_0 \) then \( \alpha + \gamma > \beta + \gamma \). Especially important example of a monomial ordering is lexicographic order: Let \( \alpha, \beta \in \mathbb{Z}^n_0 \), then \( x^\alpha >_{\text{lex}} x^\beta \) if the left-most nonzero element in the vector difference \( \alpha - \beta \in \mathbb{Z}^n \) is positive. It is obvious that the order of monomials depends also on the chosen order of variables. The importance follows from the Elimination theorem (see below).

Now, we are ready to introduce the notion of a Gröbner basis of a polynomial ideal. This notion is closely related to the so-called Hilbert basis theorem which provides the answer to the problem of description of polynomial ideal and which states that every polynomial ideal \( I \) has a finite generating set, i.e. \( I = \langle g_1, \ldots, g_t \rangle \) for some \( g_1, \ldots, g_t \in I \) (see Cox et al. (1992) for details). The basis of the polynomial ideal \( I \) used in the proof of the Hilbert basis theorem with the special property that \( \langle \text{LT}(I) \rangle = \langle \text{LT}(g_1), \ldots, \text{LT}(g_t) \rangle \) is called the Gröbner basis (where \( \text{LT}(g_1) \) means the leading term of the polynomial \( g_1 \)). Of course, for given polynomial ideal and monomial ordering there is a lot of different Gröbner bases, even with different number of generators. That is why a minimal Gröbner basis and a reduced Gröbner basis are defined. The definition of the minimal Gröbner basis follows from the fact that if \( p \) is polynomial of Gröbner basis \( G \) such that \( \text{LT}(p) \in \langle \text{LT}(G \setminus \{p\}) \rangle \) then \( G \setminus \{p\} \) is also a Gröbner basis.

**Definition 3** A minimal Gröbner basis for a polynomial ideal \( I \) is Gröbner basis \( G \) for \( I \) such that:
(1) \( \text{LC}(p) = 1 \) for all \( p \in G \),
(2) for all \( p \in G \), \( \text{LT}(p) \notin \langle \text{LT}(G \setminus \{p\}) \rangle \).

However, a polynomial ideal can still have many minimal Gröbner bases. Nevertheless, we can choose one minimal Gröbner basis which is called the reduced Gröbner basis and which is given uniquely for a given polynomial ideal \( I \) with respect to the chosen monomial ordering and order of variables.

**Definition 4** A reduced Gröbner basis for a polynomial ideal \( I \) is a Gröbner basis \( G \) for ideal \( I \) such that:

(1) \( \text{LC}(p) = 1 \) for all \( p \in G \),
(2) for all \( p \in G \), no monomial of \( p \) lies in \( \langle \text{LT}(G \setminus \{p\}) \rangle \).

The theory of Gröbner bases of a polynomial ideal is closely related to the elimination theory which studies variable elimination methods for systems of polynomial equations. The key role in the elimination theory play two basic theorems — the Elimination theorem and the Extension theorem which are proved using Gröbner bases and resultants.

One of the possible views of Gröbner bases application in solving systems of polynomial equations is as a generalization of the Gaussian elimination method for system of linear algebraic equations. This means the we can also distinguish two steps during solving system of polynomial equations using Gröbner bases:

(1) Elimination step — elimination of variables from the system of equations (see the Elimination theorem),
(2) Extension step — extension of partial solutions to complete solutions of the original system (see Extension theorem).

To be able to write down the Elimination theorem we have to define the notion of a \( k \)-th elimination ideal.

**Definition 5** Let \( I = \langle f_1, \ldots, f_s \rangle \subset k[x_1, \ldots, x_n] \) be an ideal. Then \( \ell \)-th elimination ideal \( I_\ell \) is the ideal in \( k[x_{\ell+1}, \ldots, x_n] \) defined by

\[
I_\ell = I \cap k[x_{\ell+1}, \ldots, x_n].
\]

Thus, \( I_\ell \) consists of all polynomials in ideal \( I \) which contain only variables \( x_{\ell+1}, \ldots, x_n \). Elimination of variables \( x_1, \ldots, x_\ell \) means finding nonzero elements of the \( \ell \)-th elimination ideal \( I_\ell \). Gröbner bases allow us to do it.

**Theorem 6 (The Elimination Theorem)** Let \( I \subset k[x_1, \ldots, x_n] \) be an ideal

\(^2\) where \( \text{LC}(g_1) \) means the leading coefficient of the polynomial \( g_1 \)
and let $G$ be a Gröbner basis of $I$ with respect to lexicographic order where $x_1 > x_2 > \cdots > x_n$. Then, for every $0 \leq \ell \leq n$, the set

$$G_{\ell} = G \cap k[x_{\ell+1}, \ldots, x_n]$$

is a Gröbner basis of the $\ell$th elimination ideal $I_{\ell}$.

**Proof.** See Cox et al. (1992) for more details.

The *Elimination theorem* is closely related to the lexicographic order (or, generally, to so-called elimination orders among which, of course, lexicographic order belongs) because the elimination of variables is guaranteed only for elimination orders. On the other hand, typically the so-called graded orders (e.g. graded reverse lexicographic order) do not belong among elimination orders and the *Elimination theorem* does not hold for them.

The extension step mentioned above is described by the *Extension theorem* which says when a partial solution of a polynomial system can be extended to a complete solution. In short, if $I = \langle f_1, \ldots, f_s \rangle \subset C[x_1, \ldots, x_n]$ and $\text{LT}(f_i) = g_i(x_2, \ldots, x_n)x_1^{|N_i|}$, then all partial solutions $(a_2, \ldots, a_n) \in V(I_1)$ which do not belong to $V(g_1, \ldots, g_s)$ can be extended to complete solutions (see Cox et al. (1992) for more details).

Finally, the notion of the dimension of the ideal plays an important role in the ideal theory and in its connection with algebraic geometry.

**Definition 7** Let $I$ be a proper ideal in $k[x_1, \ldots, x_n]$ and $\{x_{i_1}, \ldots, x_{i_r}\}$ a subset of $\{x_1, \ldots, x_n\}$. Then $\{x_{i_1}, \ldots, x_{i_r}\}$ is called independent modulo the ideal $I$ if $I \cap k[x_{i_1}, \ldots, x_{i_r}] = \{0\}$. Moreover, $\{x_{i_1}, \ldots, x_{i_r}\}$ is called maximally independent modulo $I$ if it is independent modulo $I$ and not properly contained in any other independent set modulo $I$. The dimension $\dim(I)$ of $I$ is defined as

$$\dim(I) = \max\{|U| | U \subseteq \{x_1, \ldots, x_n\} \text{ independent modulo } I\}.$$ 

The proper ideal $I \subset k[x_1, \ldots, x_n]$ with its dimension equal to zero is called zero-dimensional ideal.

The following two lemmas are direct consequences of the *Elimination theorem* and the definition of the dimension of the polynomial ideal. Since lexicographic order is especially important in the following sections we restate (Becker and Weispfenning, 1993, Lemmas 6.47 and 6.50) with respect to this order.

**Lemma 8** Let $I$ be a proper ideal of $k[x_1, \ldots, x_n]$ and $\{x_1, \ldots, x_r\}$ a subset of $\{x_1, \ldots, x_n\}$. Then the following are equivalent:

1. $\{x_1, \ldots, x_r\}$ is independent modulo the ideal $I$. 


(2) \( G \cap k[x_1, \ldots, x_r] = \emptyset \) for every Gröbner basis of \( I \) with respect to lexicographic order for \( x_1 < x_2 < \ldots < x_r < x_{r+1} < \ldots < x_n \).

**Lemma 9** Let \( I \) be a proper ideal of \( k[x_1, \ldots, x_n] \). Then \( I \) is zero-dimensional iff it contains a non-constant univariate polynomial in each of the variables in \( \{x_1, \ldots, x_n\} \). In this case, ideal \( I \) actually contains a unique monic\(^3\) univariate polynomial \( f_i \) of minimal degree in each variable \( x_i \), namely, the monic generator of the elimination ideal \( I \cap k[x_i] \), and \( f_i \) is the polynomial of minimal degree in \( G \cap k[x_i] \) whenever \( G \) is a Gröbner basis of \( I \) with respect to lexicographic order for \( x_i > x_1 > \ldots > x_i-1 > x_i+1 > \ldots > x_n \).

3 Hypersurfaces with rational convolutions

The most interesting hypersurfaces are those ones which admit the rational convolutions generally. The reason follows from the fact that for example offset and general offset surfaces used in machining can be computed as convolution surfaces and in this case they have rational parametrization which means that they can be expressed in a NURBS form. Thus, we can define:

**Definition 10** Let \( A \) be a rational hypersurface in \( \mathbb{R}^n \), parametrized by \( a(u_i), i = 1, \ldots, n-1 \). This parametrization is called a GRC parametrization, if and only if the convolution hypersurface \( A \ast B \) has an explicit rational parametrization for an arbitrary parametrized rational hypersurface \( B \). Further, \( A \) is called a GRC hypersurface, if and only if it possesses a GRC parametrization.

Of course, in \( \mathbb{R}^2 \) we speak about GRC curves and in \( \mathbb{R}^3 \) about GRC surfaces.

The useful theory for computing rational parametrization of convolution hypersurfaces \( C = A \ast B \) seems to be a theory of Gröbner bases mentioned in Section 2.2. Firstly, consider the ideal

\[
I = \langle \alpha_j(u_i) - \lambda \beta_j(t_i), 1 - w \lambda \rangle \subset k(t_i)[w, u_i, \lambda], j = 1, \ldots, n; i = 1, \ldots, n-1, \tag{6}
\]

see the condition (5), the last polynomial guarantees \( \lambda \neq 0 \). In addition, we assume in the remainder of this paper that the considered ideal \( I \) is zero-dimensional. Generally, for \( I \subset k[x_1, \ldots, x_n] \) the assumption \( \text{dim}(I) = 0 \) implies, see (Becker and Weispfenning, 1993, Theorem 6.54), that every Gröbner basis \( G \) of \( I \) with respect to any monomial ordering contains \( n \) polynomials \( g_i \in G \) with \( \text{LT}(g_i) = x_i^{r_i} \) for some \( 0 < r_i \in \mathbb{N} \) and \( 1 \leq i \leq n \). In fact, this follows directly from the zero-dimensionality of \( I \) and the definition of Gröbner basis. From Lemma 9, or more generally (Becker and Weispfenning, 1993, Lemma 6.50), every zero-dimensional ideal \( I \) contains non-constant univariate

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\( ^3 \) Non-zero polynomial with leading coefficient equal to 1.
polynomials $f_i \in I \cap k[x_i]$. Further, it follows from the definition of the Gröbner basis that it contains polynomials from $I$ whose leading terms divide all leading terms of all polynomials in $I$. Putting these two observations together, it is obvious that the Gröbner basis has to contain polynomials $g_i \in G$, $1 \leq i \leq n$ with $\text{LT}(g_i) = x_i^{r_i}$, $0 < r_i \in \mathbb{N}$ as was mentioned above. Moreover, (Becker and Weispfenning, 1993, Theorem 8.32) states that the number of zeros of a zero-dimensional ideal $I$ in $l^n$ where $l$ is an algebraically closed extension field of $k$ is less than or equal to the vector space dimension $\dim_k(k[x_1, \ldots, x_n]/I)$ which is equal to $r_1 \cdot r_2 \cdots r_n$. If $k$ is perfect and $I$ is radical, then equality holds.

Hence, applying mentioned facts, using the reduced Gröbner basis computation for (6) with respect to the lexicographic order for $w > u_1 > \ldots > u_{n-1} > \lambda$ and applying the Elimination theorem (see Theorem 6) we always obtain polynomials

$$g_0(w, u_1, \ldots, u_{n-1}, \lambda), g_1(u_1, \ldots, u_{n-1}, \lambda), \ldots, g_{n-1}(u_{n-1}, \lambda), g_n(\lambda) \quad (7)$$

with

$$\text{LT}(g_0) = w, \text{LT}(g_i) = u_i^{r_i}, 1 \leq i \leq n-1, \text{LT}(g_n) = \lambda^{r_n}$$

as the generators of the appropriate elimination ideals. The number of zeros of the ideal (6) is then less than or equal to $r_1 \cdot r_2 \cdots r_n$.

**Remark 11** It is clear that above mentioned assumption $\dim(I) = 0$ excludes for instance such parametrizations $a(u_i)$ where $k$ parameters, $1 \leq k \leq n - 1$, are missing in the coordinate vector of the tangent hyperplane $\alpha$. Typical examples of this situation are developable surfaces in $\mathbb{R}^3$ with 1-parametric family of tangent planes, or hyperplanes in $\mathbb{R}^n$ etc.

Generally, there is no one-to-one correspondence between points $a \in A$ and $b \in B$ such that appropriate tangent hyperplanes $\alpha(a)$ and $\beta(b)$ are parallel. This geometrical fact can be easily handled algebraically using above introduced concept.

**Definition 12** Let $G$ be a Gröbner basis of the ideal $I$ defined by (6) and let $r_i$ are degrees of leading terms of polynomials $g_1, \ldots, g_n \in G$ from (7). Then the number $\delta = r_1 \cdot r_2 \cdots r_n$ is called a degree of the construction of convolution hypersurface $A \ast B$ (in this order).

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- **Remark** A field $k$ is called **perfect** if every irreducible polynomial $f \in k[x]$ is separable. A polynomial $f \in k[x]$ is called **separable** if it is either a non-zero constant, or the factorization into non-constant linear polynomials of $f$ in $\bar{k}[x]$ where $\bar{k}$ is extension field of $k$ consists of pairwise non-associated factors. It is obvious that the field of complex number $\mathbb{C}$ is a perfect field.
The degree of the construction indicates the number of points $a \in A$ corresponding to the chosen point $b \in B$ (of course in complex extension and including the multiplicity). From the geometrical point of view, it is seen that the degree of the construction of convolution hypersurface $A \star B$ does not depend on the choice of representative of coordinate vector of tangent hyperplanes of both rational hypersurfaces $A$ and $B$ at the corresponding points $a \in A$ and $b \in B$. On the other hand, despite the fact that $A \star B = B \star A$ both constructions may be generally of different degrees. We will illustrate this fact on the computation of $A \star B$ and $B \star A$ where $A$ is the paraboloid $z = x^2 + y^2$ and $B$ is the sphere $x^2 + y^2 + z^2 - 1 = 0$ (see following example).

Example 13 Let $A$ be parametrized by

$$a(u, v) = (u, v, u^2 + v^2)$$

and $B$ by

$$b(s, t) = \left(\frac{2s}{1 + s^2 + t^2}, \frac{2t}{1 + s^2 + t^2}, \frac{1 - s^2 - t^2}{1 + s^2 + t^2}\right).$$

Two corresponding points are tied to the condition (5) where $(\alpha_1, \alpha_2, \alpha_3)(u, v) = (-2u, -2v, 1)$ and $(\beta_1, \beta_2, \beta_3)(s, t) = (2s, 2t, 1 - s^2 - t^2)$. The Gröbner basis $G$ of the ideal $I = \langle \alpha_1 - \lambda \beta_1, \alpha_2 - \lambda \beta_2, \alpha_3 - \lambda \beta_3, 1 - w \lambda \rangle \subset \mathbb{Q}(s, t)[w, u, v, \lambda]$ is given by

$$G = \left\{ w - \beta_3, u + \frac{\beta_1}{2\beta_3}, v + \frac{\beta_2}{2\beta_3}, \lambda - \frac{1}{\beta_3} \right\},$$

i.e. $\text{LT}(G) = \{w, u, v, \lambda\}$

from which it is seen that the construction of $A \star B$ is of degree 1. On the other hand, if we compute the Gröbner basis $\overline{G}$ of the ideal $J = \langle \alpha_1 - \lambda \beta_1, \alpha_2 - \lambda \beta_2, \alpha_3 - \lambda \beta_3, 1 - w \lambda \rangle \subset \mathbb{Q}(u, v)[w, s, t, \lambda]$ we get

$$\overline{G} = \left\{ \frac{4}{\alpha_1^2 + \alpha_2^2} \lambda + \frac{4\alpha_3}{\alpha_1^2 + \alpha_2^2}, s - \frac{2\alpha_1}{\alpha_1^2 + \alpha_2^2} \lambda + \frac{2\alpha_1 \alpha_3}{\alpha_1^2 + \alpha_2^2}, t - \frac{2\alpha_2}{\alpha_1^2 + \alpha_2^2} \lambda + \frac{2\alpha_2 \alpha_3}{\alpha_1^2 + \alpha_2^2}, \lambda^2 - \alpha_3 \lambda - \frac{\alpha_1^2 + \alpha_2^2}{4} \right\},$$

i.e. $\text{LT}(\overline{G}) = \{w, u, v, \lambda^2\}$

and thus the construction of convolution starting from $B$ is of degree 2.

Since the ideal $I$ is a subset of the polynomial ring $k[w, u, \lambda]$ the degree $\delta$ of the construction of convolution hypersurface $C = A \star B$ (in this order)

---

5 We always mean by Gröbner basis the reduced Gröbner basis with respect to the lexicographic order and for ordering of variables given by the definition of the ring.
depends only on the rational parametrization $a(u_i)$. Hence, we can classify parametrizations $a(u_i)$ from this point of view.

**Definition 14** Let $a(u_i), i = 1, \ldots, n - 1$ is a rational parametrization of the hypersurface $A \subset \mathbb{R}^n$. This parametrization is called a $\delta$-SRC parametrization if the construction of $C = A \star B$ is of degree $\delta$.

**Theorem 15** GRC parametrization is $1$-SRC parametrization, i.e. parametrization $a(u_i)$ is a GRC parametrization if and only if for the ideal $I$ defined by (6) and for its reduced Gröbner basis $G$ with respect to the lexicographic order holds

$$\langle \text{LT}(I) \rangle = \langle \text{LT}(G) \rangle = \langle w, u_1, \ldots, u_{n-1}, \lambda \rangle. \quad (8)$$

**Proof.** Let hypersurface $A$ is parametrized by $a(u_i)$ and hypersurface $B$ by $b(t_i), i = 1, \ldots, n - 1$.

$\Rightarrow$: Let $a(u_i)$ be a GRC parametrization. First of all, we want to prove that the system of equations (5) has one and only one solution for each parametrization $b(t_i)$ using the Gröbner basis of the ideal (6).

Since we consider only such parametrizations $a(u_i)$ providing the zero-dimensional ideals (6) which are always proper, it follows from this fact directly that the system (5) has at least one solution.

Let the system (5) has two solutions. Then from the above mentioned and Becker and Weispfenning (1993) it follows that Gröbner basis of the ideal (6) contains polynomials $g_0, \ldots, g_n$ such that (to simplify, we denote variables $x_0 = w, x_i = u_i, i = 1, \ldots, n - 1, x_n = \lambda$)

$$\text{LT}(g_n) = x_n, \ldots, \text{LT}(g_{j+1}) = x_{j+1}, \text{LT}(g_j) = x_j^2,$$

$$\text{LT}(g_{j-1}) = x_{j-1}, \ldots, \text{LT}(g_1) = x_1, \text{LT}(g_0) = 1$$

for some $j$. Therefore,

$$g_n = x_n - \varphi_{n,0}(t_1, \ldots, t_{n-1}),$$

$$\vdots$$

$$g_{j+1} = x_{j+1} - \varphi_{j+1,0}(t_1, \ldots, t_{n-1}),$$

$$g_j = x_j^2 - \varphi_{j,1}(t_1, \ldots, t_{n-1})x_j - \varphi_{j,0}(t_1, \ldots, t_{n-1}),$$

$$g_{j-1} = x_{j-1} - \varphi_{j-1,1}(t_1, \ldots, t_{n-1})x_j - \varphi_{j-1,0}(t_1, \ldots, t_{n-1}),$$

$$\vdots$$

$$g_0 = x_0 - \varphi_{0,1}(t_1, \ldots, t_{n-1})x_j - \varphi_{0,0}(t_1, \ldots, t_{n-1}).$$
where \( \varphi_{k,l}(t_i) \in k(t_i), k = 0, \ldots, n, l = 0, 1 \) which depend on tangent vector coordinates \((\beta_0, \ldots, \beta_n)(t_i)\). Since \( a(u_i) \) is a GRC parametrization, it must yield a rational reparametrization for an arbitrary rational hypersurface \( b(t_i) \). Thus, the solutions of the system (5) given by polynomials \( g_0, \ldots, g_n \) must be rational for any \( b(t_i) \) and any corresponding tangent vector coordinates \((\beta_0, \ldots, \beta_n)(t_i)\). However, there always exists a choice of \((\beta_0, \ldots, \beta_n)(t_i)\) for which a quadratic polynomial equation \( g_j = 0 \) has non-rational roots. Hence, in this case the rational reparametrization is not obtained for any arbitrary rational surface \( b(t_i) \) but only in some special cases. The similar technique can be used to prove that the system (5) cannot have more than two solutions.

Therefore, the only possibility to ensure the rational reparametrization for all rational surfaces \( b(t_i) \) is that the system (5) has one and only one solution. From this it follows directly that leading terms has to be

\[
\text{LT}(g_0) = w, \text{LT}(g_i) = u_i, \text{LT}(g_n) = \lambda, \quad i = 1, \ldots, n - 1
\]

and the corresponding polynomials in the Gröbner basis are

\[
\begin{align*}
g_0 &= w - \frac{1}{\varphi_n}(t_1, \ldots, t_{n-1}), \\
g_i &= u_i - \varphi_i(t_1, \ldots, t_{n-1}), \quad i = 1, \ldots, n - 1, \\
g_n &= \lambda - \varphi_n(t_1, \ldots, t_{n-1}).
\end{align*}
\]

It is obvious that for the ideal \( I \) then holds \( I = \langle g_0, \ldots, g_n \rangle \) and that \( G = \{g_0, \ldots, g_n\} \) forms a Gröbner basis with respect to an arbitrary monomial ordering. This brings us to the end of this part of the proof, i.e. this means that

\[
\langle \text{LT}(I) \rangle = \langle \text{LT}(G) \rangle = \langle w, u_1, \ldots, u_{n-1}, \lambda \rangle.
\]

\( \Leftarrow \): Let \( \langle \text{LT}(I) \rangle = \langle \text{LT}(G) \rangle = \langle w, u_1, \ldots, u_{n-1}, \lambda \rangle \). Then Gröbner basis \( G \) of the ideal (6) contains \( n \) polynomials (9) where \( \varphi_1, \ldots, \varphi_n \in k(t_1, \ldots, t_{n-1}) \). The rational reparametrization \( \phi \) is then

\[
\phi : (t_1, \ldots, t_{n-1}) \to (\varphi_1(t_1, \ldots, t_{n-1}), \ldots, \varphi_{n-1}(t_1, \ldots, t_{n-1})).
\]

This means that \( C = A \star B \) has an explicit rational parametric representation

\[
c(t_1, \ldots, t_{n-1}) = a(\varphi_1(t_1, \ldots, t_{n-1}), \ldots, \varphi_{n-1}(t_1, \ldots, t_{n-1}) + b(t_1, \ldots, t_{n-1})
\]

for any arbitrary hypersurface \( B \) and thus \( a(u_i) \) is the GRC parametrization. \( \Box \)

**Corollary 16** Every LN parametrization is a GRC parametrization; every LN surface is a GRC surface.
The proof is straightforward because every LN parametrization is of GRC type.

**Remark 17** Example 13 shows us another interesting fact. Starting from the paraboloid \( a(u,v) = (u, v, u^2 + v^2) \), the convolution surface of paraboloid and an arbitrary parametrized rational surface has generally the explicit rational parametric equation. Hence, the parametrization \( a(u,v) = (u, v, u^2 + v^2) \) is GRC. Conversely, starting from the sphere \( b(s,t) = (2s/(1+s^2+t^2), 2t/(1+s^2+t^2), (1-s^2-t^2)/(1+s^2+t^2)) \), the convolution surface is rational only in special situations. Furthermore, we see that convolution surface is rational in case that the discriminant of the quadratic equation \( g_3(\lambda) = 0 \) is a square of some polynomial \( \sigma \), i.e. if the other surface fulfills the condition \( \alpha_1^2(u,v) + \alpha_2^2(u,v) + \alpha_3^2(u,v) = \sigma^2(u,v) \). The paraboloid is not this type of surface but there exist many surfaces fulfilling this condition — they are known as PN surfaces in Euclidean space \( \mathbb{R}^3 \).

Of course, we are most interested in the GRC parametrizations because they possess high applications potential. Nevertheless, it is still worth to study \( \delta \)-SRC parametrization with \( \delta > 1 \). First of all, it is necessary to identify affine classes of geometric objects with which these hypersurfaces yield rational convolutions — as it was shown for instance in the Example 13 for the sphere.

**Example 18** Let \( A \) be parametrized by

\[
a(u,v) = (u + v, u^2, v^2)
\]

and let \( B \) is a LN surface described by the parametrization \( b(s,t) \) fulfilling the condition \((\beta_1, \beta_2, \beta_3)(s,t) = (s, t, 1)\). As we know two corresponding points are tied to the condition \((5)\) where \((\alpha_1, \alpha_2, \alpha_3)(u,v) = (4uv, -2v, -2u)\) and \((\beta_1, \beta_2, \beta_3)(s,t) = (s, t, 1)\). The Gröbner basis \( G \) of the ideal \( I = \langle \alpha_1 - \lambda \beta_1, \alpha_2 - \lambda \beta_2, \alpha_3 - \lambda \beta_3, 1 - w\lambda \rangle \subset \mathbb{Q}(s,t)[w, u, v, \lambda] \) is given by

\[
G = \left\{ \frac{w - t}{s}, u + \frac{s}{2t^2} v + \frac{s}{2} \lambda - \frac{s}{t} \right\},
\]

from which it is seen that \( a(u,v) = (u + v, u^2, v^2) \) is a GRC parametrization.

Moreover, using reparametrization \( \phi : (u(s,t), v(s,t)) = \left(-\frac{s}{2t}, -\frac{s}{2}\right) \) we obtain for tangent planes \( \alpha \) of \( A \) parametrized by \( a(u(s,t), v(s,t)) \)

\[
(\alpha_1, \alpha_2, \alpha_3)(s,t) = \left(\frac{s^3}{4t^3}, \frac{s^2}{4t^2}, \frac{s^2}{4t^3}\right) = \frac{s^2}{4t^3} \cdot (s,t,1) = \lambda \det(J_\phi)(s,t,1).
\]

Thus, the surface \( A \) fulfills anew the same LN condition as the surface \( B \).
Example 19 Let $A$ be again parametrized by GRC parametrization

$$a(u,v) = (u + v, u^2, v^2)$$

and let $B$ is now the sphere represented by PN parametrization

$$b(s,t) = \left( \frac{2s}{1 + s^2 + t^2}, \frac{2t}{1 + s^2 + t^2}, \frac{1 - s^2 - t^2}{1 + s^2 + t^2} \right).$$

Hence, $(\alpha_1, \alpha_2, \alpha_3)(u,v) = (4uv, -2v, -2u)$ and $(\beta_1, \beta_2, \beta_3)(s,t) = (2s, 2t, 1 - s^2 - t^2)$.

The Gröbner basis $G$ of the ideal $I = \langle \alpha_1 - \lambda \beta_1, \alpha_2 - \lambda \beta_2, \alpha_3 - \lambda \beta_3, 1 - w \lambda \rangle \subset \mathbb{Q}(s,t)[w,u,v,\lambda]$ is in this case

$$G = \left\{ w - \frac{t(1 - s^2 - t^2)}{s}, u + \frac{s}{2t}, v + \frac{s}{1 - s^2 - t^2}, \lambda - \frac{s}{t(1 - s^2 - t^2)} \right\}.$$

Using reparametrization $\phi : (u(s,t), v(s,t)) = \left( -\frac{s}{2t}, \frac{s}{\sqrt{1 - s^2 - t^2}} \right)$ we obtain for tangent planes $\alpha$ of $A$ parametrized by $a(u(s,t), v(s,t))$

$$(\alpha_1, \alpha_2, \alpha_3)(s,t) = \frac{s^2(s^2 + t^2 + 1)}{2t^3(1 - s^2 - t^2)} \left( 2s, 2t, 1 - s^2 - t^2 \right) =$$

$$= \lambda \det(J_\phi) \left( 2s, 2t, 1 - s^2 - t^2 \right).$$

Thus, the parametrization $a(u(s,t), v(s,t))$ shows now the PN property $\alpha_1^2(s,t) + \alpha_2^2(s,t) + \alpha_3^2(s,t) = \sigma^2(s,t)$.

Elaborating this generally for GRC parametrization $a(u_i)$ and other parametrization $b(t_i), i = 1, \ldots, n-1$, we obtain the reparametrization $\phi : (u_1, \ldots, u_{n-1}) = (\varphi_1(t_1, \ldots, t_{n-1}), \ldots, \varphi_{n-1}(t_1, \ldots, t_{n-1}))$ which is easy to obtain from the Gröbner basis $G = \{g_0, g_1, \ldots, g_n\}$ with

$$g_0 = w - \frac{1}{\varphi_n}(t_1, \ldots, t_{n-1}),$$

$$g_i = u_i - \varphi_i(t_1, \ldots, t_{n-1}), \quad i = 1, \ldots, n - 1,$$

$$g_n = \lambda - \varphi_n(t_1, \ldots, t_{n-1}).$$

Hence, we get for reparametrization of the coordinate vector of the tangent plane $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n)(u_1(t_1, \ldots, t_{n-1}), \ldots, u_{n-1}(t_1, \ldots, t_{n-1}))$

$$\alpha_j(t_1, \ldots, t_{n-1}) = \varphi_n(t_1, \ldots, t_{n-1}) \cdot \det(J_\phi) \cdot \beta_j(t_1, \ldots, t_{n-1}), \quad j = 1, \ldots, n$$

(10)

To sum up, it is clearly seen from the above examples that GRC parametrizations show an interesting property to be accessible to receive properties of other general parametrizations using the reparametrization $\phi$. For instance, if the
parametrization $b(s, t)$ is of LN type (see Example 18) then after reparametrization of the GRC parametrization $a(u, v)$ the new parametric representation $a'(u(s, t), v(s, t))$ also embodies the LN property. Similarly for PN– (see Example 19) or other special parametrizations.

4 RC properties of cubic curves with cubic parametrization

We start the investigation of RC properties of parametrized curves with cubics. With an $(2 \times 4)$-matrix $B$ and a vector $t = (t^3, t^2, t, 1)^T$ the general form of a polynomial cubic parametrization $\mathbb{R} \mapsto \mathbb{R}^2$ is

$$x(t) = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \end{pmatrix} t = Bt \quad \left( \begin{array}{c} b_{11} \\ b_{21} \end{array} \right) \neq 0. \quad (11)$$

We say that two parametrizations are equivalent if one can be obtained from the other by an affine change of coordinates and an affine change of parameter. And since convolutions are invariant under affine transformations, these can be used to explore RC properties of different affine types of parametrized cubics. An affine change of coordinates in $\mathbb{R}^2$ is expressed in the form

$$x' = Ax + a, \quad (12)$$

where $A$ is a regular $(2 \times 2)$-matrix and $a$ is an 2-vector. An affine transformation of parameter is

$$t = ct + d, \quad (13)$$

where $c \neq 0$. The induced change of $t$ is then

$$t = \begin{pmatrix} c^3 & 3c^2d & 3cd^2 & d^3 \\ 0 & c^2 & 2cd & d^2 \\ 0 & 0 & c & d \\ 0 & 0 & 0 & 1 \end{pmatrix} \tilde{t}. \quad (14)$$

After application of appropriate transformations (of parameter and coordinates) we get four affine classes represented by parametrizations mentioned in Table 1 and shown in Figure 2 — more details in Karger (2004). Our goal is to investigate the RC properties of these classes of cubic parametrizations. As in Example 13, we construct in all cases the ideal $I = \langle \alpha_1 - \lambda\beta_1, \alpha_2 - \lambda\beta_2, 1 - w\lambda \rangle \subset \mathbb{Q}(u)[w, t, \lambda]$, where $\alpha = (\alpha_0, \alpha_1, \alpha_2)(t)$ is a tangent of a cubically parametrized curve and $\beta = (\beta_0, \beta_1, \beta_2)(u)$ denotes a tangent of an arbitrary rational curve. For all cases we compute Gröbner basis and thus we
Table 1
Cubic curves with cubic parametrization

<table>
<thead>
<tr>
<th>Type</th>
<th>( x(t) )</th>
<th>Gröbner basis of ( I )</th>
<th>RC property</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>((t^2, t^3))</td>
<td>( { w - \frac{3\beta_1^2}{4\beta_1}, t + \frac{2\beta_1}{3\beta_2}, \lambda - \frac{4\beta_1}{3\beta_2} } )</td>
<td>GRC</td>
</tr>
<tr>
<td>(ii)</td>
<td>((t^2, t^3 - t))</td>
<td>( { w - \frac{3\beta_1^2}{4\beta_1} \lambda + \beta_1, t + \frac{\beta_1}{2}, \lambda^2 - \frac{4\beta_1}{3\beta_2} \lambda - \frac{4}{3\beta_2} } )</td>
<td>2-SRC</td>
</tr>
<tr>
<td>(iii)</td>
<td>((t^2, t^3 + t))</td>
<td>( { w + \frac{3\beta_1^2}{4\beta_1} \lambda - \beta_1, t + \frac{\beta_1}{2}, \lambda^2 - \frac{4\beta_1}{3\beta_2} \lambda + \frac{4}{3\beta_2} } )</td>
<td>2-SRC</td>
</tr>
<tr>
<td>(iv)</td>
<td>((t, t^3))</td>
<td>( { w + \beta_2, t^2 + \frac{\beta_1}{3\beta_2}, \lambda + \frac{1}{\beta_2} } )</td>
<td>2-SRC</td>
</tr>
</tbody>
</table>

can decide if the appropriate parametrization is GRC, or only SRC. Moreover, in the case of SRC parametrizations it is also very interesting to find conditions for the existence of rational convolutions.

It is seen from the Table 1 that among curves with cubic parametrization there is just one affine class of GRC cubics — type (i); other parametrizations are 2-SRC. We can also note another interesting thing — the convolution of some curve of affine type (ii) is rational just in case that the other curve fulfills the condition \( \beta_1^2(u) + 3\beta_2^2(u) = \sigma^2(u) \) for some polynomial \( \sigma \) because

\[
\lambda_{1,2} = \frac{2 \left( \beta_1 \pm \sqrt{\beta_1^2 + 3\beta_2^2} \right)}{3\beta_2^2}.
\] (15)

Thus, curves in this class yield generally the rational convolution with ellipses (and with all curves having the same tangent vector coordinates). Moreover, the affine type (ii) includes among others the famous Tschirhausen cubic which is the only example of cubic PH curve in the Euclidean plane — see Farouki and Sakkalis (1990).

Similarly, the convolution of cubics of type (iii) is rational just in case that the other curve fulfills the condition \( \beta_1^2(u) - 3\beta_2^2(u) = \sigma^2(u) \) for some polynomial \( \sigma \) because

\[
\lambda_{1,2} = \frac{2 \left( \beta_1 \pm \sqrt{\beta_1^2 - 3\beta_2^2} \right)}{3\beta_2^2}.
\] (16)

Therefore, curves in this class yield generally the rational convolution with hyperbolas and with all curves having the same tangent vector coordinates).

And finally, for cubics of type (iv) we obtain the RC condition \( -\frac{\beta_1(u)}{3\beta_2(u)} = \sigma^2(u) \).

**Remark 20** Though the above mentioned classification is done from the affine point of view, we can also use our approach to prove that there is only one type of polynomial PH cubic in the Euclidean plane \( \mathbb{R}^2 \). Of course, we have to look for this curve among cubics represented by 2-SRC parametrizations of type (ii) because the property of PH curves is the possibility to yield rational convolution.
Fig. 2. Representatives of four affine classes of cubics.

curve with circles and we know that the convolution of every curve of affine type
(ii) is rational iff the other curve fulfills the condition $\beta_1^2(u) + 3\beta_2^2(u) = \sigma^2(u)$.

All we have to do is to apply the transformation

$$\beta_1(u) = a\beta_1^*(u), \quad a \in \mathbb{R}$$
$$\beta_2(u) = \frac{a}{\sqrt{3}}\beta_2^*(u)$$

mapping ellipses onto circles on the curve $x(t) = (t^2, t^3 - t)^T$, i.e.

$$x(t) \mapsto x'(t) = \left(at^2, \frac{a}{\sqrt{3}}(t^3 - t)\right)^T.$$  \hspace{1cm} (18)

To sum up, it is well-known that (18) describes so-called Tschirhausen cubic.

Remark 21 Of course, using the method from Example 19, we can get another PH parametrization from the GRC parametrization $x(t) = (t^2, t^3)^T$. 

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However, this one is not polynomial but rational

\[ x(\tau) = \left( \frac{16\tau^2}{9(\tau^2 - 1)^2}, \frac{64\tau^3}{27(\tau^2 - 1)^3} \right)^T. \]

5 RC properties of quadratically parametrized surfaces

With an \((3 \times 6)\)-matrix \(B\) and a vector \(u = (u^2, uv, v^2, u, v, 1)^T\) the general form of a polynomial quadratic parametrization \(\mathbb{R}^2 \mapsto \mathbb{R}^3\) is

\[
x(u, v) = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} & b_{26} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} & b_{36} \end{pmatrix} u = Bu, \quad \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \neq 0. \quad (19)
\]

We say that two parametrizations are equivalent if one can be obtained from the other by an affine change of coordinates and an affine change of parameters. An affine change of coordinates in \(\mathbb{R}^3\) is expressed in the form

\[
x' = Ax + a, \quad (20)
\]

where \(A\) is a regular \((3 \times 3)\)-matrix and \(a\) is an 3-vector. An affine transformation of parameter is given by an \((2 \times 3)\)-matrix \(C\)

\[
(u, v)^T = C(\tilde{u}, \tilde{v}, 1)^T \quad (21)
\]

where \(C = (a, b, c | d, e, f)\) and \((a, b | d, e) \neq 0\). The induced change of \(u\) is then

\[
u = \begin{pmatrix} a^2 & b^2 & 2ab & 2ac & 2bc & c^2 \\ d^2 & e^2 & 2de & 2df & 2ef & f^2 \\ ad & be & ae + bd & a + cd & bf + ce & cf \end{pmatrix} \tilde{u}. \quad (22)
\]

After application of appropriate transformations (of parameters and coordinates) we get 15 affine classes of quadratically parametrized surfaces — more details in Peters and Reif (2004). Table 2 shows all non-quadratic ones (i.e. surfaces of degree three or four) — 6 are elliptic and hyperbolic paraboloids (LN surfaces), cone and parabolic cylinders (developable surface).
In all cases, we compute the Gröbner basis of the ideal $I = \langle \alpha_1 - \lambda \beta_1, \alpha_2 - \lambda \beta_2, \alpha_3 - \lambda \beta_3, 1 - w \rangle \subset \mathbb{Q}(s, t)[w, u, v, \lambda]$, where $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)(u, v)$ is a tangent plane of a quadratically parametrized surface of the type (i)–(ix) and $\beta = (\beta_0, \beta_1, \beta_2, \beta_3)(s, t)$ denotes a tangent plane of an arbitrary rational surface. We see from the computation that all mentioned surfaces yield generally a convolution which has an explicit rational parametrization for an arbitrary parametrized rational surface. We have proved the following surprising theorem.

**Theorem 22** All non-quadric quadratically parametrized surfaces are GRC surfaces.

Application of the above result is straightforward. If we use for modelling quadratic triangular Bézier patches

$$x(u, v, w) = \sum_{i+j+k=0}^{i,j,k>0} p_{ijk} B_{i,j,k}^2(u, v, w) \quad u, v, w \geq 0, u + v + w = 1 \quad (23)$$

with quadratic blending functions

$$B_{i,j,k}^2(u, v, w) = \frac{2}{i!j!k!} u^i v^j w^k \quad (24)$$

yielding zero-dimensional ideal (6), we have guaranteed that their parametrizations are GRC, i.e. their offset surfaces (either classical or general) are rational.

**Corollary 23** The non-developable quadratic triangular Bézier surfaces always admit a rational convolution with any arbitrary rational surface.
Example 24 We can recall the method from the Example 18 to show how to transform the quadratic parametrizations from the Table 2 on LN type.

Table 3
Reparametrization of non-quadric quadratically parametrized surfaces

<table>
<thead>
<tr>
<th>Type</th>
<th>( \phi: (s, t) \rightarrow (u(s, t), v(s, t)) )</th>
<th>((\alpha_1, \alpha_2, \alpha_3)(s, t))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>(( -\frac{s}{2}, -\frac{t}{2} ))</td>
<td>(\frac{s^2}{4t^3}(s, t, 1))</td>
</tr>
<tr>
<td>(ii)</td>
<td>(( -\frac{s}{2}, -\frac{t}{2} ))</td>
<td>(\frac{1}{4t}(s, t, 1))</td>
</tr>
<tr>
<td>(iii)</td>
<td>(( \frac{2s}{t^2}, -\frac{s}{t} ))</td>
<td>(-\frac{4s^2}{t^3}(s, t, 1))</td>
</tr>
<tr>
<td>(iv)</td>
<td>(( -\frac{1}{4}, -\frac{st-2}{t^2} ))</td>
<td>(-\frac{1}{t^2}(s, t, 1))</td>
</tr>
<tr>
<td>(v)</td>
<td>(( -\frac{2st}{4t^2+1}, -\frac{s}{4t^2+1} ))</td>
<td>(-\frac{4s^2}{(4t^2+1)^2}(s, t, 1))</td>
</tr>
<tr>
<td>(vi)</td>
<td>(( \frac{2s}{s^2-4t}, -\frac{s^2}{s^2-4t} ))</td>
<td>(\frac{64s^4}{(s^2-4t)^2}(s, t, 1))</td>
</tr>
<tr>
<td>(vii)</td>
<td>(( -\frac{s(s-2)}{s^2-4t}, -\frac{s(s-2t)}{s^2-4t} ))</td>
<td>(-\frac{16s^2(s-2)^2(s-2t)^2}{(s^2-4t)^3}(s, t, 1))</td>
</tr>
<tr>
<td>(viii)</td>
<td>(( \frac{2t^2}{s^2-4t}, -\frac{st}{s^2-4t} ))</td>
<td>(-\frac{64t^6}{(s^2-4t)^3}(s, t, 1))</td>
</tr>
<tr>
<td>(ix)</td>
<td>(( \frac{s^2+2t^2}{s^2-4t}, -\frac{s(t+2)}{s^2-4t} ))</td>
<td>(-\frac{16(s^2+2t^2)^2(t+2)^2}{s^2-4t})</td>
</tr>
</tbody>
</table>

Fig. 3. Affine types (i) - (vi) of quadratically parametrized surfaces.
6 Conclusion

We have investigated such parametrized rational hypersurfaces $A \subset \mathbb{R}^n$ which admit rational convolution hypersurface $A \ast B$ generally, or in some special cases under specific conditions. In connection with this, we have introduced a special class of rational parametrizations, so called GRC parametrizations, which always lead to the explicit rational description of convolution hypersurfaces for any arbitrary rational hypersurface $B$. The useful theoretical apparatus for this approach seems to be the algebraic theory of Gröbner bases that was applied on zero-dimensional ideal which reflects the parametrization of hypersurface $A$ and yields information about RC properties of given parametrization. In addition, the computed Gröbner basis also shows the way...
how to obtain the rational reparametrization $\phi$ which leads to rational representation of convolution hypersurface $A \ast B$.

Further, we have examined some concrete examples of so-called GRC hypersurfaces (surfaces with GRC parametrizations). The theory was presented in $\mathbb{R}^2$ on the case of cubics with cubic parametrizations and in $\mathbb{R}^3$ on non-quadric quadratically parametrized surfaces. The surprising result (and one of main results of this paper) is the fact that all non-quadric quadratically parametrized surfaces are of GRC type, i.e. they always posses rational offsets (classical or general). It is obvious that this result can be immediately applied if we use quadratic triangular Bézier patches. Despite the fact that we have concentrated in this paper mainly on quadratic parametrizations of surfaces in $\mathbb{R}^3$, of course, they exist other GRC parametrization of higher degrees (see example of the cubically parametrized polynomial surface in Figure 5).

Finally, the links between well-known parametrizations of curves and surfaces (PH/PN and LN) were examined and we have also shown the property of GRC parametrizations to adopt properties of general parametrizations and thus to behave in different situations in different forms.

References


Fig. 6. Convolution surface of the affine type (ii) surface $a(u, v) = (u, u^2 + v, v^2)$ and paraboloid $b(s, t) = (s, t, s^2 + t^2)$. 
Fig. 7. Convolution surface of the affine type (ii) surface \( a(u, v) = -(u, u^2 + v, v^2) \) and paraboloid \( b(s, t) = -(s, t, s^2 + t^2) \).