

Příklady z integrálního počtu v reálném oboru

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Plzeň
2004

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1 Primitivní funkce

Ve všech výsledcích je vynechána integrační konstanta.

Příklad 1 (základní vlastnosti). Vypočtěte integrály

$$(a) \int \sqrt{x} \, dx, \quad \left[\frac{2}{3} \sqrt{x^3} \right]$$

$$(b) \int \frac{dx}{x^2}, \quad \left[-\frac{1}{x} \right]$$

$$(c) \int a^x e^x \, dx, \quad \left[\frac{(ae)^x}{1+\ln a} \right]$$

$$(d) \int \left(\frac{1-x}{x} \right)^2 dx, \quad \left[x - 2 \ln x - \frac{1}{x} \right]$$

$$(e) \int \frac{3 \cdot 2^x - 2 \cdot 3^x}{2^x} dx, \quad \left[3x - \frac{2\left(\frac{3}{2}\right)^x}{\ln \frac{3}{2}} \right]$$

$$(f) \int \frac{\sqrt{x} - 2\sqrt[3]{x^2} + 1}{\sqrt[4]{x}} dx, \quad \left[\frac{4}{5}x\sqrt[4]{x} - \frac{24}{17}x\sqrt[12]{x^5} + \frac{4}{3}\sqrt[4]{x^3} \right]$$

$$(g) \int \frac{\sqrt{x^4 + x^{-4} + 2}}{x^3} dx, \quad \left[\ln |x| - \frac{1}{4x^4} \right]$$

$$(h) \int \frac{x^2}{1+x^2} dx, \quad \left[x - \operatorname{arctg} x \right]$$

$$(i) \int \frac{e^{3x} + 1}{e^x + 1} dx, \quad \left[\frac{1}{2} e^{2x} - e^x + x \right]$$

$$(j) \int \frac{\cos 2x}{\cos^2 x \sin^2 x} dx, \quad \left[-\cot g x - \operatorname{tg} x = -\frac{1}{\sin x \cos x} \right]$$

$$(k) \int \operatorname{tg}^2 x \, dx, \quad \left[\operatorname{tg} x - x \right]$$

$$(l) \int \operatorname{cotg}^2 x \, dx, \quad \left[-x - \operatorname{cotg} x \right]$$

$$(m) \int 2 \sin^2 \frac{x}{2} dx, \quad \left[x - \sin x \right]$$

$$(n) \int \frac{dx}{\cos 2x + \sin^2 x}, \quad \left[\operatorname{tg} x \right]$$

$$(o) \int \operatorname{tgh}^2 x \, dx, \quad \left[x - \operatorname{tgh} x \right]$$

$$(p) \int \operatorname{cotgh}^2 x \, dx, \quad \left[x - \operatorname{cotgh} x \right]$$

$$(q) \int \frac{1 + 2x^2}{x^2(1+x^2)} dx, \quad \left[\operatorname{arctg} x - \frac{1}{x} \right]$$

- (r) $\int \frac{(1+x)^2}{x(1+x^2)} dx,$ $[\ln|x| + 2 \operatorname{arctg} x]$
- (s) $\int \frac{1 + \cos^2 x}{1 + \cos 2x} dx,$ $[\frac{1}{2}(\operatorname{tg} x + x)]$
- (t) $\int \frac{x^4}{1+x^2} dx.$ $[\frac{x^3}{3} - x + \operatorname{arctg} x]$

Příklad 2. Buď $\int f(x) dx = F(x) + C$. Ukažte, že platí

$$\int f(ax+b) dx = \frac{1}{a}F(ax+b) + C, \quad a, b \in \mathbb{R}, \quad a \neq 0.$$

Příklad 3 (substituce). Vypočtěte integrály

- (a) $\int \frac{dx}{2x+a},$ $[\frac{1}{2} \ln|2x+a|]$
- (b) $\int (x+1)^{15} dx,$ $[\frac{1}{16}(x+1)^{16}]$
- (c) $\int (3-2x)^4 dx,$ $[-\frac{(3-2x)^5}{10}]$
- (d) $\int (e^{-x} + e^{-2x}) dx,$ $[-e^{-x} - \frac{1}{2}e^{-2x}]$
- (e) $\int (\sin 5x - \sin 5\alpha) dx,$ $[-\frac{1}{5} \cos 5x - x \sin 5\alpha]$
- (f) $\int \frac{dx}{\sqrt{3-2x}},$ $[-\sqrt{3-2x}]$
- (g) $\int \frac{dx}{\cos^2 5x},$ $[\frac{1}{5} \operatorname{tg} 5x]$
- (h) $\int \frac{dx}{1-10x},$ $[-\frac{1}{10} \ln|1-10x|]$
- (i) $\int \frac{dx}{\sin^2(2x + \frac{\pi}{4})},$ $[-\frac{1}{2} \operatorname{cotg}(2x + \frac{\pi}{4}) \text{ nebo } \frac{\sin 2x}{\sin 2x + \cos 2x}]$
- (j) $\int \frac{dx}{1 + \cos x},$ $[\operatorname{tg} \frac{x}{2}]$
- (k) $\int \frac{dx}{1 + \sin x},$ $[-\operatorname{tg}(\frac{\pi}{4} - \frac{x}{2}) \text{ nebo } \frac{1 - \cos x + \sin x}{1 + \sin x}]$
- (l) $\int [\sinh(2x+1) + \cosh(2x-1)] dx,$ $[\frac{1}{2e} \cosh 2x + \frac{e}{2} \sinh 2x]$
- (m) $\int \frac{dx}{\cosh^2 \frac{x}{2}},$ $[2 \operatorname{tgh} \frac{x}{2}]$

$$(n) \int \frac{dx}{\sinh^2 \frac{x}{2}}, \quad \left[-2 \operatorname{cotgh} \frac{x}{2}\right]$$

$$(o) \int \frac{dx}{\sqrt{4-9x^2}}, \quad \left[\frac{1}{3} \arcsin \frac{3x}{2}\right]$$

Příklad 4 (substituce). Vypočtěte integrály

$$(a) \int \frac{e^{2x}}{1-3e^{2x}} dx, \quad \left[-\frac{1}{6} \ln |1-3e^{2x}|\right]$$

$$(b) \int \operatorname{cotg} x dx, \quad [\ln |\sin x|]$$

$$(c) \int \operatorname{tg} x dx, \quad [-\ln |\cos x|]$$

$$(d) \int \sin^2 x \cos x dx, \quad \left[\frac{1}{3} \sin^3 x\right]$$

$$(e) \int \frac{dx}{x(1+\ln x)}, \quad [\ln |1+\ln x|]$$

$$(f) \int \frac{\cos 2x}{\sin x \cos x} dx, \quad [\ln |\sin 2x|]$$

$$(g) \int \frac{\cos x}{1+2\sin x} dx, \quad \left[\frac{1}{2} \ln |1+2\sin x|\right]$$

$$(h) \int \frac{x}{\sqrt{x^2+1}} dx, \quad [\sqrt{x^2+1}]$$

$$(i) \int \frac{x^3}{\sqrt[3]{x^4+1}} dx, \quad \left[\frac{3}{8} \sqrt[3]{(x^4+1)^2}\right]$$

$$(j) \int \frac{x}{\sqrt{1-x^2}} dx, \quad [-\sqrt{1-x^2}]$$

$$(k) \int x e^{-x^2} dx, \quad \left[-\frac{1}{2} e^{-x^2}\right]$$

$$(l) \int \frac{dx}{e^x + e^{-x}}, \quad [\operatorname{arctg} e^x]$$

$$(m) \int e^{\sin x} \cos x dx, \quad [e^{\sin x}]$$

$$(n) \int \frac{\ln^2 x}{x} dx, \quad \left[\frac{1}{3} \ln^3 x\right]$$

$$(o) \int \frac{x}{x^4+1} dx, \quad \left[\frac{1}{2} \operatorname{arctg} x^2\right]$$

$$(p) \int \frac{2^x}{\sqrt{1-4^x}} dx, \quad \left[\frac{\arcsin 2^x}{\ln 2}\right]$$

$$(q) \int \frac{\sin x}{\sqrt{\cos^3 x}} dx, \quad \left[\frac{2}{\sqrt{\cos x}}\right]$$

- (r) $\int \frac{\sin x + \cos x}{\sqrt[3]{\sin x - \cos x}} dx,$ $[\frac{3}{2}\sqrt[3]{1 - \sin 2x}]$
- (s) $\int \frac{dx}{\sin^2 x \sqrt[4]{\cot g x}},$ $[-\frac{4}{3}\sqrt[4]{\cot g^3 x}]$
- (t) $\int \frac{\operatorname{arctg} x}{1+x^2} dx,$ $[\frac{1}{2} \operatorname{arctg}^2 x]$
- (u) $\int \frac{x^4}{(x^5+1)^4} dx,$ $[-\frac{1}{15(x^5+1)^3}]$
- (v) $\int \frac{dx}{(x-1)^2+4},$ $[\frac{1}{2} \operatorname{arctg} \frac{x-1}{2}]$
- (w) $\int \frac{2x - \sqrt{\arcsin x}}{\sqrt{1-x^2}} dx,$ $[-2\sqrt{1-x^2} - \frac{2}{3}\sqrt{\arcsin^3 x}]$
- (x) $\int \frac{dx}{4x^2+4x+5},$ $[\frac{1}{4} \operatorname{arctg} x + \frac{1}{2}]$
- (y) $\int \frac{dx}{\sqrt{2-6x-9x^2}}.$ $[\frac{1}{3} \arcsin \frac{3x+1}{\sqrt{3}}]$

Příklad 5 (per-partes). Vypočtete integrály

- (a) $\int x \cos x dx,$ $[x \sin x + \cos x]$
- (b) $\int x \sin 2x dx,$ $[\frac{1}{4} \sin 2x - \frac{1}{2} x \cos 2x]$
- (c) $\int x e^{-x} dx,$ $[-e^{-x}(x+1)]$
- (d) $\int \ln x dx,$ $[x(\ln x - 1)]$
- (e) $\int x^n \ln x dx \quad (n \neq -1),$ $[\frac{x^{n+1}}{n+1} (\ln x - \frac{1}{n+1})]$
- (f) $\int 3^x x dx,$ $[\frac{3^x}{\ln^2 3} (x \ln 3 - 1)]$
- (g) $\int \operatorname{arctg} x dx,$ $[x \operatorname{arctg} x - \frac{1}{2} \ln(1+x^2)]$
- (h) $\int \arcsin x dx,$ $[x \arcsin x + \sqrt{1-x^2}]$
- (i) $\int x^2 e^{-2x} dx,$ $[-\frac{1}{2} e^{-2x} (x^2 + x + \frac{1}{2})]$
- (j) $\int x^3 e^{-x^2} dx,$ $[-\frac{x^2+1}{2} e^{-x^2}]$
- (k) $\int \ln(x + \sqrt{1+x^2}) dx,$ $[x \ln(x + \sqrt{1+x^2}) - \sqrt{1+x^2}]$

- (l) $\int \ln^2 x \, dx,$ $[x(\ln^2 x - 2 \ln x + 2)]$
- (m) $\int (\arcsin x)^2 \, dx,$ $[x \arcsin^2 x + 2\sqrt{1-x^2} \arcsin x - 2x]$
- (n) $\int e^{3x}(\sin 2x - \cos 2x) \, dx,$ $[\frac{e^{3x}}{13}(\sin 2x - 5 \cos 2x)]$
- (o) $\int \frac{x}{\cos^2 x} \, dx,$ $[x \operatorname{tg} x + \ln |\cos x|]$
- (p) $\int \operatorname{arctg} \sqrt{x} \, dx,$ $[-\sqrt{x} + (1+x) \operatorname{arctg} \sqrt{x}]$
- (q) $\int \sqrt{a^2 - x^2} \, dx,$ $[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a} = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \operatorname{arctg} \frac{x}{\sqrt{a^2 - x^2}}]$
- (r) $\int x \sin^2 x \, dx,$ $[\frac{x^2}{4} - \frac{x}{4} \sin 2x - \frac{\cos 2x}{8}]$
- (s) $\int \frac{dx}{(a^2 + x^2)^2},$ $[\frac{x}{2a^2(a^2 + x^2)} + \frac{1}{2a^3} \operatorname{arctg} \frac{x}{a}]$
- (t) $\int \frac{x e^{\operatorname{arctg} x}}{(1+x^2)^{\frac{3}{2}}} \, dx.$ $[\frac{(x-1)e^{\operatorname{arctg} x}}{2\sqrt{1+x^2}}]$

Příklad 6 (rozklad na parciální zlomky). Vypočtěte integrály

- (a) $\int \frac{x \, dx}{(x+1)(2x+1)},$ $[\ln \frac{|x+1|}{\sqrt{|2x+1|}}]$
- (b) $\int \frac{2x^2 + 41x - 91}{(x-1)(x+3)(x-4)} \, dx,$ $[\ln \left| \frac{(x-1)^4(x-4)^5}{(x+3)^7} \right|]$
- (c) $\int \frac{x^5 + x^4 - 8}{x^3 - 4x} \, dx,$ $[\frac{x^3}{3} + \frac{x^2}{2} + 4x + \ln \left| \frac{x^2(x-2)^5}{(x+2)^3} \right|]$
- (d) $\int \frac{32x \, dx}{(2x-1)(4x^2-16x+15)},$ $[\ln \left| \frac{(2x-1)(2x-5)^5}{(2x-3)^6} \right|]$
- (e) $\int \frac{2x^2 - 5}{x^4 - 5x^2 + 6} \, dx,$ $[\frac{1}{2\sqrt{2}} \ln \left| \frac{x-\sqrt{2}}{x+\sqrt{2}} \right| + \frac{1}{2\sqrt{3}} \ln \left| \frac{x-\sqrt{3}}{x+\sqrt{3}} \right|]$
- (f) $\int \frac{x^3 - 6x^2 + 11x - 5}{(x-2)^4} \, dx,$ $[\ln |x-2| + \frac{3x-8}{6(x-2)^3}]$
- (g) $\int \frac{x^2 \, dx}{(x+2)^2(x+4)^2},$ $[2 \ln \left| \frac{x+4}{x+2} \right| - \frac{5x+12}{x^2+6x+8}]$
- (h) $\int \frac{1}{8} \left(\frac{x-1}{x+1} \right)^4 \, dx,$ $[\frac{x}{8} - \ln |x+1| - \frac{9x^2+12x+5}{3(x+1)^3}]$
- (i) $\int \frac{x^2 - 2x + 3}{(x-1)(x^3 - 4x^2 + 3x)} \, dx,$ $[\frac{1}{x-1} + \ln \frac{\sqrt{|(x-1)(x-3)|}}{|x|}]$
- (j) $\int \frac{x^3 - 2x^2 + 4}{x^3(x-2)^2} \, dx,$ $[\frac{1}{4} \ln \left| \frac{x}{x-2} \right| - \frac{3x^2+3x+2}{2x^2(x-2)}]$

- (k) $\int \frac{dx}{x(x^2+1)},$ $[\ln \frac{|x|}{\sqrt{x^2+1}}]$
- (l) $\int \frac{dx}{1+x^3},$ $[\frac{1}{6} \ln \frac{(x+1)^2}{x^2-x+1} + \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{2x-1}{\sqrt{3}}]$
- (m) $\int \frac{x^4+1}{x^3-x^2+x-1} dx,$ $[\frac{(x+1)^2}{2} + \ln \frac{|x-1|}{\sqrt{x^2+1}} - \operatorname{arctg} x]$
- (n) $\int \frac{dx}{(x^2+1)(x^2+x)},$ $[\frac{1}{4} \ln \frac{x^4}{(x+1)^2(x^2+1)} - \frac{1}{2} \operatorname{arctg} x]$
- (o) $\int \frac{3x^2+x+3}{(x-1)^3(x^2+1)} dx,$ $[\frac{1}{4} [\ln \frac{\sqrt{x^2+1}}{|x-1|} + \operatorname{arctg} x - \frac{7}{(x-1)^2}]]$
- (p) $\int \frac{x^3+x-1}{(x^2+2)^2} dx,$ $[\frac{2-x}{4(x^2+2)} + \frac{1}{2} \ln(x^2+2) - \frac{1}{4\sqrt{2}} \operatorname{arctg} \frac{x}{\sqrt{2}}]$
- (q) $\int \frac{5x^2-12}{(x^2-6x+13)^2} dx,$ $[\frac{13x-159}{8(x^2-6x+13)} + \frac{53}{16} \operatorname{arctg} \frac{x-3}{2}]$
- (r) $\int \frac{dx}{(x^2+9)^3},$ $[\frac{x(x^2+15)}{216(x^2+9)^2} + \frac{1}{648} \operatorname{arctg} \frac{x}{3}]$
- (s) $\int \frac{2x}{(1+x)(1+x^2)^2} dx,$ $[\frac{x-1}{2(x^2+1)} + \frac{1}{4} \ln \frac{1+x^2}{(1+x)^2}]$
- (t) $\int \frac{x^9}{(x^4-1)^2} dx,$ $[\frac{x^2}{4} \frac{2x^4-3}{x^4-1} + \frac{3}{8} \ln |\frac{x^2-1}{x^2+1}|]$
- (u) $\int \frac{x^8}{x^8-1} dx.$ $[x - \frac{1}{4} \operatorname{arctg} x + \frac{1}{4\sqrt{2}} [\operatorname{arctg}(1-\sqrt{2}x) - \operatorname{arctg}(1+\sqrt{2}x)]] +$
 $+ \frac{1}{8} [\ln |\frac{x-1}{x+1}| + \frac{1}{\sqrt{2}} \ln \frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}]]$

Příklad 7 (odmocniny). Vypočtete integrály

- (a) $\int \frac{x+1}{\sqrt[3]{3x+1}} dx,$ $[\frac{x+2}{5} \sqrt[3]{(3x+1)^2}]$
- (b) $\int \frac{1}{x} \sqrt{\frac{1-x}{1+x}} dx,$ $[\ln |\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}| + 2 \operatorname{arctg} \sqrt{\frac{1-x}{1+x}}]$
- (c) $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}} dx,$ $[(\sqrt{x}-2)\sqrt{1-x} - \arcsin \sqrt{x}]$
- (d) $\int \frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}} dx,$ $[\frac{1}{2} [x^2 - x\sqrt{x^2-1} + \ln(x+\sqrt{x^2-1})]]$
- (e) $\int \frac{x\sqrt[3]{2+x}}{x+\sqrt[3]{2+x}} dx,$ $[\frac{3}{4}t^4 - \frac{3}{2}t^2 - \frac{3}{4} \ln |t-1| + \frac{15}{8} \ln(t^2+t+2) - \frac{27}{4\sqrt{7}} \operatorname{arctg} \frac{2t+1}{\sqrt{7}};$
 $t = \sqrt[3]{2+x}]$
- (f) $\int \frac{dx}{x(\sqrt{x}+\sqrt[5]{x^2})},$ $[\ln \frac{x}{(1+\sqrt[10]{x})^{10}} + \frac{10}{\sqrt[10]{x}} - \frac{5}{\sqrt[5]{x}} + \frac{10}{3\sqrt[10]{x^3}} - \frac{5}{2\sqrt[5]{x^2}}]$

- (g) $\int \frac{x}{\sqrt{x+1} + \sqrt[3]{x+1}} dx,$ $[-x - \frac{3}{2}(x+1)^{\frac{2}{3}} + \frac{6}{5}(x+1)^{\frac{5}{6}} + \frac{6}{7}(x+1)^{\frac{7}{6}} - \frac{3}{4}(x+1)^{\frac{4}{3}} + \frac{2}{3}(x+1)^{\frac{3}{2}}]$
- (h) $\int \frac{dx}{(1 + \sqrt[4]{x})^3 \sqrt{x}},$ $[-2 \frac{1+2\sqrt[4]{x}}{(1+\sqrt[4]{x})^2}]$
- (i) $\int \frac{1 - \sqrt{x+1}}{1 + \sqrt[3]{x+1}} dx,$ $[6t - 3t^2 - 2t^3 + \frac{3}{2}t^4 + \frac{6}{5}t^5 - \frac{6}{7}t^7 + 3 \ln(1+t^2) - 6 \operatorname{arctg} t;$
 $t = \sqrt[6]{x+1}]$
- (j) $\int \frac{dx}{\sqrt[3]{(x+1)^2(x-1)^4}},$ $[-\frac{3}{2} \sqrt[3]{\frac{x+1}{x-1}}]$
- (k) $\int \frac{dx}{x + \sqrt{x^2 + x + 1}},$ $[\frac{3}{2(2z+1)} + \frac{1}{2} \ln \frac{z^4}{|2z+1|^3}; z = x + \sqrt{x^2 + x + 1}]$
- (l) $\int \frac{dx}{1 + \sqrt{1 - 2x - x^2}},$ $[\ln |\frac{z-1}{z}| - 2 \operatorname{arctg} z; z = \frac{1+\sqrt{1-2x-x^2}}{x}]$
- (m) $\int \frac{dx}{x\sqrt{2+x-x^2}},$ $[-\frac{1}{\sqrt{2}} \ln |\frac{\sqrt{2+x-x^2} + \sqrt{2}}{x} + \frac{1}{2\sqrt{2}}|]$
- (n) $\int \frac{\sqrt{2x+x^2}}{x^2} dx,$ $[\ln |x+1 + \sqrt{2x+x^2}| - \frac{4}{x+\sqrt{2x+x^2}}]$
- (o) $\int \frac{dx}{(x-1)\sqrt{x^2+x+1}},$ $[-\frac{1}{\sqrt{3}} \ln |\frac{3+3x+2\sqrt{3(x^2+x+1)}}{x-1}|]$
- (p) $\int \frac{dx}{\sqrt{(4+x^2)^3}},$ $[\frac{x}{4\sqrt{4+x^2}}]$
- (q) $\int \frac{x^2}{\sqrt{(2-x^2)^3}} dx,$ $[\frac{x}{\sqrt{2-x^2}} - \arcsin \frac{x}{\sqrt{2}}]$
- (r) $\int \frac{1 + \sqrt{1-x^2}}{1 - \sqrt{1-x^2}} dx,$ $[-\frac{2+x^2}{x} - \frac{2}{x} \sqrt{1-x^2} - 2 \arcsin x]$
- (s) $\int \frac{\sqrt{1+x^2}}{2+x^2} dx,$ $[\frac{1}{2\sqrt{2}} \ln \frac{\sqrt{2x^2+2}-x}{\sqrt{2x^2+2+x}} + \ln(x + \sqrt{x^2+1})]$
- (t) $\int x^{-1}(1 + \sqrt[3]{x})^{-3} dx,$ $[3 [\ln |\frac{\sqrt[3]{x}}{1+\sqrt[3]{x}}| + \frac{2\sqrt[3]{x}+3}{2(1+\sqrt[3]{x})^2}]]$
- (u) $\int x^5 \sqrt[3]{(1+x^3)^2} dx,$ $[\frac{5x^6+2x^3-3}{40} \sqrt[3]{(1+x^3)^2}]$
- (v) $\int \frac{dx}{x\sqrt[3]{1+x^5}},$ $[\frac{1}{5} \ln \frac{|u-1|}{\sqrt{u^2+u+1}} + \frac{\sqrt{3}}{5} \operatorname{arctg} \frac{1+2u}{\sqrt{3}}; u = \sqrt[3]{1+x^5}]$
- (w) $\int \frac{dx}{\sqrt[4]{1+x^4}},$ $[\frac{1}{4} \ln |\frac{z+1}{z-1}| - \frac{1}{2} \operatorname{arctg} z; z = \frac{\sqrt[4]{1+x^4}}{x}]$
- (x) $\int \sqrt[3]{3x-x^3} dx.$ $[\frac{3z}{2(z^3+1)} - \frac{1}{4} \ln \frac{(z+1)^2}{z^2-z+1} - \frac{\sqrt{3}}{2} \operatorname{arctg} \frac{2z-1}{\sqrt{3}}; z = \frac{\sqrt[3]{3x-x^3}}{x}]$

Příklad 8 (goniometrické funkce). Vypočtěte integrály

- (a) $\int \sin^3 x \cos^2 x \, dx,$ $[\frac{1}{5} \cos^5 x - \frac{1}{3} \cos^3 x]$
- (b) $\int \frac{\sin^3 x}{\cos^4 x} \, dx,$ $[\frac{1}{3 \cos^3 x} - \frac{1}{\cos x}]$
- (c) $\int \frac{dx}{\cos x \sin^3 x},$ $[\ln |\operatorname{tg} x| - \frac{1}{2 \sin^2 x}]$
- (d) $\int \frac{\sin^4 x}{\cos^2 x} \, dx,$ $[\operatorname{tg} x + \frac{1}{4} \sin 2x - \frac{3}{2} x]$
- (e) $\int \frac{dx}{\sin^4 x \cos^4 x},$ $[\frac{(\operatorname{tg}^2 x - 1)(\operatorname{tg}^4 x + 10 \operatorname{tg}^2 x + 1)}{3 \operatorname{tg}^3 x}]$
- (f) $\int \cos^5 x \, dx,$ $[\sin x - \frac{2}{3} \sin^3 x + \frac{1}{5} \sin^5 x]$
- (g) $\int \sin^6 x \, dx,$ $[\frac{5}{16} x - \frac{1}{4} \sin 2x + \frac{3}{64} \sin 4x + \frac{1}{48} \sin^3 2x]$
- (h) $\int \operatorname{cotg}^4 x \, dx,$ $[x - \frac{1}{3} \operatorname{cotg}^3 x + \operatorname{cotg} x]$
- (i) $\int \operatorname{tg}^5 x \, dx,$ $[\frac{1}{4} \operatorname{tg}^4 x - \frac{1}{2} \operatorname{tg}^2 x - \ln |\cos x|]$
- (j) $\int \frac{\cos^4 x + \sin^4 x}{\cos^2 x - \sin^2 x} \, dx,$ $[\frac{1}{4} (\ln |\frac{1+\operatorname{tg} x}{1-\operatorname{tg} x}| + \sin 2x)]$
- (k) $\int \sin 5x \cos x \, dx,$ $[-\frac{1}{8} \cos 4x - \frac{1}{12} \cos 6x]$
- (l) $\int \cos x \cos 2x \cos 3x \, dx,$ $[\frac{x}{4} + \frac{\sin 2x}{8} + \frac{\sin 4x}{16} + \frac{\sin 6x}{24}]$
- (m) $\int \sin^3 2x \cos^2 3x \, dx,$ $[-\frac{3}{16} \cos 2x + \frac{3}{64} \cos 4x + \frac{1}{48} \cos 6x - \frac{3}{128} \cos 8x + \frac{1}{192} \cos 12x]$
- (n) $\int \frac{dx}{\sin x + \cos x},$ $[\frac{\sqrt{2}}{2} \ln |\operatorname{tg}(\frac{\pi}{8} + \frac{x}{2})|]$
- (o) $\int \frac{dx}{1 + \operatorname{tg} x},$ $[\frac{1}{2}(x + \ln |\sin x + \cos x|)]$
- (p) $\int \frac{dx}{5 - 3 \cos x},$ $[\frac{1}{2} \operatorname{arctg}(2 \operatorname{tg} \frac{x}{2})]$
- (q) $\int \frac{dx}{5 + 4 \sin x},$ $[\frac{2}{3} \operatorname{arctg} \frac{5 \operatorname{tg} \frac{x}{2} + 4}{3}]$
- (r) $\int \frac{\sin^2 x}{1 + \sin^2 x} \, dx,$ $[x - \frac{1}{\sqrt{2}} \operatorname{arctg}(\sqrt{2} \operatorname{tg} x)]$
- (s) $\int \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x}, \quad a, b \in \mathbb{R},$ $[\frac{1}{ab} \operatorname{arctg}(\frac{a \operatorname{tg} x}{b})]$
- (t) $\int \frac{dx}{1 + \varepsilon \cos x}, \quad \varepsilon > 0,$ $[\frac{2}{\sqrt{1-\varepsilon^2}} \operatorname{arctg}(\sqrt{\frac{1-\varepsilon}{1+\varepsilon}} \operatorname{tg} \frac{x}{2}); \quad 0 < \varepsilon < 1]$

$$\begin{aligned}
 & \frac{1}{\sqrt{\varepsilon^2-1}} \ln \frac{\varepsilon+\cos x+\sqrt{\varepsilon^2-1} \sin x}{1+\varepsilon \cos x}; \quad \varepsilon > 1] \\
 \text{(u)} \quad & \int \frac{\sin x \cos x}{\sin x + \cos x} dx, & \left[\frac{1}{2}(\sin x - \cos x) - \frac{1}{2\sqrt{2}} \ln \left| \operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{8} \right) \right| \right] \\
 \text{(v)} \quad & \int \frac{dx}{\sin^4 x + \cos^4 x}, & \left[\frac{1}{\sqrt{2}} \operatorname{arctg} \left(\frac{\operatorname{tg} 2x}{\sqrt{2}} \right) \right] \\
 \text{(w)} \quad & \int \frac{dx}{\sin^3 x + \cos^3 x}, & \left[-\frac{2}{3} \operatorname{arctg} (\cos x - \sin x) + \frac{\sqrt{2}}{3} \ln \frac{\sqrt{2}-1+\operatorname{tg} \frac{x}{2}}{\sqrt{2}+1-\operatorname{tg} \frac{x}{2}} \right] \\
 \text{(x)} \quad & \int \frac{\sin^2 x - \cos^2 x}{\sin^4 x + \cos^4 x} dx, & \left[\frac{1}{2\sqrt{2}} \ln \frac{\sqrt{2}-\sin 2x}{\sqrt{2}+\sin 2x} \right] \\
 \text{(y)} \quad & \int \frac{dx}{a \sin x + b \cos x}, \quad a, b \in \mathbb{R}. & \left[\frac{1}{\sqrt{a^2+b^2}} \ln \left| \frac{\operatorname{tg} \frac{x}{2}-t_1}{\operatorname{tg} \frac{x}{2}-t_2} \right|; \quad t_1 = \frac{a-\sqrt{a^2+b^2}}{b}; \quad t_2 = \frac{a+\sqrt{a^2+b^2}}{b} \right]
 \end{aligned}$$

Příklad 9. Označme

$$I_n = \int \sin^n x dx, \quad K_n = \int \cos^n x dx, \quad L_n = \int \frac{dx}{\sin^n x}, \quad M_n = \int \frac{dx}{\cos^n x}; \quad n \geq 2.$$

Ukažte, že platí

$$\begin{aligned}
 I_n &= -\frac{\cos x \sin^{n-1} x}{n} + \frac{n-1}{n} I_{n-2}, & K_n &= \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n} K_{n-2}, \\
 L_n &= -\frac{\cos x}{(n-1) \sin^{n-1} x} + \frac{n-2}{n-1} L_{n-2}, & M_n &= \frac{\sin x}{(n-1) \cos^{n-1} x} + \frac{n-2}{n-1} M_{n-2}.
 \end{aligned}$$

Příklad 10 (hyperbolické funkce). Vypočtete integrály

$$\begin{aligned}
 \text{(a)} \quad & \int \sinh^2 x dx, & \left[\frac{\sinh 2x}{4} - \frac{x}{2} \right] \\
 \text{(b)} \quad & \int \operatorname{tgh}^2 x dx, & [x - \operatorname{tgh} x] \\
 \text{(c)} \quad & \int \sinh^3 x dx, & \left[\frac{1}{3} \cosh^3 x - \cosh x \right] \\
 \text{(d)} \quad & \int \frac{dx}{\sinh x}, & [\ln |\operatorname{tg} \frac{x}{2}|] \\
 \text{(e)} \quad & \int \frac{x}{\cosh^2 x} dx. & [x \operatorname{tgh} x - \ln \cosh x]
 \end{aligned}$$

Příklad 11.

$$\begin{aligned}
 \text{(a)} \quad & \int x^3 e^{3x} dx, & \left[e^{3x} \left(\frac{x^3}{3} - \frac{x^2}{3} + \frac{2x}{9} - \frac{2}{27} \right) \right] \\
 \text{(b)} \quad & \int x^5 \sin 5x dx, & \left[-\left(\frac{x^5}{5} - \frac{4x^3}{25} + \frac{24x}{625} \right) \cos 5x + \left(\frac{x^4}{5} - \frac{12x^2}{125} + \frac{24}{3125} \right) \sin 5x \right] \\
 \text{(c)} \quad & \int x^7 e^{-x^2} dx, & \left[-\frac{e^{-x^2}}{2} (x^6 + 3x^4 + 6x^2 + 6) \right]
 \end{aligned}$$

- (d) $\int e^{ax} \cos^2 bx \, dx,$ $[e^{ax} [\frac{1}{2a} + \frac{a \cos 2bx + 2b \sin 2bx}{2(a^2 + 4b^2)}]]$
- (e) $\int x e^x \sin x \, dx,$ $[\frac{e^x}{2} [x(\sin x - \cos x) + \cos x]]$
- (f) $\int x^2 e^x \cos x \, dx,$ $[\frac{e^x}{2} [(x-1)^2 \sin x + (x^2-1) \cos x]]$

Příklad 12. Vypočtete integrály

- (a) $\int \frac{e^{2x}}{1+e^x} \, dx,$ $[e^x - \ln(1+e^x)]$
- (b) $\int \frac{dx}{1+e^{\frac{x}{2}}+e^{\frac{x}{3}}+e^{\frac{x}{6}}},$ $[x - 3 \ln [(1+e^{\frac{x}{6}})\sqrt{1+e^{\frac{x}{3}}}] - 3 \operatorname{arctg} e^{\frac{x}{6}}]$
- (c) $\int \sqrt{\frac{e^x-1}{e^x+1}} \, dx,$ $[\ln(e^x + \sqrt{e^{2x}-1}) + \arcsin e^{-x}]$
- (d) $\int \frac{dx}{\sqrt{1+e^x+e^{2x}}},$ $[x - \ln |2+e^x+2\sqrt{1+e^x+e^{2x}}|]$
- (e) $\int \ln^2(x + \sqrt{1+x^2}) \, dx,$ $[x \ln^2(x + \sqrt{1+x^2}) - 2\sqrt{1+x^2} \ln(x + \sqrt{1+x^2}) + 2x]$
- (f) $\int \ln(\sqrt{1-x} + \sqrt{1+x}) \, dx,$ $[-\frac{x}{2} + x \ln(\sqrt{1-x} + \sqrt{1+x}) + \frac{1}{2} \arcsin x]$
- (g) $\int \frac{\ln x}{(1+x^2)^{\frac{3}{2}}} \, dx,$ $[\frac{x \ln x}{\sqrt{1+x^2}} - \ln(x + \sqrt{1+x^2}) = \frac{x \ln x}{\sqrt{1+x^2}} - \operatorname{argsinh} x]$
- (h) $\int \sqrt{x} \operatorname{arctg} \sqrt{x} \, dx,$ $[-\frac{x}{3} + \frac{1}{3} \ln(1+x) + \frac{2x\sqrt{x}}{3} \operatorname{arctg} \sqrt{x}]$
- (i) $\int x \ln \frac{1+x}{1-x} \, dx,$ $[x + \frac{x^2-1}{2} \ln \frac{1+x}{1-x}]$
- (j) $\int \frac{dx}{a^2 - b^2 \cos^2 x}, \quad a, b \in \mathbb{R},$ $[\frac{1}{a\sqrt{a^2-b^2}} \operatorname{arctg} \frac{a \operatorname{tg} x}{\sqrt{a^2-b^2}}; a^2 - b^2 > 0,$
 $\frac{1}{2a\sqrt{b^2-a^2}} \ln \frac{a \operatorname{tg} x - \sqrt{b^2-a^2}}{a \operatorname{tg} x + \sqrt{b^2-a^2}}; a^2 - b^2 < 0]$
- (k) $\int \frac{x e^x}{\sqrt{1+e^x}} \, dx,$ $[(2x-4)\sqrt{1+e^x} + 2 \ln \frac{\sqrt{1+e^x}+1}{\sqrt{1+e^x}-1}]$
- (l) $\int \frac{x \operatorname{arctg} x}{(1+x^2)^2} \, dx,$ $[\frac{x+(x^2-1) \operatorname{arctg} x}{4(x^2+1)}]$
- (m) $\int \frac{x e^x}{(1+x)^2} \, dx,$ $[\frac{e^x}{1+x}]$
- (n) $\int \frac{\operatorname{tg} x}{1 + \operatorname{tg} x + \operatorname{tg}^2 x} \, dx,$ $[x - \frac{2}{\sqrt{3}} \operatorname{arctg} \frac{1+2 \operatorname{tg} x}{\sqrt{3}}]$
- (o) $\int \sinh ax \cos bx \, dx,$ $[\frac{a \cosh ax \cos bx + b \sinh ax \sin bx}{a^2 + b^2}]$
- (p) $\int |x| \, dx,$ $[\frac{|x|x|}{2}]$

- (q) $\int e^{-|x|} dx$, [$e^x - 1$ pro $x < 0$; $1 - e^{-x}$ pro $x \geq 0$]
 (r) $\int \max\{1, x^2\} dx$. [x pro $|x| \leq 1$; $\frac{x^3}{3} + \frac{2}{3} \operatorname{sgn} x$ pro $|x| > 1$]

2 Určitý integrál

Příklad 1.

- (a) $\int_{-1}^8 \sqrt[3]{x} dx$, [$\frac{45}{4}$]
 (b) $\int_2^{-13} \frac{dx}{\sqrt[5]{(3-x)^4}}$, [$-5(\sqrt[5]{16} - 1)$]
 (c) $\int_0^{16} \frac{dx}{\sqrt{x+9} - \sqrt{x}}$, [12]
 (d) $\int_1^e \frac{dx}{x\sqrt{1 - \ln^2 x}}$, [$\frac{\pi}{2}$]
 (e) $\int_0^{\sqrt{\frac{a}{2}}} \frac{x^{n-1}}{\sqrt{a^2 - x^{2n}}} dx$, $a \in \mathbb{R}$, $n \in \mathbb{N}$, [$\frac{\pi}{6n}$]
 (f) $\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{x^3}{\sqrt{(\frac{5}{8} - x^4)^3}} dx$, [$\frac{4}{3}$]
 (g) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dx}{1 + \cos x}$, [2]
 (h) $\int_0^{\frac{\pi}{2}} \cos^5 x \sin 2x dx$, [$\frac{2}{7}$]
 (i) $\int_{-\frac{\pi}{2}}^{-\frac{\pi}{4}} \frac{\cos^3 x}{\sqrt[3]{\sin x}} dx$, [$\frac{21}{16\sqrt[3]{2}} - \frac{9}{8}$]
 (j) $\int_1^2 x \log_2 x dx$, [$2 - \frac{3}{4 \ln 2}$]
 (k) $\int_0^{e-1} \ln(x+1) dx$, [1]
 (l) $\int_0^a \sqrt{a^2 - x^2} dx$, $a \in \mathbb{R}$, [$\frac{\pi a^2}{4}$]
 (m) $\int_0^{\frac{\pi}{2}} e^{2x} \cos x dx$, [$\frac{e^{\pi} - 2}{5}$]
 (n) $\int_4^9 \frac{\sqrt{x}}{\sqrt{x} - 1} dx$, [$7 + 2 \ln 2$]

- (o) $\int_0^1 \frac{\sqrt{e^x}}{\sqrt{e^x + e^{-x}}} dx,$ $[\ln \frac{e + \sqrt{1+e^2}}{1+\sqrt{2}} = \operatorname{argsinh} e - \operatorname{argsinh} 1]$
- (p) $\int_0^1 x^2 \sqrt{1-x^2} dx,$ $[\frac{\pi}{16}]$
- (q) $\int_1^{\sqrt{3}} \frac{\sqrt{1+x^2}}{x^2} dx,$ $[\sqrt{2} - \frac{2}{\sqrt{3}} + \ln \frac{2+\sqrt{3}}{1+\sqrt{2}}]$
- (r) $\int_0^{-\ln 2} \sqrt{1-e^{2x}} dx,$ $[\frac{\sqrt{3}}{2} + \ln(2-\sqrt{3})]$
- (s) $\int_{\sqrt{2}}^{2\sqrt{5}} \frac{dx}{x \sqrt{(x^2+5)^5}},$ $[\frac{1}{50\sqrt{5}} \ln \frac{3+\sqrt{5}}{2} - \frac{2^4 \cdot 23}{3^4 \cdot 5^4}]$
- (t) $\int_0^1 \sqrt{2x+x^2} dx,$ $[\sqrt{3} - \frac{1}{2} \ln(2+\sqrt{3})]$
- (u) $\int_0^{\frac{\pi}{2}} \frac{dx}{2 \cos x + 3},$ $[\frac{2}{\sqrt{5}} \operatorname{arctg} \frac{1}{\sqrt{5}}]$
- (v) $\int_0^{\frac{\pi}{2}} \frac{dx}{1 + \frac{1}{6} \sin^2 x},$ $[\frac{\pi}{2} \sqrt{\frac{6}{7}}]$
- (w) $\int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{a^2 \cos^2 x + b^2 \sin^2 x} dx, \quad a, b \in \mathbb{R},$ $[\frac{1}{a^2-b^2} \ln |\frac{a}{b}|]$
- (x) $\int_0^{2\pi} \frac{dx}{1 + \varepsilon \cos x}, \quad 0 \leq \varepsilon < 1,$ $[\frac{2\pi}{\sqrt{1-\varepsilon^2}}]$
- (y) $\int_0^{\frac{\pi}{2}} \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x}, \quad a, b \in \mathbb{R} \setminus \{0\}.$ $[\frac{\pi}{2|ab|}]$

Příklad 2. Bud'

$$I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m x \cos^n x dx, \quad m, n \geq 0.$$

Ukažte, že platí

$$I_{m,n} = \frac{n-1}{m+n} I_{m,n-2} = \frac{m-1}{m+n} I_{m-2,n}.$$

Užitím této formule vypočtete

$$\int_0^1 x^p (1-x)^q dx, \quad p, q \in \mathbb{N}.$$

$$[\frac{p!q!}{(p+q+1)!}; \text{ pološte } x = \sin^2 t]$$

Příklad 3. Vypočtěte následující integrály tak, že je vyjádříte jako limitu integrálních součtů

(a) $\int_{-1}^2 x^2 dx,$ [3]

(b) $\int_0^{\frac{\pi}{2}} \sin x dx.$ [1]

Příklad 4. Užitím určitého integrálu najděte

(a) $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n+i},$ [ln 2; užiňte $\int_1^2 \frac{dx}{x}$]

(b) $\lim_{n \rightarrow \infty} n \sum_{k=1}^n \frac{1}{n^2+k^2},$ [$\frac{\pi}{4}$; užiňte $\int_0^1 \frac{dx}{1+x^2}$]

(c) $\lim_{n \rightarrow \infty} \frac{1}{n^{p+1}} \sum_{k=1}^n k^p, \quad p > 0.$ [$\frac{1}{p+1}$; užiňte $\int_0^1 x^p dx$]

Příklad 5 (derivace integrálu). Najděte

(a) $\frac{d}{dx} \int_a^b \sin x^2 dx,$ [0]

(b) $\frac{d}{da} \int_a^b \sin x^2 dx,$ [$-\sin a^2$]

(c) $\frac{d}{db} \int_a^b \sin x^2 dx.$ [$\sin b^2$]

Příklad 6 (derivace podle meze). Vypočtěte derivace

(a) $\frac{d}{dx} \int_0^x \frac{1-t+t^2}{1+t+t^2} dt, \quad \text{pro } x=1,$ [$\frac{1}{3}$]

(b) $\frac{d}{dx} \int_x^5 \sqrt{1+t^2} dt, \quad \text{pro } x=0; \text{ pro } x=\frac{3}{4}.$ [$-1; -\frac{5}{4}$]

Příklad 7. Vypočtěte

(a) $\lim_{x \rightarrow 0} \frac{1}{x} \int_0^x \cos t^2 dt,$ [1]

(b) $\lim_{x \rightarrow \infty} \frac{1}{\sqrt{x^2+1}} \int_0^x \operatorname{arctg}^2 t dt,$ [$\frac{\pi^2}{4}$]

(c) $\lim_{x \rightarrow \infty} \frac{\left(\int_0^x e^{t^2} dt \right)^2}{\int_0^x e^{2t^2} dt}.$ [0; užiňte l'Hospitalova pravidla]

Příklad 8. Najděte $\int_a^b f(x) dx$, je-li

$$(a) \quad a = 0, \quad b = 2, \quad f(x) = \begin{cases} x^2 & \text{pro } 0 \leq x \leq 1, \\ 2 - x & \text{pro } 1 < x \leq 2, \end{cases} \quad \left[\frac{5}{6}\right]$$

$$(b) \quad a = 0, \quad b = 1, \quad f(x) = \begin{cases} x & \text{pro } 0 \leq x \leq t, \\ t \frac{1-x}{1-t} & \text{pro } t < x \leq 1, \end{cases} \quad \left[\frac{t}{2}\right]$$

$$(c) \quad a = -2, \quad b = 2, \quad f(x) = \begin{cases} -x & \text{pro } x \in \langle -2, -1 \rangle, \\ 0 & \text{pro } x \in (-1, 0), \\ \sqrt{1-x^2} & \text{pro } x \in \langle 0, 1 \rangle, \\ 2-x & \text{pro } x \in (1, 2). \end{cases} \quad \left[2 + \frac{\pi}{4}\right]$$

Příklad 9. Buďte $f(x)$ sudá a $g(x)$ lichá funkce, které jsou integrovatelné na intervalu $\langle -a, a \rangle$. Ukažte, že

$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx; \quad \int_{-a}^a g(x) dx = 0.$$

Příklad 10. Pomocí vět o střední hodnotě odhadněte integrál I

$$(a) \quad \int_0^{2\pi} \frac{dx}{1 + \frac{1}{2} \cos x}, \quad \left[\frac{4\pi}{3} \leq I \leq 4\pi\right]$$

$$(b) \quad \int_0^1 \frac{x^9}{\sqrt{1+x}} dx, \quad \left[\frac{1}{10\sqrt{2}} \leq I \leq \frac{1}{10}\right]$$

$$(c) \quad \int_0^{100} \frac{e^{-x}}{x+100} dx, \quad \left[\frac{1}{200} \leq I \leq \frac{1}{100}\right]$$

$$(d) \quad \int_{100\pi}^{200\pi} \frac{\sin x}{x} dx, \quad \left[|I| \leq \frac{1}{50\pi}\right]$$

$$(e) \quad \int_a^b \frac{e^{-\alpha x}}{x} dx. \quad \left[|I| \leq \frac{2e^{-\alpha a}}{a}\right]$$

Příklad 11 (nevlastní integrál). Vypočtěte integrály

$$(a) \quad \int_1^{\infty} \frac{dx}{x^4}, \quad \left[\frac{1}{3}\right]$$

$$(b) \quad \int_0^{\infty} e^{-ax} dx, \quad a > 0, \quad \left[\frac{1}{a}\right]$$

$$(c) \quad \int_{-\infty}^{+\infty} \frac{2x}{x^2+1} dx, \quad [\text{diverguje}]$$

$$(d) \quad \int_{-\infty}^{+\infty} \frac{dx}{x^2+2x+2}, \quad [\pi]$$

$$(e) \quad \int_2^{\infty} \frac{\ln x}{x} dx, \quad [\text{diverguje}]$$

- (f) $\int_1^{\infty} \frac{dx}{x^2(x+1)},$ [1 - ln 2]
- (g) $\int_0^{\infty} x e^{-x^2} dx,$ [$\frac{1}{2}$]
- (h) $\int_0^{\infty} x \sin x dx,$ [diverguje]
- (i) $\int_0^{\infty} e^{-ax} \cos bx dx,$ [$\frac{a}{a^2+b^2}$ pro $a > 0$; diverguje pro $a \leq 0$]
- (j) $\int_1^{\infty} \frac{\operatorname{arctg} x}{x^2} dx,$ [$\frac{\pi}{4} + \frac{1}{2} \ln 2$]
- (k) $\int_0^{\infty} \frac{dx}{1+x^3},$ [$\frac{2\pi}{3\sqrt{3}}$]
- (l) $\int_{-\infty}^{+\infty} \frac{dx}{(x^2+x+1)^2},$ [$\frac{4\pi}{3\sqrt{3}}$]
- (m) $\int_0^{\infty} \frac{\operatorname{arctg} x}{(1+x^2)^{\frac{3}{2}}} dx,$ [$\frac{\pi}{2} - 1$]
- (n) $\int_0^1 \frac{dx}{\sqrt{1-x^2}},$ [$\frac{\pi}{2}$]
- (o) $\int_1^2 \frac{x}{\sqrt{x-1}} dx,$ [$\frac{8}{3}$]
- (p) $\int_1^e \frac{dx}{x\sqrt{\ln x}},$ [2]
- (q) $\int_0^1 \frac{dx}{1-x^2+2\sqrt{1-x^2}},$ [$\frac{\pi}{3\sqrt{3}}$]
- (r) $\int_{-1}^1 \frac{x-1}{\sqrt[3]{x^5}} dx,$ [diverguje]
- (s) $\int_a^b \frac{dx}{\sqrt{(x-a)(b-x)}}, \quad a < b,$ [π]
- (t) $\int_a^b \frac{x}{\sqrt{(x-a)(b-x)}} dx, \quad a < b,$ [$\frac{\pi(a+b)}{2}$]
- (u) $\int_1^{\infty} \frac{dx}{x\sqrt{x^2-1}},$ [$\frac{\pi}{2}$]
- (v) $\int_0^{\infty} x^n e^{-x} dx, \quad n \in \mathbb{N},$ [$n!$]
- (w) $\int_0^{\infty} x^{2n+1} e^{-x^2} dx, \quad n \in \mathbb{N},$ [$\frac{n!}{2}$]
- (x) $\int_0^{\infty} \frac{dx}{(a^2+x^2)^n}, \quad n \in \mathbb{N},$ [$\frac{(2n-3)!!}{(2n-2)!!} \frac{\pi}{2a^{2n-1}}$]

$$(y) \int_0^1 \frac{x^n}{\sqrt{1-x^2}} dx, \quad n \in \mathbb{N}. \quad \left[\frac{(2k-1)!!}{(2k)!!} \frac{\pi}{2} \text{ pro } n = 2k; \frac{(2k)!!}{(2k+1)!!} \text{ pro } n = 2k + 1 \right]$$

Příklad 12. Vyšetřete konvergenci integrálů

$$(a) \int_1^\infty \frac{e^{-x}}{x} dx, \quad [\text{konverguje}]$$

$$(b) \int_0^\infty \frac{dx}{\sqrt{1+x^3}}, \quad [\text{konverguje}]$$

$$(c) \int_2^\infty \frac{dx}{\sqrt[3]{x^3-1}}, \quad [\text{diverguje}]$$

$$(d) \int_1^\infty \frac{\sin x}{x^4} dx, \quad [\text{konverguje}]$$

$$(e) \int_0^\infty \frac{x^2}{x^4-x^2+1} dx, \quad [\text{konverguje}]$$

$$(f) \int_0^2 \frac{dx}{\ln x}, \quad [\text{diverguje}]$$

$$(g) \int_0^\infty x^{p-1} e^{-x} dx, \quad [\text{konverguje pro } p > 0]$$

$$(h) \int_0^\infty \frac{x^m}{1+x^n} dx, \quad m, n \in \mathbb{N}, \quad [\text{konverguje pro } n - m > 1]$$

$$(i) \int_0^1 \frac{\sqrt{x}}{\sqrt{1-x^4}} dx, \quad [\text{konverguje}]$$

$$(j) \int_0^1 \frac{x^2}{\sqrt[3]{(1-x^2)^5}} dx, \quad [\text{diverguje}]$$

$$(k) \int_0^1 \frac{\sqrt{x}}{e^{\sin x} - 1} dx, \quad [\text{konverguje}]$$

$$(l) \int_0^1 \frac{dx}{e^x - \cos x}. \quad [\text{diverguje}]$$

3 Užití určitého integrálu

Příklad 1. Najděte obsah rovinného oboru, omezeného křivkami

$$(a) \quad ax = y^2; \quad ay = x^2, \quad \left[\frac{a^2}{3} \right]$$

$$(b) \quad y = 2x = x^2; \quad x + y = 0, \quad \left[\frac{9}{2} \right]$$

$$(c) \quad y = x; \quad y = x + \sin^2 x, \quad 0 \leq x \leq \pi, \quad \left[\frac{\pi}{2} \right]$$

$$(d) \quad y = -x^2 + 4x - 3 \text{ a tečnami k této křivce v bodech } [0, -3] \text{ a } [3, 0], \quad \left[\frac{9}{4} \right]$$

$$(e) \quad x^2 + y^2 = 8; \quad y = \frac{x^2}{2} \text{ (dva obory),} \quad \left[2\pi + \frac{4}{3}; 6\pi - \frac{4}{3} \right]$$

- (f) $x^2 + y^2 = a^2$; $x^2 - 2y^2 = \frac{a^2}{4}$ (tři obory), $[a^2 \left[\frac{\pi}{6} - \frac{\sqrt{2}}{8} \ln(\sqrt{3} + \sqrt{2}) \right] \text{ dva obory};$
 $a^2 \left[\frac{2\pi}{3} + \frac{\sqrt{2}}{4} \ln(\sqrt{3} + \sqrt{2}) \right]]$
- (g) $y^2 = x^3$; $y = 8$; $x = 0$, [19,2]
- (h) $y^2 = x^2(a^2 - x^2)$, [$\frac{4}{3}a^3$]
- (i) $4(y^2 - x^2) + x^3 = 0$, [$\frac{128}{15}$]
- (j) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ (astroida). [$\frac{3\pi a^2}{8}$]

Příklad 2. Najděte obsah rovinného oboru, omezeného parametricky zadanou křivkou $x = \psi(t)$, $y = \varphi(t)$, $t \in \langle \alpha, \beta \rangle$

- (a) $x = a(t - \sin t)$, $y = a(1 - \cos t)$ (cykloida) $t \in \langle 0, 2\pi \rangle$; $y = 0$, [$3\pi a^2$]
- (b) $x = 3t^2$, $y = 3t - t^3$, [$\frac{72}{5}\sqrt{3}$]
- (c) $x = t^2 - 1$, $y = t^3 - t$, [$\frac{8}{15}$]
- (d) $x = a(2 \cos t - \cos 2t)$, $y = a(2 \sin t - \sin 2t)$ (kardioida), [$6\pi a^2$]
- (e) $x^3 + y^3 - 3axy = 0$ (Descartův list). [$\frac{3}{2}a^2$; předpokládejte, že $y = tx$]

Návod. Užijte vzorců

$$S = \left| \int_{\alpha}^{\beta} \psi(t)\varphi'(t) dt \right|$$

nebo

$$(*) \quad S = \frac{1}{2} \left| \int_{\alpha}^{\beta} (\varphi'(t)\psi(t) - \varphi(t)\psi'(t)) dt \right|.$$

Příklad 3. Užitím vzorce (*) ukažte, že pro křivku zadanou v polárních souřadnicích platí

$$(\dagger) \quad S = \frac{1}{2} \int_{\varphi_0}^{\varphi_1} r^2(\varphi) d\varphi.$$

Příklad 4. Užitím vzorce (†) najděte obsah rovinného oboru, omezeného křivkou v polárním tvaru $r = r(\varphi)$

- (a) $r^2 = a^2 \cos 2\varphi$ (lemniskata), [a^2]
- (b) $r = a \cos 5\varphi$ (pětিলistä růže), [$\frac{\pi a^2}{4}$]
- (c) $r = a \sin \varphi$ (čtyřlístá růže), [$\frac{\pi a^2}{4}$]
- (d) $r = a |\sin 2\varphi|$ (čtyřlístek), [$\frac{\pi a^2}{2}$]
- (e) $r = 2 + \cos 2\varphi$ vně křivky $r = 2 + \sin \varphi$. [$\frac{51\sqrt{3}}{16}$]

Příklad 5. Najděte délku oblouku křivky

- (a) $y^2 = x^3$ pro přímku $x = \frac{4}{3}$, [$\frac{112}{27}$]
- (b) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, [6a]
- (c) $y = a \cosh \frac{x}{a}$, $0 \leq x \leq b$, [$a \sinh \frac{b}{a}$]
- (d) $y^2 = 2px$, $0 \leq x \leq x_0$, [$2\sqrt{x_0(x_0 + \frac{p}{2})} + p \ln \frac{\sqrt{x_0 + \sqrt{x_0 + \frac{p}{2}}} + \sqrt{\frac{p}{2}}}{\sqrt{\frac{p}{2}}}$]
- (e) $y = \sqrt{x - x^2} + \arcsin \sqrt{x}$, [2]
- (f) $x = a(t - \sin t)$, $y = a(1 - \cos t)$, $0 \leq t \leq 2\pi$, [8a]
- (g) $x = a(\cos t + t \sin t)$, $y = a(\sin t - t \cos t)$, $t \in (0, 2\pi)$, [$2\pi^2 a$]
- (h) $\left(\frac{x}{a}\right)^{\frac{4}{3}} + \left(\frac{y}{b}\right)^{\frac{4}{3}} = 1$, $a, b > 0$, [$4\frac{a^2+ab+b^2}{a+b}$]
- (i) $r = a(1 + \cos \varphi)$, [8a]
- (j) $r = a\varphi$ (Archimedova spirála) $0 \leq \varphi \leq 2\pi$, [$\pi a\sqrt{1 + 4\pi^2} + \frac{a}{2} \ln(2\pi + \sqrt{1 + 4\pi^2})$]

Příklad 6. Najděte objem rotačního tělesa, které vznikne rotací oboru

- (a) $y = \sin x$, $y = 0$, $0 \leq x \leq \pi$, kolem osy x ; kolem osy y , [$\frac{\pi^2}{2}$; $2\pi^2$]
- (b) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$; $y = \pm b$, kolem osy x ; kolem osy y , [$\frac{4\pi ab^2}{3}(2\sqrt{2} - 1)$; $\frac{8\pi a^2 b}{3}$]
- (c) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ kolem osy symetrie, [$\frac{32\pi a^3}{105}$]
- (d) $x^2 + (y - b)^2 = a^2$, $0 < a \leq b$, kolem osy x , [$2\pi^2 a^2 b$]
- (e) $x = a(t - \sin t)$, $y = a(1 - \cos t)$, $0 \leq t \leq 2\pi$; $y = 0$, kolem osy x ; kolem osy y ; kolem přímky $y = 2a$. [$5\pi^2 a^3$; $6\pi^3 a^3$; $7\pi^2 a^3$]

Příklad 7. Najděte obsah pláště rotačního tělesa, které vznikne rotací křivky

- (a) $y = a \cosh \frac{x}{a}$, $-a \leq x \leq a$, kolem osy x , [$\pi a^2(\sinh 2 + 2)$]
- (b) $4x^2 + y^2 = 4$ kolem osy x ; kolem osy y , [$4\pi \left(2 + \frac{\ln(2+\sqrt{3})}{\sqrt{3}}\right)$; $2\pi \left(1 + \frac{4\pi}{3\sqrt{3}}\right)$]
- (c) $y = x\sqrt{\frac{x}{a}}$, $0 \leq x \leq a$, kolem osy x , [$\frac{4\pi a^2}{243} \left(21\sqrt{13} + 2 \ln \frac{3+\sqrt{13}}{2}\right)$]
- (d) $y = \operatorname{tg} x$, $0 \leq x \leq \frac{\pi}{4}$, kolem osy x , [$\pi \left[\sqrt{5} - \sqrt{2} + \ln \frac{(\sqrt{2}+1)(\sqrt{5}-1)}{2}\right]$]
- (e) $x^2 + (y - b)^2 = a^2$, $b \geq a$, kolem osy x , [$4\pi^2 ab$]
- (f) $x = a(t - \sin t)$, $y = a(1 - \cos t)$, $0 \leq t \leq 2\pi$, kolem osy x ; kolem osy y ; kolem přímky $y = 2a$, [$\frac{64}{3}\pi a^2$; $16\pi^2 a^2$; $\frac{32}{3}\pi a^2$]
- (g) $x = a \cos^3 t$, $y = a \sin^3 t$ kolem osy x ; kolem přímky $y = x$, [$\frac{12}{5}\pi a^2$; $\frac{3\pi}{5} a^2(4\sqrt{2} - 1)$]
- (h) $r = a(1 + \cos \varphi)$ kolem polární osy, [$\frac{32}{5}\pi a^2$]
- (i) $r^2 = a^2 \cos 2\varphi$ kolem polární osy; kolem osy $\varphi = \frac{\pi}{2}$; kolem osy $\varphi = \frac{\pi}{4}$. [$2\pi a^2(2 - \sqrt{2})$; $2\pi a^2\sqrt{2}$; $4\pi a^2$]