\( \mathcal{I} \)-ultrafilters
and summable ideals

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\(\mathcal{I}\)-ultrafilters

Definition A. (Baumgartner)

Let \(\mathcal{I}\) be a family of subsets of a set \(X\) such that \(\mathcal{I}\) contains all singletons and is closed under subsets. An ultrafilter \(\mathcal{U}\) on \(\omega\) is called an \(\mathcal{I}\)-ultrafilter if for every \(F: \omega \to X\) there exists \(A \in \mathcal{U}\) such that \(F[A] \in \mathcal{I}\).
**I-ultrafilters**

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- if $\mathcal{I} \subseteq \mathcal{J}$ then every $\mathcal{I}$-ultrafilter is a $\mathcal{J}$-ultrafilter
\(\mathcal{I}\)-ultrafilters

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- if \(\mathcal{I} \subseteq \mathcal{J}\) then every \(\mathcal{I}\)-ultrafilter is a \(\mathcal{J}\)-ultrafilter
- if \(\mathcal{U} \leq_{RK} \mathcal{V}\) and \(\mathcal{V}\) is an \(\mathcal{I}\)-ultrafilter then \(\mathcal{U}\) is also an \(\mathcal{I}\)-ultrafilter
I-ultrafilters

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Let $\mathcal{I}$ be a family of subsets of a set $X$ such that $\mathcal{I}$ contains all singletons and is closed under subsets. An ultrafilter $\mathcal{U}$ on $\omega$ is called an $\mathcal{I}$-ultrafilter if for every $F : \omega \to X$ there exists $A \in \mathcal{U}$ such that $F[A] \in \mathcal{I}$.

- if $\mathcal{I} \subseteq \mathcal{J}$ then every $\mathcal{I}$-ultrafilter is a $\mathcal{J}$-ultrafilter
- if $\mathcal{U} \leq_{RK} \mathcal{V}$ and $\mathcal{V}$ is an $\mathcal{I}$-ultrafilter then $\mathcal{U}$ is also an $\mathcal{I}$-ultrafilter
- $\mathcal{I}$-ultrafilters and $\langle \mathcal{I} \rangle$-ultrafilters coincide

where $\langle \mathcal{I} \rangle$ is the ideal generated by $\mathcal{I}$
Weak $\mathcal{I}$-ultrafilters

Definition B.
Let $\mathcal{I}$ be a family of subsets of a set $X$ such that $\mathcal{I}$ contains all singletons and is closed under subsets. An ultrafilter $\mathcal{U}$ on $\omega$ is called

**weak $\mathcal{I}$-ultrafilter** if for every *finite-to-one* function $F : \omega \to X$ there exists $A \in \mathcal{U}$ such that $F[A] \in \mathcal{I}$. 
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weak $\mathcal{I}$-ultrafilter if for every finite-to-one function $F : \omega \to X$ there exists $A \in \mathcal{U}$ such that $F[A] \in \mathcal{I}$.

$\mathcal{I}$-friendly ultrafilter if for every one-to-one function $F : \omega \to X$ there exists $A \in \mathcal{U}$ such that $F[A] \in \mathcal{I}$. 
Ideals on $\omega$
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Definition.

An ideal $\mathcal{I}$ on $\omega$ is tall (dense) if for every infinite set $A \subseteq \omega$ there exists infinite $B \subseteq A$ such that $B \in \mathcal{I}$.
Ideals on $\omega$

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An ideal $\mathcal{I}$ on $\omega$ is **tall** (dense) if for every infinite set $A \subseteq \omega$ there exists infinite $B \subseteq A$ such that $B \in \mathcal{I}$.

Lemma 1.
If $\mathcal{I}$ is not a tall ideal then there are no $\mathcal{I}$-ultrafilters.
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**Lemma 1.**
If $\mathcal{I}$ is not a tall ideal then there are no $\mathcal{I}$-ultrafilters.

**Proposition 2.**
$(p = c)$ If $\mathcal{I}$ is a tall ideal then $\mathcal{I}$-ultrafilters exist.
Summable ideals

Definition.

Given a function $g : \omega \to [0, \infty)$ such that $\sum_{n \in \omega} g(n) = \infty$ then the family $I_g = \{ A \subseteq \omega : \sum_{a \in A} g(a) < +\infty \}$

is a proper ideal which we call summable ideal determined by function $g$. 
Summable ideals

- Every summable ideal is a $P$-ideal.
- Every summable ideal is an $F_\sigma$ ideal.
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Observation.
A summable ideal is tall if and only if $\lim_{n \to \infty} g(n) = 0$. 
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We consider only tall summable ideals.
Rapid ultrafilters

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Theorem (Hrušák)
An ultrafilter $\mathcal{U} \in \omega^*$ is rapid if and only if $\mathcal{I}_g^* \not\leq_{KB} \mathcal{U}$ for every summable ideal $\mathcal{I}_g$. 
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An ultrafilter $\mathcal{U}$ on $\omega$ is called a rapid ultrafilter if the enumeration functions of its sets form a dominating family in $(\omega^\omega, \leq^*)$.

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An ultrafilter $\mathcal{U} \in \omega^*$ is rapid if and only if $\mathcal{I}_g^* \not\leq_{KB} \mathcal{U}$ for every summable ideal $\mathcal{I}_g$.

Theorem (Vojtáš)
An ultrafilter $\mathcal{U} \in \omega^*$ is rapid if and only if $\mathcal{U} \cap \mathcal{I}_g \neq \emptyset$ for every summable ideal $\mathcal{I}_g$. 
Rapid ultrafilters

Theorem 3.
For an ultrafilter $\mathcal{U} \in \omega^*$ the following are equivalent:

- $\mathcal{U}$ is rapid
- $\mathcal{U}$ is a weak $\mathcal{I}_g$-ultrafilter for every summable ideal $\mathcal{I}_g$
- $\mathcal{U}$ is an $\mathcal{I}_g$-friendly ultrafilter for every summable ideal $\mathcal{I}_g$
- $\mathcal{U} \cap \mathcal{I}_g \neq \emptyset$ for every summable ideal $\mathcal{I}_g$
Rapid ultrafilters

Theorem 4.

($\text{MA}_{\text{ctble}}$) There is an $\mathcal{I}_g$-ultrafilter which is not a rapid ultrafilter.
Rapid ultrafilters

Theorem 4.

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Theorem 5.

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Rapid ultrafilters

Theorem 4.
(MA$_{\text{ctble}}$) There is an $\mathcal{I}_g$-ultrafilter which is not a rapid ultrafilter.

Theorem 5.
(MA$_{\text{ctble}}$) There is a rapid ultrafilter which is not an $\mathcal{I}_g$-ultrafilter.

Theorem 5$^*$.
(MA$_{\text{ctble}}$) For every tall ideal $\mathcal{I}$ there is a $Q$-point which is not an $\mathcal{I}$-ultrafilter.
$\mathcal{I}_g$-ultrafilters

$\mathcal{U}$ is an $\mathcal{I}_g$-ultrafilter

$\Downarrow$

$\mathcal{U}$ is a weak $\mathcal{I}_g$-ultrafilter

$\Downarrow$

$\mathcal{U}$ is an $\mathcal{I}_g$-friendly ultrafilter

$\Downarrow$

$\mathcal{U} \cap \mathcal{I}_g \neq \emptyset$
$\mathcal{I}_g$-ultrafilters

$\mathcal{U}$ is an $\mathcal{I}_g$-ultrafilter

$\downarrow$

$\mathcal{U}$ is a weak $\mathcal{I}_g$-ultrafilter

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$\mathsf{ZFC} \downarrow \uparrow \mathsf{ZFC}$

$\mathcal{U} \cap \mathcal{I}_g \neq \emptyset$
$\mathcal{I}_g$-ultrafilters

$\mathcal{U}$ is an $\mathcal{I}_g$-ultrafilter

\[
\text{ZFC} \Downarrow \Updownarrow \text{MA}_{\text{ctble}}
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$\mathcal{U}$ is a weak $\mathcal{I}_g$-ultrafilter

\[
\Downarrow
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\( \mathcal{I}_g \)-ultrafilters

\( \mathcal{U} \) is an \( \mathcal{I}_g \)-ultrafilter

\[ \text{ZFC} \downarrow \uparrow \text{MA_{ctble}} \]

\( \mathcal{U} \) is a weak \( \mathcal{I}_g \)-ultrafilter

\[ \text{ZFC} \downarrow \uparrow ??? \]

\( \mathcal{U} \) is an \( \mathcal{I}_g \)-friendly ultrafilter

\[ \text{ZFC} \downarrow \uparrow \text{ZFC} \]

\( \mathcal{U} \cap \mathcal{I}_g \neq \emptyset \)
$\mathcal{I}_g$-ultrafilters

$\mathcal{U}$ is an $\mathcal{I}_g$-ultrafilter
  $\Downarrow$

$\mathcal{U}$ is a weak $\mathcal{I}_g$-ultrafilter
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$\mathcal{U} \cap \mathcal{I}_g \neq \emptyset$
$\mathcal{I}_g$-ultrafilters

$\mathcal{U}$ is an $\mathcal{I}_g$-ultrafilter

$\implies$

$\mathcal{U}$ is a weak $\mathcal{I}_g$-ultrafilter

$\implies$

$\mathcal{U}$ is an $\mathcal{I}_g$-friendly ultrafilter

$\implies$

$\mathcal{U} \cap \mathcal{I}_g \neq \emptyset$

Proposition 2.

$(p = c)$ For every tall ideal $\mathcal{I}$ there is an $\mathcal{I}$-ultrafilter.
\[ \mathcal{I}_g \text{-ultrafilters} \]

\[ \mathcal{U} \text{ is an } \mathcal{I}_g \text{-ultrafilter} \]
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\[ \Downarrow \]
\[ \mathcal{U} \cap \mathcal{I}_g \neq \emptyset \]

**Theorem 6.**

(MA\text{ctble}) For every summable ideal \( \mathcal{I}_g \) there is an \( \mathcal{I}_g \)-ultrafilter.
$\mathcal{I}_g$-ultrafilters

$\mathcal{U}$ is an $\mathcal{I}_g$-ultrafilter

$\Downarrow$

$\mathcal{U}$ is a weak $\mathcal{I}_g$-ultrafilter

$\Downarrow$

$\mathcal{U}$ is an $\mathcal{I}_g$-friendly ultrafilter

$\Downarrow$

$\mathcal{U} \cap \mathcal{I}_g \neq \emptyset$

**Observation**

For every summable ideal $\mathcal{I}_g$ there is an ultrafilter $\mathcal{U} \in \omega^*$ such that $\mathcal{U} \cap \mathcal{I}_g \neq \emptyset$. 
$\mathcal{I}_g$-ultrafilters

$\mathcal{U}$ is an $\mathcal{I}_g$-ultrafilter

$\Rightarrow$

$\mathcal{U}$ is a weak $\mathcal{I}_g$-ultrafilter

$\Rightarrow$

$\mathcal{U}$ is an $\mathcal{I}_g$-friendly ultrafilter

$\Rightarrow$

$\mathcal{U} \cap \mathcal{I}_g \neq \emptyset$
$\mathcal{I}_g$-ultrafilters

$\mathcal{U}$ is an $\mathcal{I}_g$-ultrafilter
\[\Downarrow\]
$\mathcal{U}$ is a weak $\mathcal{I}_g$-ultrafilter
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$\mathcal{U}$ is an $\mathcal{I}_g$-friendly ultrafilter
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$\mathcal{U} \cap \mathcal{I}_g \neq \emptyset$

At least for some summable ideals $\mathcal{I}_g$-friendly ultrafilters exist in $\text{ZFC}$. 
Some results

Theorem 7.

$\mathcal{I}_{1/n}$-friendly ultrafilters exist in ZFC.
Some results

Theorem 7.
$I_{1/n}$-friendly ultrafilters exist in ZFC.

Corollary 8.
If $I_{1/n} \subseteq I_g$ then $I_g$-friendly ultrafilters exist in ZFC.

Examples: $g(n) = \frac{1}{n \ln n}$
Some results

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$I_{1/n}$-friendly ultrafilters exist in ZFC.

Corollary 8.
If $I_{1/n} \subseteq I_g$ then $I_g$-friendly ultrafilters exist in ZFC.

Examples: $g(n) = \frac{1}{n \ln n}$

Theorem 9.
If $g(n) = \frac{\ln p n}{n}$, $p \in \omega$, then $I_g$-friendly ultrafilters exist in ZFC.
Construction

Theorem 7.

$I_{1/n}$-friendly ultrafilters exist in ZFC.
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Theorem 7.
\(\mathcal{I}_{1/n}\)-friendly ultrafilters exist in ZFC.

Definition.
A family \(\mathcal{F} \subseteq \mathcal{P}(\omega)\) is called

- a **\(k\)-linked family** if \(F_1 \cap \ldots \cap F_k\) is infinite whenever \(F_i \in \mathcal{F}, i \leq k\).

- a **centered system** if \(\mathcal{F}\) is \(k\)-linked for every \(k\) i.e., if any finite subfamily of \(\mathcal{F}\) has an infinite intersection.
Construction

We say that $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is a summable family if for every one-to-one function $f : \omega \to \mathbb{N}$ there is $A \in \mathcal{F}$ such that $f[A] \in \mathcal{I}_{1/n}$. 
Construction

We say that $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is a **summable family** if for every one-to-one function $f : \omega \to \mathbb{N}$ there is $A \in \mathcal{F}$ such that $f[A] \in \mathcal{I}_{1/n}$.

**Proposition 10.**
For every $k \in \mathbb{N}$ there exists a summable $k$-linked family $\mathcal{F}_k \subseteq \mathcal{P}(\omega)$. 
Lemma 11.

If $\mathcal{F}_k \subseteq \mathcal{P}(\omega)$ is a $k$-linked family then

$\mathcal{F} = \{ F \subseteq \omega : (\forall k)(\exists U^k \in \mathcal{F}_k) U^k \subseteq^* F \}$

is a centered system.
Construction

Lemma 11.
If $\mathcal{F}_k \subseteq \mathcal{P}(\omega)$ is a $k$-linked family then
$\mathcal{F} = \{ F \subseteq \omega : (\forall k)(\exists U^k \in \mathcal{F}_k) U^k \subseteq^* F \}$
is a centered system.

If every $\mathcal{F}_k$ is summable then $\mathcal{F}$ is summable.
Construction

Lemma 11.
If \( \mathcal{F}_k \subseteq \mathcal{P}(\omega) \) is a \( k \)-linked family then
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\mathcal{F} = \{ F \subseteq \omega : (\forall k)(\exists U^k \in \mathcal{F}_k)\ U^k \subseteq^* F \}
\]
is a centered system.

If every \( \mathcal{F}_k \) is summable then \( \mathcal{F} \) is summable.

More generally, if \( \mathcal{I} \) is a \( P \)-ideal and for every one-to-one function \( f \in \omega \mathbb{N} \) and for every \( k \in \mathbb{N} \) there exists \( U^k \in \mathcal{F}_k \) such that \( f[U^k] \in \mathcal{I} \) then there exists \( U \in \mathcal{F} \) such that \( f[U] \in \mathcal{I} \).
Some questions

Question.
Do $\mathcal{I}_g$-friendly ultrafilters exist in ZFC for every summable ideal $\mathcal{I}_g$? What about $g(n) = \frac{1}{\sqrt{n}}$ or $\frac{1}{\ln n}$?
Some questions

Question.
Do $\mathcal{I}_g$-friendly ultrafilters exist in ZFC for every summable ideal $\mathcal{I}_g$? What about $g(n) = \frac{1}{\sqrt{n}}$ or $\frac{1}{\ln n}$?

Question.
Is there an $\mathcal{I}_g$-friendly ultrafilter which is not an $\mathcal{I}_h$-friendly ultrafilter whenever $\mathcal{I}_g \not\subseteq \mathcal{I}_h$?
References

