Thin Ultrafilters

Jana Flašková

Abstract. Two ideals on \( \omega \) are presented which determine the same class of \( \mathcal{I} \)-ultrafilters. It is proved that the existence of these ultrafilters is independent of ZFC and the relation between this class and other well-known classes of ultrafilters is shown.

Let \( \mathcal{I} \) be a family of subsets of a set \( X \) such that \( \mathcal{I} \) contains all singletons and is closed under subsets. Given an ultrafilter \( \mathcal{U} \) on \( \omega \), we say that \( \mathcal{U} \) is an \( \mathcal{I} \)-ultrafilter if for any mapping \( F : \omega \to X \) there is \( A \in \mathcal{U} \) such that \( F(A) \in \mathcal{I} \).

Concrete examples of \( \mathcal{I} \)-ultrafilters are nowhere dense ultrafilters, measure zero ultrafilters or countably closed ultrafilters defined by taking \( X = 2^\omega \) and \( \mathcal{I} \) to contain all the nowhere dense sets, the sets with closure of measure zero, or the sets with countable closure, respectively. The class of \( \alpha \)-ultrafilters was defined for an indecomposable countable ordinal \( \alpha \) by taking \( X = \omega_1 \) and \( \mathcal{I} \) to consist of the subsets of \( \omega_1 \) with order type less than \( \alpha \).

Consistency results about existence of these ultrafilters and some inclusions among the appropriate classes of ultrafilters were obtained by Baumgartner [1], Brendle [4], Barney [2]. It was proved by Shelah [8] that consistently there are no nowhere dense ultrafilters, consequently all mentioned ultrafilters (except the \( \alpha \)-ultrafilters for which the question is still open) may not exist.
In this paper, we focus on free ultrafilters on \( \omega \) defined by taking \( X = \omega \) and \( \mathcal{I} \) to be two different collections of subsets of natural numbers.

**I. Basic facts and definitions.**

It was noticed in [1] that for a given family \( \mathcal{I} \) the \( \mathcal{I} \)-ultrafilters are closed downward under the Rudin-Keisler ordering \( \leq_{RK} \) (recall that \( \mathcal{U} \leq_{RK} \mathcal{V} \) if there is a function \( f : \omega \to \omega \) whose Stone extension \( \beta f : \beta \omega \to \beta \omega \) maps \( \mathcal{V} \) on \( \mathcal{U} \), see [5]).

Replacing \( f[U] \in \mathcal{I} \) by \( f[U] \in \langle \mathcal{I} \rangle \) in the definition of \( \mathcal{I} \)-ultrafilter, where \( \langle \mathcal{I} \rangle \) is the ideal generated by \( \mathcal{I} \), we get the same concept (see [2]), i.e. \( \mathcal{I} \)-ultrafilters and \( \langle \mathcal{I} \rangle \)-ultrafilters coincide. Obviously, if \( \mathcal{U} \) is an \( \mathcal{I} \)-ultrafilter then \( \mathcal{U} \cap \mathcal{I} \neq \emptyset \) (the converse is not true) and if \( \mathcal{I} \subseteq \mathcal{J} \) then every \( \mathcal{I} \)-ultrafilter is a \( \mathcal{J} \)-ultrafilter.

**Lemma.** If \( C \) is a class of ultrafilters closed downward under \( \leq_{RK} \) and \( \mathcal{I} \) an ideal on \( \omega \) then the following are equivalent:

(i) There exists \( \mathcal{U} \in C \) which is not an \( \mathcal{I} \)-ultrafilter

(ii) There exists \( \mathcal{V} \in C \) which extends \( \mathcal{I}^* \), the dual filter of \( \mathcal{I} \)

**Proof.** No ultrafilter extending \( \mathcal{I}^* \) is an \( \mathcal{I} \)-ultrafilter, so (ii) implies (i) trivially. To prove (i) implies (ii) assume that \( \mathcal{U} \in C \) is not an \( \mathcal{I} \)-ultrafilter. Hence there is a function \( f \in \omega^\omega \) such that \( (\forall A \in \mathcal{I}) f^{-1}[A] \notin \mathcal{U} \). Let \( \mathcal{V} = \{ V \subseteq \omega : f^{-1}[V] \in \mathcal{U} \} \). Obviously \( \mathcal{V} \) extends \( \mathcal{I}^* \) and \( \mathcal{V} \leq_{RK} \mathcal{U} \). Since \( C \) is closed downward under \( \leq_{RK} \) and \( \mathcal{U} \in C \) we get \( \mathcal{V} \in C \). \( \square \)

An infinite set \( A \subseteq \omega \) with enumeration \( A = \{ a_n : n \in \omega \} \) is called *almost thin* if \( \limsup_n \frac{a_n}{a_{n+1}} < 1 \) and *thin* (see [3]) if \( \lim_n \frac{a_n}{a_{n+1}} = 0 \). (Notice that by enumeration of a set of natural numbers we always mean an order preserving enumeration.)

We will denote the ideal generated by finite and thin sets by \( \mathcal{I} \) and the ideal generated by finite and almost thin sets by \( \mathcal{A} \). The corresponding \( \mathcal{I} \)-ultrafilters will be called *thin ultrafilters* and *almost thin ultrafilters* respectively. We prove in Section I. that in fact these two classes of ultrafilters coincide, although the ideals \( \mathcal{I} \) and \( \mathcal{A} \) differ as the set \( \{ 2^n : n \in \omega \} \in \mathcal{A} \setminus \mathcal{I} \).
We show in Section II. the relation between thin ultrafilters and selective ultrafilters and the relation between thin ultrafilters and $Q$-points in Section III. Let us recall the definitions:

A free ultrafilter $U$ is called a selective ultrafilter if for all partitions of $\omega$, $\{R_i : i \in \omega\}$, either for some $i$, $R_i \in U$, or $\exists U \in \mathcal{U}$ such that $(\forall i \in \omega) |U \cap R_i| \leq 1$.

A free ultrafilter $U$ is called a $Q$-point if for all partitions of $\omega$ consisting of finite sets, $\{Q_i : i \in \omega\}$, $\exists U \in \mathcal{U}$ such that $(\forall i \in \omega) |U \cap Q_i| \leq 1$.

II. Thin ultrafilters and almost thin ultrafilters.

The existence of thin ultrafilters is independent of ZFC. It is easy to construct a thin ultrafilter if we assume the Continuum Hypothesis (in Section II. we prove that the strictly weaker assumption (see [6]) that selective ultrafilters exist is sufficient) and we prove in Section III. that every thin ultrafilter is a $Q$-point whose existence is not provable in ZFC (see [7]).

Proposition 1. (CH) There is a thin ultrafilter.

Proof. Enumerate $\omega = \{f_\alpha : \alpha < \omega_1\}$. By transfinite induction on $\alpha < \omega_1$ we construct countable filter bases $\mathcal{F}_\alpha$ satisfying

(i) $\mathcal{F}_0$ is the Fréchet filter
(ii) $\mathcal{F}_\alpha \subseteq \mathcal{F}_\beta$ whenever $\alpha \leq \beta$
(iii) $\mathcal{F}_\gamma = \bigcup_{\alpha < \gamma} \mathcal{F}_\alpha$ for $\gamma$ limit
(iv) $(\forall \alpha) (\exists F \in \mathcal{F}_{\alpha+1}) f_\alpha[F] \in \mathcal{F}$

Suppose we have constructed $\mathcal{F}_\alpha$. If there exists a set $F \in \mathcal{F}_\alpha$ such that $f_\alpha[F] \in \mathcal{F}$ then put $\mathcal{F}_{\alpha+1} = \mathcal{F}_\alpha$. If $f_\alpha[F] \notin \mathcal{F}$ (in particular, $f_\alpha[F]$ is infinite) for every $F \in \mathcal{F}_\alpha$ then enumerate $\mathcal{F}_\alpha = \{F_n : n \in \omega\}$ and construct by induction a set $U = \{u_n : n \in \omega\}$ which we extend the filter base by:

Choose arbitrary $u_0 \in F_0$ such that $f_\alpha(u_0) > 0$ (such an element exists since $f_\alpha[F_0]$ is infinite). If $u_0, u_1, \ldots, u_{k-1}$ are already known we can choose $u_k \in \bigcap_{i \leq k} F_i$ so that $f_\alpha(u_k) > k \cdot f_\alpha(u_{k-1})$.

It is obvious that $U \subseteq F_n$, i.e. all but finitely many elements of $U$ are contained in $F_n$, for all $n \in \omega$. We can check immediately that
$f_\alpha[U]$ is thin:

$$\lim_{n \to \infty} \frac{f_\alpha(u_n)}{f_\alpha(u_{n+1})} \leq \lim_{n \to \infty} \frac{f_\alpha(u_n)}{n+1} = \lim_{n \to \infty} \frac{1}{n+1} = 0$$

To complete the induction step let $F_{\alpha+1}$ be the countable filter base generated by $F_\alpha$ and the set $U$.

It is clear that any ultrafilter extending $\bigcup_{\alpha<\omega_1} F_\alpha$ is thin. □

Since $\mathcal{I} \subset \mathcal{A}$ every thin ultrafilter has to be almost thin. The following proposition states that the converse also holds true and the classes of almost thin and thin ultrafilters coincide.

**Proposition 2.** Every almost thin ultrafilter is a thin ultrafilter.

**Proof.** Because of the Lemma it suffices to prove that every almost thin ultrafilter contains a thin set. To that end assume that $\mathcal{U}$ is an almost thin ultrafilter and $U_0 \in \mathcal{U}$ is an almost thin set which is not thin with enumeration $U_0 = \{u_n : n \in \omega\}$. Denote $\limsup_n \frac{u_n}{u_{n+1}} = q_0 < 1$. We may assume that the set of even numbers belongs to $\mathcal{U}$ (otherwise the role of even and odd numbers interchange).

Define $g : \omega \to \omega$ so that $g(u_n) = 2n$, $g[\omega \setminus U_0] = \{2n+1 : n \in \omega\}$.

Since $\mathcal{U}$ is an almost thin ultrafilter there exists $U_1 \in \mathcal{U}$ such that $g[U_1]$ is almost thin. Let $U = U_0 \cap U_1 = \{u_{nk} : k \in \omega\}$. Almost thin sets are closed under subsets, therefore $g[U] = \{g(u_{nk}) : k \in \omega\} \subseteq g[U_1]$ is almost thin and $1 > \limsup_k \frac{g(u_{nk})}{g(u_{nk+1})} = \limsup_k \frac{2n_k}{2n_{k+1}}$.

We know that there is $n_0$ such that $(\forall n \geq n_0) \frac{u_n}{u_{n+1}} \leq \frac{q_0+1}{2}$ and that there is $k_0$ such that $(\forall k \geq k_0) n_k \geq n_0$. Hence for $k \geq k_0$ we have

$$\frac{u_{nk}}{u_{nk+1}} \cdots \frac{u_n}{u_{n+1}} \leq \left(\frac{q_0+1}{2}\right)^{n_{k+1}-n_k}$$

It follows from $\limsup_k \frac{n_k}{n_{k+1}} < 1$ that $\lim_k (n_{k+1} - n_k) = +\infty$. Hence
\[
\lim_{k \to \infty} \frac{u_{nk}}{u_{nk+1}} \leq \lim_{k \to \infty} \left( \frac{q_0 + 1}{2} \right)^{n_{k+1} - n_k} = 0
\]
and the set \( U \in \mathcal{U} \) is thin. \( \square \)

**III. Thin ultrafilters and selective ultrafilters.**

**Proposition 3.** Every selective ultrafilter is thin.

**Proof.** Selective ultrafilters are minimal points in Rudin-Keisler ordering (see [5]), hence the class is downward closed under \( \leq_{RK} \) and we may apply the Lemma. It suffices to prove that there is no selective ultrafilter extending the dual filter of \( \mathcal{T} \).

Claim: Every selective ultrafilter contains a thin set.

Assume \( U \) is a selective ultrafilter and consider the partition of \( \omega \), \( \{R_n : n \in \omega \} \), where \( R_0 = \{0\} \) and \( R_n = [n!, (n + 1)!) \) for \( n > 0 \). Since \( U \) is selective there exists \( U_0 \in \mathcal{U} \) such that \( |U_0 \cap R_n| \leq 1 \) for every \( n \in \omega \). Since \( U \) is an ultrafilter either \( A_0 = \bigcup \{R_n : n \text{ is even}\} \) or \( A_1 = \bigcup \{R_n : n \text{ is odd}\} \) belongs to \( U \). Without loss of generality, assume \( A_0 \in \mathcal{U} \). Enumerate \( U = U_0 \cap A_0 \in \mathcal{U} \) as \( \{u_k : k \in \omega \} \).

If \( u_k \in [(2m_k)!, (2m_k + 1)!] \) then \( u_{k+1} \geq (2m_k + 2)! \) and we have
\[
\frac{u_k}{u_{k+1}} = \frac{(2m_k + 1)!}{(2m_k + 2)!} = \frac{1}{2m_k + 2} \leq \frac{1}{2k + 2}. \quad \text{Hence } U \text{ is thin.} \quad \square
\]

**Proposition 4.** (CH) Not every thin ultrafilter is selective.

**Proof.** Fix a partition \( \{R_n : n \in \omega \} \) of \( \omega \) into infinite sets and enumerate \( \omega \omega = \{f_\alpha : \alpha < \omega_1\} \). By transfinite induction on \( \alpha < \omega_1 \) we will construct countable filter bases \( \mathcal{F}_\alpha \) satisfying

(i) \( \mathcal{F}_0 \) is the countable filter base generated by Fréchet filter and \( \{\omega \setminus R_n : n \in \omega \} \)

(ii) \( \mathcal{F}_\alpha \subseteq \mathcal{F}_\beta \) whenever \( \alpha \leq \beta \)

(iii) \( \mathcal{F}_\gamma = \bigcup_{\alpha < \gamma} \mathcal{F}_\alpha \) for \( \gamma \) limit

(iv) (\( \forall \alpha \)) (\( \forall F \in \mathcal{F}_\alpha \)) \( \{n : |F \cap R_n| = \omega \} \) is infinite

(v) (\( \forall \alpha \)) (\( \exists F \in \mathcal{F}_{\alpha+1} \)) \( f_\alpha[F] \in \mathcal{T} \)

Suppose we know already \( \mathcal{F}_\alpha \). If there is a set \( F \in \mathcal{F}_\alpha \) such that \( f_\alpha[F] \in \mathcal{T} \) then put \( \mathcal{F}_{\alpha+1} = \mathcal{F}_\alpha \). If \( (\forall F \in \mathcal{F}_\alpha) f_\alpha[F] \notin \mathcal{T} \) then one of the following cases occurs.
Case A. $(\forall F \in \mathcal{F}_\alpha) \{ n : |f_\alpha[F \cap R_n]| = \omega \}$ is infinite

Fix an enumeration $\{F_m : m \in \omega \}$ of $\mathcal{F}_\alpha$. According to the assumption the set $I_m = \{ n : |f_\alpha[\bigcap_{i \leq m} F_i \cap R_n]| = \omega \}$ is infinite for all $m \in \omega$. Let us enumerate the set $\{ (m, n) : n \in I_m, m \in \omega \}$ as $\{ p_k : k \in \omega \}$ so that each ordered pair $(m, n)$ occurs infinitely many times. By induction we will construct a set $U = \{ u_k : k \in \omega \}$ which we may add to $\mathcal{F}_\alpha$.

Consider $p_0 = (m, n)$. Since $|f_\alpha[\bigcap_{i \leq m} F_i \cap R_n]| = \omega$ we can choose $u_0 \in \bigcap_{i \leq m} F_i \cap R_n$ such that $f_\alpha(u_0) > 0$. Suppose we know $u_0, \ldots, u_{k-1}$ such that $u_{i+1} > u_i$, $u_i \in \bigcap_{j \leq m} F_j \cap R_n$ where $p_i = (m, n)$ and $f_\alpha(u_{i+1}) > (i + 1) \cdot f_\alpha(u_i)$ for $i < k - 1$. Consider $p_k = (m, n)$. Since $\bigcap_{i \leq m} F_i \cap R_n$ and its image under $f_\alpha$ is infinite we may choose $u_k \in \bigcap_{i \leq m} F_i \cap R_n$ so that $u_k > u_{k-1}$ and $f_\alpha(u_k) > k \cdot f_\alpha(u_{k-1})$.

It remains to verify that $f_\alpha[U]$ is thin and $\mathcal{F}_\alpha \cup \{ U \}$ generates a filter base satisfying (iv).

- $f_\alpha[U]$ is thin:

\[
\lim_{k \to \infty} \frac{f_\alpha(u_k)}{f_\alpha(u_{k+1})} \leq \lim_{k \to \infty} \frac{f_\alpha(u_k)}{k \cdot f_\alpha(u_k)} = \lim_{k \to \infty} \frac{1}{(k + 1)} = 0
\]

- $(\forall F \in \mathcal{F}_\alpha) \{ n : |U \cap F \cap R_n| = \omega \}$ is infinite:

For every $F \in \mathcal{F}_\alpha$ there is $m_F \in \omega$ such that $U \cap F \cap R_n \supseteq U \cap \bigcap_{i \leq m_F} F_i \cap R_n$ which is infinite whenever $n \in I_{m_F}$ since $U \cap \bigcap_{i \leq m_F} F_i \cap R_n \supseteq \{ u_k : p_k = (m_F, n) \}$.

To complete the induction step let $\mathcal{F}_{\alpha+1}$ be the countable filter base generated by $\mathcal{F}_\alpha$ and $U$.

Case B. $(\exists F_0 \in \mathcal{F}_\alpha) \{ n : |f_\alpha[F_0 \cap R_n]| = \omega \}$ is finite

Enumerate $\mathcal{F}_\alpha \setminus \{ F_0 \} = \{ F_m : m > 0 \}$. Since $F_0$ satisfies (iv) there is $n_0 \in \omega$ such that $|f_\alpha[F_0 \cap R_{n_0}]| < \omega$ and $|F_0 \cap R_{n_0}| = \omega$.

It follows that there is $z_0 \in \omega$ such that $f_\alpha^{-1}([z_0]) \cap F_0 \cap R_{n_0}$ is infinite. The set $\bigcap_{i \leq m} F_i$ satisfies (iv) for any $m \in \omega$ and for all but finitely many $n$ the set $f_\alpha[\bigcap_{i \leq m} F_i \cap R_n]$ is finite. So we may choose $n_m > n_{m-1}$ such that $f_\alpha[\bigcap_{i \leq m} F_i \cap R_{n_m}]$ is finite and $\bigcap_{i \leq m} F_i \cap R_{n_m}$ infinite. We find $z_m$ such that $f_\alpha^{-1}([z_m]) \cap \bigcap_{i \leq m} F_i \cap R_{n_m}$ is infinite.
Consider the sequence ⟨zₘ : m ∈ ω⟩. We can find a subsequence ⟨zₘₖ : j ∈ ω⟩ which is either constant or satisfies zₘₖ₊₁ > (j+1)·zₘₖ for every j ∈ ω. Set U = ⋃ₖ f⁻¹[Rₖ]. It is obvious that f[α][U] ∈ F and (∀F ∈ F′) {n : |U ∩ F ∩ Rₙ| = ω} is infinite.

To complete the induction step let Fα₊₁ be the countable filter base generated by Fα and U.

The filter base F = ⋃α<ω₁ Fα satisfies (iv) and every ultrafilter which extends F is thin because of condition (v).

Claim: Every filter satisfying (iv) may be extended to an ultrafilter satisfying (iv).
Whenever F is a filter satisfying (iv) and A ⊆ ω then either for every F ∈ F exist infinitely many n ∈ ω such that |A ∩ F ∩ Rₙ| = ω, so the filter generated by F and A satisfies (iv) or there is F₀ ∈ F such that for all but finitely many n ∈ ω we have |A ∩ F₀ ∩ Rₙ| < ω. Then since for every F ∈ F exist infinitely many n ∈ ω for which |F ∩ F₀ ∩ Rₙ| = ω the filter generated by F and ω \ A satisfies (iv). Hence for every subset of ω we may extend F either by the set itself or its complement. Consequently, F may be extended to an ultrafilter satisfying (iv).

**IV. Thin ultrafilters and Q-points.**

**Proposition 5.** Every thin ultrafilter is a Q-point.

**Proof.** Let U be a thin ultrafilter and Q = {Qₙ : n ∈ ω} a partition of ω into finite sets. Enumerate Qₙ = {qᵢⁿ : i = 0,...,kₙ} (where kₙ = |Qₙ| − 1).

Define a one-to-one function f : ω → ω in the following way:

\[ f(q₀ⁿ) = 0, \quad f(q⁺ⁿ) = (n + 2) \cdot \max\{f(qᵢⁿ), kₙ₊₁\} \quad \text{for} \quad n ∈ ω \]

\[ f(qᵢⁿ) = f(q₀ⁿ) + i \quad \text{for} \quad i ≤ kₙ, n ∈ ω \]

Notice that f | Qₙ is strictly increasing for every n.

Since U is a thin ultrafilter there exists U₀ ∈ U such that f[U₀] = {vₘ : m ∈ ω} is a thin set. Hence there is m₀ ∈ ω such that \( \frac{vₘ}{vₘ₊₁} < \frac{1}{2} \)

for every m ≥ m₀.
Since \( f \) is one-to-one and \( \mathcal{Q} \) a partition of \( \omega \) we find \( K \subseteq \omega \) of size at most \( m_0 \) such that \( \{ f^{-1}(v_i) : i < m_0 \} \subseteq \bigcup_{n \in K} Q_n \). The latter set is finite. Therefore \( U = U_0 \setminus \bigcup_{n \in K} Q_n \in \mathcal{U} \).

Claim: \( (\forall n \in \omega) \ |U \cap Q_n| \leq 1 \)

The intersection is clearly empty for \( n \in K \). Assume for the contrary that for some \( n \notin K \) there are two distinct elements \( u_1, u_2 \in U \cap Q_n \), \( u_1 < u_2 \). Then \( f(u_1) = v_m \) for some \( m \geq m_0 \) and \( f(u_2) = v_n \) for some \( n \geq m + 1 \). We get \( \frac{v_m}{v_m+1} \geq \frac{v_m}{v_n} = \frac{f(u_1)}{f(u_2)} \geq \frac{f(q_n^{n-1})}{f(q_n^{n+2})+k_n} \geq \frac{(n+1)-M}{(n+1)-M+M} = \frac{n+1}{n+2} \) where \( M = \max\{f(q_{k_n-1}^{n-1}), k_n\} \). But \( \frac{n+1}{n+2} \geq \frac{1}{2} \), a contradiction. \( \square \)

Note that while proving that every selective ultrafilter contains a thin set we actually proved that every \( Q \)-point contains a thin set as the partition under consideration consists of finite sets. However, \( Q \)-points need not be thin ultrafilters.

**Proposition 6.** \( (CH) \) Not every \( Q \)-point is a thin ultrafilter.

**Proof.** Fix \( \{ R_\alpha : n \in \omega \} \) a partition of \( \omega \) into infinite sets. Let \( \{ \mathcal{Q}_\alpha : \alpha < \omega_1 \} \) be the list of all partitions of \( \omega \) into finite sets. By transfinite induction on \( \alpha < \omega_1 \) we will construct countable filter bases \( \mathcal{F}_\alpha \) so that the following are satisfied:

(i) \( \mathcal{F}_0 \) is the Fréchet filter

(ii) \( \mathcal{F}_\alpha \subseteq \mathcal{F}_{\beta} \) whenever \( \alpha < \beta \)

(iii) \( \mathcal{F}_\gamma = \bigcup_{\alpha < \gamma} \mathcal{F}_\alpha \) for \( \gamma \) limit

(iv) \( (\forall \alpha) \ (\forall F \in \mathcal{F}_\alpha) \ (\forall n) \ |F \cap R_n| = \omega \)

(v) \( (\forall \alpha) \ (\exists F \in \mathcal{F}_{\alpha+1}) \ (\forall Q \in \mathcal{Q}_\alpha) \ |F \cap Q| \leq 1 \)

Suppose we already know \( \mathcal{F}_\alpha \). If there is a set \( F \in \mathcal{F}_\alpha \) such that \( |F \cap Q| \leq 1 \) for each \( Q \in \mathcal{Q}_\alpha \) then let \( \mathcal{F}_{\alpha+1} = \mathcal{F}_\alpha \). If \( (\forall F \in \mathcal{F}_\alpha) \ (\exists Q \in \mathcal{Q}_\alpha) \ |F \cap Q| > 1 \), we construct by induction a set \( U \) compatible with \( \mathcal{F}_\alpha \) such that \( |U \cap Q| \leq 1 \) for each \( Q \in \mathcal{Q}_\alpha \).

Enumerate \( \mathcal{F}_\alpha = \{ F_k : k \in \omega \} \), \( \mathcal{Q}_\alpha = \{ Q_n : n \in \omega \} \) and list \( \{ R_n : n \in \omega \} = \{ M_k : k \in \omega \} \) so that each \( R_n \) is listed infinitely often. To start the construction of \( U \) choose \( u_0 \in M_0 \) arbitrarily. Since \( \bigcup \mathcal{Q}_\alpha = \omega \) there exists \( n_0 \in \omega \) such that \( u_0 \in Q_{n_0} \).
Suppose we already know \( u_0, \ldots, u_{k-1} \) and \( n_0, \ldots, n_{k-1} \) such that \( u_i \in Q_n \) for \( i < k \). Since \( \bigcup_{i < k} Q_n \) is finite and \( \bigcap_{i < k} F_i \cap M_k \) is infinite according to (iv) we may choose \( u_k \in (\bigcap_{i < k} F_i \cap M_k) \setminus \bigcup_{i < k} Q_n \). Let \( Q_{n_k} \) be the unique element of \( \mathcal{Q} \) which contains \( u_k \).

Finally, let \( U = \{ u_k : k \in \omega \} \). For every \( n \in \omega \) we have either \( U \cap Q_n = \{ u_k \} \) (if \( n = n_k \)) or \( U \cap Q_n = \emptyset \). It remains to check that \( U \) is compatible with \( \mathcal{F}_\alpha \) and satisfies (iv). However, for every \( F \in \mathcal{F}_\alpha \) and for every \( R_n \) there is \( n_F \in \omega \) such that \( U \cap F \cap R_n \supseteq U \cap \bigcap_{i < n_F} F_i \cap R_n \supseteq \{ u_k : k \geq n_F, M_k = R_n \} \). The latter set is infinite. Hence the countable filter base generated by \( \mathcal{F}_\alpha \) and \( U \) satisfies (ii), (iv), (v) as required and it may be taken as \( \mathcal{F}_{\alpha + 1} \).

Because of condition (v) every ultrafilter which extends the filter base \( \mathcal{F} = \bigcup_{\alpha < \omega_1} \mathcal{F}_\alpha \) is a \( \mathcal{Q} \)-point. Let \( \mathcal{G} = \{ \bigcup_{n \in M} R_n : \omega \setminus M \in \mathcal{F} \} \). Condition (iv) in induction assumption guarantees that \( \mathcal{F} \cup \mathcal{G} \) generates a free filter on \( \omega \). Every ultrafilter \( \mathcal{U} \) which extends \( \mathcal{F} \cup \mathcal{G} \) is a \( \mathcal{Q} \)-point but not a thin ultrafilter because for every \( U \in \mathcal{U} \) \( f[U] \notin \mathcal{F} \) where \( f : \omega \to \omega \) is defined by \( f \restriction R_n = n \). \( \square \)

**References**