Hindman spaces and summable ultrafilters

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Van der Waerden spaces

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- (van der Waerden theorem)
  Sets that are not AP-sets form an ideal
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**Definition A. (Kojman)**

A topological space \( X \) is called van der Waerden if for every sequence \( \langle x_n \rangle_{n \in \omega} \) in \( X \) there exists a converging subsequence \( \langle x_{n_k} \rangle_{k \in \omega} \) so that \( \{ n_k : k \in \omega \} \) is an AP-set.
Hindman spaces

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**Definition A. (Kojman)**

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A topological space \( X \) is called \textbf{Hindman} if for every sequence \( \langle x_n \rangle_{n \in \omega} \) in \( X \) there exists a converging subsequence \( \langle x_{n_k} \rangle_{k \in \omega} \) so that \( \{n_k : k \in \omega\} \) is an \textbf{IP-set}.
Hindman spaces

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A topological space X is called **Hindman** if for every sequence ⟨xₙ⟩ₙ∈ω in X there exists a converging subsequence ⟨xₙₖ⟩ₖ∈ω so that {nₖ : k ∈ ω} is an **IP-set**.

!!! only finite T₂ spaces fullfill the condition!!!
Hindman spaces

An **IP-sequence** in a topological space is a sequence indexed by $FS(D)$ for some infinite $D \subseteq \mathbb{N}$. 
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An IP-sequence $\langle x_n \rangle_{n \in FS(D)}$ in a topological space $X$ **IP-converges** to a point $x \in X$ if for every neighborhood $U$ of $x$ there exists $m \in \mathbb{N}$ such that $\{x_n : n \in FS(D \setminus m)\} \subseteq U$. 
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Definition B. (Kojman)

A topological space $X$ is called Hindman if for every sequence $\langle x_n \rangle_{n \in \omega}$ in $X$ there exists an infinite set $D \subseteq \mathbb{N}$ such that $\langle x_n \rangle_{n \in FS(D)}$ IP-converges to some $x \in X$. 
Known facts

Theorem (Kojman)

- There exists a sequentially compact space which is not van der Waerden.
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**Proof.** Consider the one-point compactification of $\Psi(\mathcal{A})$ for a suitable MAD family $\mathcal{A}$. 
**Ψ-spaces**

For a given maximal almost disjoint (MAD) family $A$ of infinite subsets of $\mathbb{N}$ we define the space $\Psi(A)$ as follows:
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- Every point $p_A$ has neighborhood base of all sets $\{p_A\} \cup A \setminus K$ where $K$ is a finite subset of $A$. 
\textbf{Ψ-spaces}

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Note: $\Psi(\mathcal{A})$ is regular, first countable and separable.
Known facts

Theorem (Kojman)
If a Hausdorff space $X$ satisfies the following condition

(*) The closure of every countable set in $X$ is compact and first-countable.

Then $X$ is both van der Waerden and Hindman.
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For example, compact metric spaces or every successor ordinal with the order topology satisfy (*).
Known facts + questions

Theorem (Kojman, Shelah)
(CH) There exists a van der Waerden space which is not Hindman.
**Known facts + questions**

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(CH) There exists a van der Waerden space which is not Hindman.

**Theorem (Jones)**
(MA$_{\sigma-\text{cent.}}$) There exists a van der Waerden space which is not Hindman.
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Theorem (Jones)
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(Jones) Is it consistent that there is a Hindman space which is not a van der Waerden space?
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Theorem (Kojman, Shelah)
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Theorem (Jones)
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Is it possible to strengthen the result of Jones?
\( \mathcal{I}_{1/n} \)-spaces

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\mathcal{I}_{1/n} = \{ A \subseteq \mathbb{N} : \sum_{a \in A} \frac{1}{a} < \infty \}
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$\mathcal{I}_{1/n}$-spaces

$$\mathcal{I}_{1/n} = \{ A \subseteq \mathbb{N} : \sum_{a \in A} \frac{1}{a} < \infty \}$$

- $\mathcal{I}_{1/n}$ is an $F_\sigma$-ideal like van der Waerden ideal.
$I_{1/n}$-spaces

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\[ \mathcal{I}_{1/n} = \{ A \subseteq \mathbb{N} : \sum_{a \in A} \frac{1}{a} < \infty \} \]

- \( \mathcal{I}_{1/n} \) is an \( F_\sigma \)-ideal like van der Waerden ideal.

**Definition C.**

A topological space \( X \) is called \( \mathcal{I}_{1/n} \)-space if for every sequence \( \langle x_n \rangle_{n \in \omega} \) in \( X \) there exists a converging subsequence \( \langle x_{n_k} \rangle_{k \in \omega} \) so that \( \{ n_k : k \in \omega \} \) does not belong to \( \mathcal{I}_{1/n} \).
Theorem 1.
If a Hausdorff space \( X \) satisfies the following condition
\((\ast)\) The closure of every countable set in \( X \) is compact and first-countable.

Then \( X \) is an \( I_{1/n} \)-space.
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If a Hausdorff space \( X \) satisfies the following condition

(\( \ast \)) The closure of every countable set in \( X \) is compact and first-countable.

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Theorem 2.
There exists a sequentially compact space which is not an \( I_{1/n} \)-space.
$\mathcal{I}_{1/n} \& \text{van der Waerden spaces}$

Erdős-Turán Conjecture.
Every set $A \notin \mathcal{I}_{1/n}$ is an AP-set.

If Erdős-Turán Conjecture is true then every $\mathcal{I}_{1/n}$-space is van der Waerden.
\( \mathcal{I}_{1/n} \) & van der Waerden spaces

Erdős-Turán Conjecture.
Every set \( A \notin \mathcal{I}_{1/n} \) is an AP-set.

If Erdős-Turán Conjecture is true then every \( \mathcal{I}_{1/n} \)-space is van der Waerden.

Theorem 3.
\((\text{MA}_{\sigma-\text{cent.}})\) There exists a van der Waerden space which is not an \( \mathcal{I}_{1/n} \)-space.
\[ \mathcal{I}_{1/n} \] & Hindman spaces

There is no inclusion between \( \mathcal{I}_{1/n} \) and Hindman ideal.
$I_{1/n}$ & Hindman spaces

There is no inclusion between $I_{1/n}$ and Hindman ideal.

Theorem 4.

$(\text{MA}_{\sigma-\text{cent.}})$ There exists an $I_{1/n}$-space which is not Hindman.
\( \mathcal{I}_{1/n} \) & Hindman spaces

There is no inclusion between \( \mathcal{I}_{1/n} \) and Hindman ideal.

**Theorem 4.**

(MA\(\sigma\)-cent.) There exists an \( \mathcal{I}_{1/n} \)-space which is not Hindman.

**Proposition**

(MA\(\sigma\)-cent.) There exists a MAD family \( A \) consisting of non-IP-sets so that for every \( B \subseteq \mathbb{N}, B \notin \mathcal{I}_{1/n} \) and every finite-to-one function \( f : B \to \mathbb{N} \) there exists \( C \subseteq B, C \notin \mathcal{I}_{1/n} \) and \( A \in \mathcal{A} \) so that \( f[C] \subseteq A \).
$\mathcal{I}_{1/n}$ & Hindman spaces

**Question**
Is it consistent that there is a Hindman space which is not an $\mathcal{I}_{1/n}$-space?
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$A \subseteq \mathbb{N}$ is an **ip-rich set** if $A$ contains all finite sums of elements of arbitrarily large finite sets.
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$A \subseteq \mathbb{N}$ is an **ip-rich set** if $A$ contains all finite sums of elements of arbitrarily large finite sets.

- Every IP-set is by definition ip-rich
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\( A \subseteq \mathbb{N} \) is an **ip-rich set** if \( A \) contains all finite sums of elements of arbitrarily large finite sets.

- Every IP-set is by definition ip-rich
- (Folkman-Rado-Sanders)
  Sets that are not ip-rich form an ideal
\( \mathcal{I}_{1/n} \) & Hindman spaces

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Is it consistent that there is a Hindman space which is not an \( \mathcal{I}_{1/n} \)-space?

\( A \subseteq \mathbb{N} \) is an **ip-rich set** if \( A \) contains all finite sums of elements of arbitrarily large finite sets.

- Every IP-set is by definition ip-rich
- (Folkman-Rado-Sanders) Sets that are not ip-rich form an ideal
- Ideal \( \mathcal{I}_{ipr} \) is an \( F_\sigma \)-ideal
\textbf{Definition D.}

A topological space \( X \) is called an \( \mathcal{I}_{ipr} \)-space if for every sequence \( \langle x_n \rangle_{n \in \omega} \) in \( X \) there exists a converging subsequence \( \langle x_{n_k} \rangle_{k \in \omega} \) so that \( \{n_k : k \in \omega\} \) is an ip-rich set.
\( \mathcal{I}_{1/n} \) & \( \mathcal{I}_{ipr} \)-spaces

**Definition D.**
A topological space \( X \) is called \( \mathcal{I}_{ipr} \)-space if for every sequence \( \langle x_n \rangle_{n \in \omega} \) in \( X \) there exists a converging subsequence \( \langle x_{n_k} \rangle_{k \in \omega} \) so that \( \{ n_k : k \in \omega \} \) is an ip-rich set.

**Theorem 7.**
(\( \text{MA}_{\sigma-\text{cent.}} \)) There exists an \( \mathcal{I}_{ipr} \)-space which is not an \( \mathcal{I}_{1/n} \)-space.
Weak $\mathcal{I}$-ultrafilters

Definition (Baumgartner)
Let $\mathcal{I}$ be a family of subsets of a set $X$ such that $\mathcal{I}$ contains all singletons and is closed under subsets. An ultrafilter $\mathcal{U}$ on $\mathbb{N}$ is called an $\mathcal{I}$-ultrafilter if for every $F: \mathbb{N} \to X$ there exists $A \in \mathcal{U}$ such that $F[A] \in \mathcal{I}$. 
Weak $\mathcal{I}$-ultrafilters

Definition E.
Let $\mathcal{I}$ be a family of subsets of a set $X$ such that $\mathcal{I}$ contains all singletons and is closed under subsets. An ultrafilter $\mathcal{U}$ on $\mathbb{N}$ is called an weak $\mathcal{I}$-ultrafilter if for every finite-to-one $F : \mathbb{N} \to X$ there exists $A \in \mathcal{U}$ such that $F[A] \in \mathcal{I}$. 
Weak $\mathcal{I}$-ultrafilters

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Definition (Hindman)
An ultrafilter $\mathcal{U}$ on $\mathbb{N}$ is called weakly summable if every $U \in \mathcal{U}$ is an IP-set.
\( \mathcal{I}_{1/n} \)-ultrafilters

Theorem 8.

(MA\text{ctble}) There exists an \( \mathcal{I}_{1/n} \)-ultrafilter \( \mathcal{U} \in \mathbb{N}^* \) such that every \( U \in \mathcal{U} \) is an ip-rich set.
$\mathcal{I}_{1/n}$-ultrafilters

Theorem 8.
$(\text{MA}_{\text{ctble}})$ There exists an $\mathcal{I}_{1/n}$-ultrafilter $\mathcal{U} \in \mathbb{N}^*$ such that every $U \in \mathcal{U}$ is an ip-rich set.

Theorem 9.
$(\text{MA}_{\text{ctble}})$ There exists a weak $\mathcal{I}_{1/n}$-ultrafilter $\mathcal{U} \in \mathbb{N}^*$ which is weakly summable ultrafilter.
**$\mathcal{I}_{1/n}$-ultrafilters**

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**Theorem 9.**
\((\text{MA}_{\text{ctble}})\) There exists a weak $\mathcal{I}_{1/n}$-ultrafilter $\mathcal{U} \in \mathbb{N}^*$ which is weakly summable ultrafilter.

**Question**
Is it consistent that there is a weakly summable $\mathcal{I}_{1/n}$-ultrafilter?
References


