Some special points in Čech-Stone compactification of natural numbers

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Filters

Definition.

For a non-empty set $X$, a filter on $X$ is a family $\mathcal{F} \subseteq \mathcal{P}(X)$ such that:

- $\mathcal{F} \neq \emptyset$ and $\emptyset \notin \mathcal{F}$
- if $F_1, F_2 \in \mathcal{F}$ then $F_1 \cap F_2 \in \mathcal{F}$
- if $F \in \mathcal{F}$ and $F \subseteq G \subseteq X$ then $G \in \mathcal{F}$. 
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If moreover $\mathcal{F}$ satisfies
- for every $M \subseteq X$ either $M \in \mathcal{F}$ or $X \setminus M \in \mathcal{F}$
then $\mathcal{F}$ is called an ultrafilter.
Ideals

Definition.
For a non-empty set $X$, an ideal on $X$ is a family $\mathcal{I} \subseteq \mathcal{P}(X)$ such that:

- $\mathcal{I} \neq \mathcal{P}(X)$ and $\emptyset \in \mathcal{I}$
- if $A_1, A_2 \in \mathcal{I}$ then $A_1 \cup A_2 \in \mathcal{I}$
- if $A \in \mathcal{I}$ and $B \subseteq A \subseteq X$ then $B \in \mathcal{I}$.
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Examples: 

$\mathcal{Z}_0 = \{ A \subseteq \mathbb{N} : \limsup_{n \to \infty} \frac{|A \cap n|}{n} = 0 \}$

$\mathcal{I}_{1/n} = \{ A \subseteq \mathbb{N} : \sum_{a \in A} \frac{1}{a} < \infty \}$
0-points

Definition A. (Gryzlov)

An ultrafilter $\mathcal{U}$ on $\omega$ is called a 0-point if for every one-to-one function $f : \omega \rightarrow \mathbb{N}$ there exists $U \in \mathcal{U}$ such that $f[U] \in \mathcal{Z}_0$. 
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• Every $Q$-point is a 0-point.
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Theorem (Gryzlov)
0-points exist in ZFC.
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Theorem (Gryzlov)
There are $2^c$ many distinct 0-points.
Topological consequences

Problem 235. (Hart, van Mill)
For what nowhere dense sets $A \subseteq \omega^*$ do we have $\bigcup_{\pi \in S_{\omega}} \pi[A] \neq \omega^*$?
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Some consistent answers:
- For all if $\pi > c$.
- Not for all if $P$-points exist.
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Examples in ZFC:

- singletons
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Examples in ZFC:
- Singletons
- (Gryzlov) $A = \{ \mathcal{U} \in \omega^* : \mathcal{Z}_0^* \subseteq \mathcal{U} \}$. 
Summable ultrafilters

Definition B.
An ultrafilter $\mathcal{U}$ on $\omega$ is called a **summable ultrafilter** if for every one-to-one function $f : \omega \rightarrow \mathbb{N}$ there exists $U \in \mathcal{U}$ such that $f[U] \in \mathcal{I}_{1/n}$.
Summable ultrafilters

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An ultrafilter $\mathcal{U}$ on $\omega$ is called a *summable ultrafilter* if for every one-to-one function $f : \omega \to \mathbb{N}$ there exists $U \in \mathcal{U}$ such that $f[U] \in \mathcal{I}_{1/n}$.

- Every summable ultrafilter is a 0-point.
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Definition B.

An ultrafilter \( \mathcal{U} \) on \( \omega \) is called a **summable ultrafilter** if for every one-to-one function \( f : \omega \to \mathbb{N} \) there exists \( U \in \mathcal{U} \) such that \( f[U] \in \mathcal{I}_{1/n} \).

- Every summable ultrafilter is a 0-point.
- Every \( Q \)-point is a summable ultrafilter.
Summable ultrafilters

Theorem 1.

(MA_{ctble}) There exists a $P$-point which is not a summable ultrafilter.
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Corollary 2.
It is consistent that there exists a 0-point which is not a summable ultrafilter.
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Corollary 2.

It is consistent that there exists a 0-point which is not a summable ultrafilter.

Question

Is there a 0-point which is not a summable ultrafilter in ZFC?
Summable ultrafilters

Theorem 3.
Summable ultrafilters exist in ZFC.
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Corollary 4.
The set \( A = \{ U \in \omega^* : \mathcal{I}_{1/n}^* \subseteq U \} \) is a nowhere dense subset of \( \omega^* \) such that \( \bigcup_{\pi \in S_\omega} \pi[A] \neq \omega^* \).
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The set \( A = \{ U \in \omega^* : \mathcal{I}_{1/n}^* \subseteq U \} \) is a nowhere dense subset of \( \omega^* \) such that \( \bigcup_{\pi \in S_\omega} \pi[A] \neq \omega^* \).

Proposition 5.
There exist \( 2^c \) many distinct summable ultrafilters.
Construction

Definition.
A family $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is called

- a $k$-linked family if $F_0 \cap F_1 \cap \ldots \cap F_k$ is infinite whenever $F_i \in \mathcal{F}$, $i \leq k$.

- a centered system if $\mathcal{F}$ is $k$-linked for every $k$ i.e., if any finite subfamily of $\mathcal{F}$ has an infinite intersection.
Construction

We say that $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is a \textbf{summable family} if for every one-to-one function $f : \omega \to \mathbb{N}$ there is $A \in \mathcal{F}$ such that $f[A] \in \mathcal{I}_{1/n}$. 
Construction

We say that $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is a **summable family** if for every one-to-one function $f : \omega \rightarrow \mathbb{N}$ there is $A \in \mathcal{F}$ such that $f[A] \in \mathcal{I}_{1/n}$.

**Proposition 6.**
For every $k \in \mathbb{N}$ there exists a summable $k$-linked family $\mathcal{F}_k \subseteq \mathcal{P}(\omega)$. 
Construction

Lemma 7.
If $F_k \subseteq \mathcal{P}(\omega)$ is a $k$-linked family then
\[
\mathcal{F} = \{ F \subseteq \omega : (\forall k)(\exists U^k \in \mathcal{F}_k) U^k \subseteq^* F \}
\]
is a centered system.

If every $\mathcal{F}_k$ is summable then $\mathcal{F}$ is summable.

More generally, if $\mathcal{I}$ is a $P$-ideal and for every one-to-one function $f \in \omega \mathbb{N}$ and for every $k \in \mathbb{N}$ there exists $U^k \in \mathcal{F}_k$ such that $f[U^k] \in \mathcal{I}$ then there exists $U \in \mathcal{F}$ such that $f[U] \in \mathcal{I}$.
Problems

\[ \mathcal{I}_g = \{ A \subseteq \mathbb{N} : \sum_{a \in A} g(a) < \infty \} \]

**Question**

Does there exist an ultrafilter \( \mathcal{U} \) on \( \omega \) such that for every one-to-one function there exists a set \( U \in \mathcal{U} \) such that \( f[U] \in \mathcal{I}_g \)?

In particular, for \( g(n) = \frac{1}{\sqrt{n}} \)?
Problems

\[ \mathcal{I}_g = \{ A \subseteq \mathbb{N} : \sum_{a \in A} g(a) < \infty \} \]

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Does there exist an ultrafilter \( \mathcal{U} \) on \( \omega \) such that for every one-to-one function there exists a set \( U \in \mathcal{U} \) such that \( f[U] \in \mathcal{I}_g \)?
In particular, for \( g(n) = \frac{1}{\sqrt{n}} \)?

Question
Does there exist an ultrafilter \( \mathcal{U} \) on \( \omega \) such that for every finite-to-one function there exists a set \( U \in \mathcal{U} \) such that \( f[U] \in \mathcal{I}_{1/n}(\mathbb{Z}_0) \)?
References


