Remarks about van der Waerden ideal

Jana Flašková\textsuperscript{1}

\textsuperscript{1}Department of Mathematics
University of West Bohemia in Pilsen

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An arithmetic progression of length $l$ is the finite sequence \( \{a + id : i = 0, 1, \ldots, l - 1\} \) where $a, d \in \mathbb{N}$.

Van der Waerden Theorem (finite version).
For any given natural numbers $k$ and $l$, there is some natural number $W(k, l)$ such that if the integers \( \{1, 2, \ldots, W(k, l)\} \) are colored, each with one of $k$ different colors, then there exists an arithmetic progression of length at least $l$, all of which elements are of the same color.
Van der Waerden theorem and AP-sets

Definition.
A set $A \subseteq \mathbb{N}$ is called an AP-set if it contains arbitrary long arithmetic progressions.

Van der Waerden Theorem (infinite version).
If an AP-set is partitioned into finitely many pieces then at least one of them is again an AP-set.

Sets which are not AP-sets form a proper ideal on $\mathbb{N}$ — van der Waerden ideal denoted by $\mathcal{W}$
Van der Waerden ideal and other ideals

Szemerédi Theorem.

$$\mathcal{W} \subseteq \mathcal{Z} \quad \text{where} \quad \mathcal{Z} = \{ A \subseteq \mathbb{N} : \limsup_{n \to \infty} \frac{|A \cap n|}{n} = 0 \}$$

Erdős Conjecture.

$$\mathcal{W} \subseteq \mathcal{I}_{1/n} \quad \text{where} \quad \mathcal{I}_{1/n} = \{ A \subseteq \mathbb{N} : \sum_{a \in A} \frac{1}{a} < \infty \}$$
What sets belong to $\mathcal{W}$?

Example A. $\{n! : n \in \omega\}$ or $\{2^n : n \in \omega\}$ do not contain arithmetic progressions of length 3.
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Example B. $\{n^2 : n \in \omega\}$ contains infinitely many arithmetic progressions of length 3 (known by Pythagoras), but no arithmetic progression of length 4 (proved by Euler).
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Example B. $\{n^2 : n \in \omega\}$ contains infinitely many arithmetic progressions of length 3 (known by Pythagoras), but no arithmetic progression of length 4 (proved by Euler).

Example C. The set of the prime numbers does not belong to the van der Waerden ideal (Green-Tao).
Van der Waerden ideal $\mathcal{W}$

The van der Waerden ideal $\mathcal{W}$ is

- **a tall ideal** — because every infinite $A \subseteq \mathbb{N}$ contains an infinite subset with no arithmetic progressions of length 3

- **not a $P$-ideal** — consider for example the sets

$$A_k = \{2^n + k : n \in \omega\} \text{ for } k \in \omega$$
Van der Waerden ideal $\mathcal{W}$

The van der Waerden ideal $\mathcal{W}$ is

- $F_\sigma$-ideal — because $\mathcal{W} = \bigcup_{n \in \mathbb{N}} \mathcal{W}_n$ where

  $\mathcal{W}_n = \{ A \subseteq \mathbb{N} : A \text{ contains no a. p. of length } n \}$
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The family $\mathcal{W}_n$

- is not an ideal for every $n \in \mathbb{N}$
- generates a proper ideal $\langle \mathcal{W}_n \rangle$
Strictly increasing sequence of ideals

The ideal $\langle \mathcal{W}_n \rangle$ is a tall $F_\sigma$-ideal for every $n \geq 3$.

Fact.

$$\mathcal{W} = \bigcup_{n \geq 3} \langle \mathcal{W}_n \rangle$$

and $\langle \mathcal{W}_n \rangle \subseteq \langle \mathcal{W}_{n+1} \rangle$ for every $n \in \mathbb{N}$. 
Proposition 1.

For every $n \geq 3$ there exists $A \subset \mathbb{N}$ such that

$$A \in \mathcal{W}_{n+1} \setminus \langle \mathcal{W}_n \rangle$$
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$$A \in \mathcal{W}_{n+1} \setminus \langle \mathcal{W}_n \rangle$$

Proof. Consider

$$A = \left\{ \sum_{i=0}^{k} c_i \cdot n^{2i} : k \in \omega, c_i = 0, \ldots, n - 1, c_k \neq 0 \right\}$$
Strictly increasing sequence of ideals

Claim 1. Show $A \in \mathcal{W}_{n+1}$ (straightforward calculation)

Claim 2. Show $A \notin \langle \mathcal{W}_n \rangle$ (use Hales-Jewett theorem)
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Let $L(n) \ldots$ be the set of finite words in the alphabet $\{0, 1, \ldots, n - 1\}$.

A variable word $w(x)$ is a finite word in the alphabet $\{0, 1, \ldots, n - 1, x\}$ in which the variable $x$ occurs at least once.
Hales-Jewett theorem.

For every $n, r \in \mathbb{N}$ there exists a number $HJ(n, r)$ such that if words in $L(n)$ of length $HJ(n, r)$ are colored by $r$ colors then there exists a variable word $w(x)$ such that $w(0), w(1), \ldots, w(n - 1)$ have the same color.

The symbol $w(i)$ denotes the word in $L(n)$ which is produced from $w(x)$ by replacing all the occurrences of the variable $x$ by the letter of the alphabet in brackets.
Some questions

Conjecture. \( A \in \langle \mathcal{W}_n \rangle \) if and only if there exists \( k \in \mathbb{N} \) such that \( A \) does not contain a copy of \( n^k \).
Some questions

**Conjecture.** $A \in \langle \mathcal{W}_n \rangle$ if and only if there exists $k \in \mathbb{N}$ such that $A$ does not contain a copy of $n^k$.

**Question 1.** Is it true that whenever a set $A$ does not contain a copy of $3^2$ then $A \in \langle \mathcal{W}_3 \rangle$?
Some questions

Conjecture. $A \in \langle W_n \rangle$ if and only if there exists $k \in \mathbb{N}$ such that $A$ does not contain a copy of $n^k$.

Question 1. Is it true that whenever a set $A$ does not contain a copy of $3^2$ then $A \in \langle W_3 \rangle$?

Question 2. Does the set $\{n^2 : n \in \omega\}$ belong to the ideal $\langle W_3 \rangle$?
The square of squares...

\[ a^2 - d_1 - d_2 \quad a^2 - d_2 \quad a^2 + d_1 - d_2 \]

\[ a^2 - d_1 \quad a^2 \quad a^2 + d_1 \]

\[ a^2 - d_1 + d_2 \quad a^2 + d_2 \quad a^2 + d_1 + d_2 \]
...transforms...

\[ a^2 + d_1 - d_2 \]

\[ a^2 - d_2 \quad \quad a^2 + d_1 \]

\[ a^2 - d_1 - d_2 \quad a^2 \quad a^2 + d_1 + d_2 \]

\[ a^2 - d_1 \quad a^2 + d_2 \]

\[ a^2 - d_1 + d_2 \]
...transforms...

<table>
<thead>
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<tr>
<td>$a^2 - d_2$</td>
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<td>$a^2 - d_1 - d_2$</td>
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<td>$a^2 - d_1$</td>
<td>$a^2 - d_1 + d_2$</td>
<td>$a^2 + d_2$</td>
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...into a magic square of squares

\[
\begin{array}{ccc}
    a^2 + d_1 & a^2 + d_1 - d_2 & a^2 - d_2 \\
    a^2 - d_1 - d_2 & a^2 & a^2 + d_1 + d_2 \\
    a^2 + d_2 & a^2 - d_1 + d_2 & a^2 - d_1
\end{array}
\]
Problem 1. Can a 3x3 magic square be constructed with nine distinct square numbers?

Asked by Martin LaBar (1984), republished by Martin Gardner (1996) who offered $100 to the first person to construct such a square (or to prove its impossibility).

Problem 2. Provide a new example of 3x3 magic square with seven distinct square entries different from rotations, symmetries and $k^2$ multiples of the known example or provide any example with eight square entries.

Christian Boyer offers 1000€ prize + a bottle of champagne since 2010 (was only 100€ from 2005 to 2009).