Ultrafilters and divergent series

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Ultrafilters on $\omega$

Definition.

$\mathcal{U} \subseteq \mathcal{P}(\omega)$ is an ultrafilter if

- $\mathcal{U} \neq \emptyset$ and $\emptyset \notin \mathcal{U}$
- if $U_1, U_2 \in \mathcal{U}$ then $U_1 \cap U_2 \in \mathcal{U}$
- if $U \in \mathcal{U}$ and $U \subseteq V \subseteq \omega$ then $V \in \mathcal{U}$.
- for every $M \subseteq \omega$ either $M$ or $\omega \setminus M$ belongs to $\mathcal{U}$

Example. fixed (or principal) ultrafilter $\{A \subseteq \omega : n \in A\}$

Free ultrafilters may be viewed as points in the remainder $\omega^*$ of the Čech-Stone compactification $\beta\omega$. 
Divergent series

The harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$
Divergent series

The harmonic series

\[ \sum_{n=1}^{\infty} \frac{1}{n} \]

Some divergent series grow to infinity slower:

\[ \sum_{n \in \mathbb{N}} \frac{1}{2n}, \quad \sum_{n \in \mathbb{N}} \frac{1}{n \cdot \ln n}, \quad \sum_{n \in \mathbb{N}} \frac{1}{n \cdot \ln n \cdot \ln \ln n} \]

Some divergent series grow to infinity faster:

\[ \sum_{n \in \mathbb{N}} \frac{\ln n}{n}, \quad \sum_{n \in \mathbb{N}} \frac{1}{\sqrt{n}}, \quad \sum_{n \in \mathbb{N}} \frac{1}{\ln n} \]
Summable ideals

Definition.
For a given a divergent series \( \sum_{n \in \mathbb{N}} g(n) = +\infty \) such that \( \lim_{n \to \infty} g(n) = 0 \) the family

\[
\mathcal{I}_g = \{ A \subseteq \mathbb{N} : \sum_{n \in A} g(n) < +\infty \}
\]

is a tall proper ideal which we call summable ideal determined by the series \( \sum_{n \in \mathbb{N}} g(n) \) (by the function \( g \)).
Summable ideals

Lemma.

Assume $g_1$ and $g_2$ are two sequences of positive real numbers such that $\sum_{n \in \mathbb{N}} g_i(n) = +\infty$ and $\lim_{n \to \infty} g_i(n) = 0$ for $i = 1, 2$. Then

1. If $g_1 \leq^* g_2$ then $I_{g_1} \supset I_{g_2}$.

2. If $\lim_{n \to \infty} \frac{g_1(n)}{g_2(n)} \in \mathbb{R}$, then $I_{g_1} \supset I_{g_2}$. 
Rapid ultrafilters

Definition.

A free ultrafilter $\mathcal{U}$ on $\omega$ is called rapid if the enumeration functions of its sets form a dominating family in $(\omega^\omega, \leq^*)$. 

Theorem (Booth?). (CH) Rapid ultrafilters exist.

Theorem (Miller). In Laver’s model there are no rapid ultrafilters.
Rapid ultrafilters

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Characterization of rapid ultrafilters

Theorem (Vojtáš).

The following are equivalent for an ultrafilter $\mathcal{U} \in \omega^*$:

- $\mathcal{U}$ is rapid
- $\mathcal{U} \cap \mathcal{I}_g \neq \emptyset$ for every tall summable ideal $\mathcal{I}_g$
Characterization of rapid ultrafilters

Theorem (Vojtáš).

The following are equivalent for an ultrafilter $U \in \omega^*$:

- $U$ is rapid
- $U \cap I_g \neq \emptyset$ for every tall summable ideal $I_g$

One can add two more equivalent conditions:

- $(\forall f : \omega \to \mathbb{N} \text{ one-to-one}) \ (\exists U \in U) \text{ such that } f[U] \in I_g$
  for every tall summable ideal $I_g$
Characterization of rapid ultrafilters

Theorem (Vojtáš).

The following are equivalent for an ultrafilter \( \mathcal{U} \in \omega^* \):

- \( \mathcal{U} \) is rapid
- \( \mathcal{U} \cap \mathcal{I}_g \neq \emptyset \) for every tall summable ideal \( \mathcal{I}_g \)

One can add two more equivalent conditions:

- \( \forall f : \omega \to \mathbb{N} \) one-to-one) \( \exists \mathcal{U} \in \mathcal{U} \) such that \( f[\mathcal{U}] \in \mathcal{I}_g \) for every tall summable ideal \( \mathcal{I}_g \)
- \( \forall f : \omega \to \mathbb{N} \) finite-to-one) \( \exists \mathcal{U} \in \mathcal{U} \) such that \( f[\mathcal{U}] \in \mathcal{I}_g \) for every tall summable ideal \( \mathcal{I}_g \)

(\( \mathcal{U} \) is a weak \( \mathcal{I}_g \)-ultrafilter for every \( \mathcal{I}_g \))
How many summable ideals decide?

Proposition.

There is a family $\mathcal{D}$ of tall summable ideals such that $|\mathcal{D}| = \omega$ and an ultrafilter $\mathcal{U} \in \omega^*$ is rapid if and only if it has a nonempty intersection with every tall summable ideal in $\mathcal{D}$. 
Proposition.

There is a family $\mathcal{D}$ of tall summable ideals such that $|\mathcal{D}| = \varpi$ and an ultrafilter $\mathcal{U} \in \omega^*$ is rapid if and only if it has a nonempty intersection with every tall summable ideal in $\mathcal{D}$.

Is it possible that an ultrafilter has a nonempty intersection with "many" tall summable ideals simultaneously and it is not a rapid ultrafilter?
How many summable ideals decide?

Proposition (Cancino Manríquez).

For any family $\mathcal{D}$ of tall summable ideals such that $|\mathcal{D}| < \omega$ there is an ultrafilter $\mathcal{U} \in \omega^*$ which meets all ideals $\mathcal{I} \in \mathcal{D}$, but $\mathcal{U}$ is not a rapid ultrafilter.
How many summable ideals decide?

Proposition (Cancino Manríquez).

For any family $\mathcal{D}$ of tall summable ideals such that $|\mathcal{D}| < \mathfrak{d}$ there is an ultrafilter $\mathcal{U} \in \omega^*$ which meets all ideals $\mathcal{I} \in \mathcal{D}$, but $\mathcal{U}$ is not a rapid ultrafilter.

Can we find a family of tall summable ideals of cardinality at least $\mathfrak{d}$ such that an ultrafilter has a nonempty intersection with every ideal in the family, but is not a rapid ultrafilter?
The $\ell^p$-hierarchy

For $1 \leq p < \infty$ the $\ell^p$-space is defined as follows

$$\ell^p = \{(x_n)_{n \in \mathbb{N}} \in \mathbb{N} \oplus \mathbb{R} : \sum_{n \in \mathbb{N}} |x_n|^p < \infty\}$$
The $\ell^p$-hierarchy

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The space $\ell^1$ is the space of all absolutely convergent series.
The \( \ell^p \)-hierarchy

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The space \( \ell^1 \) is the space of all absolutely convergent series.

\[
\left( \frac{1}{n} \right)_n \in \bigcap_{k \in \mathbb{N}} \ell^{1+\frac{1}{k}} \setminus \ell^1 \text{ and also } \left( \frac{\ln n}{n} \right)_n \in \bigcap_{k \in \mathbb{N}} \ell^{1+\frac{1}{k}} \setminus \ell^1
\]

\[
\left( \frac{1}{\sqrt{n}} \right)_n \in \bigcap_{k \in \mathbb{N}} \ell^{2+\frac{1}{k}} \setminus \ell^2, \text{ but } \left( \frac{1}{\ln n} \right)_n \not\in \ell^p \text{ for any } 1 < p < +\infty
\]
Rapid ultrafilters once more

Proposition 1.
The following are equivalent for an ultrafilter $\mathcal{U} \in \omega^*$:

- $\mathcal{U}$ is rapid
- $\mathcal{U} \cap \mathcal{I}_g \neq \emptyset$ for every tall summable ideal $\mathcal{I}_g$
- $\mathcal{U} \cap \mathcal{I}_g \neq \emptyset$ for every tall summable ideal $\mathcal{I}_g$ with $g \notin \ell^2$
- $\mathcal{U} \cap \mathcal{I}_g \neq \emptyset$ for every tall summable ideal $\mathcal{I}_g$ with $g \notin \ell^k$ for a given $k > 2$
Conjecture 2.

For any $k \in \mathbb{N}$ there is an ultrafilter $\mathcal{U}$ in ZFC such that $\mathcal{U} \cap \mathcal{I}_g \neq \emptyset$ for every tall summable ideal $\mathcal{I}_g$ with $g \in \ell^k$. 
The bottom part of the hierarchy

Conjecture 2.
For any \( k \in \mathbb{N} \) there is an ultrafilter \( \mathcal{U} \) in ZFC such that
\( \mathcal{U} \cap I_g \neq \emptyset \) for every tall summable ideal \( I_g \) with \( g \in \ell^k \).

- if \( g \in \ell^k \) and \( h \not\in \ell^k \) then \( h \not\preceq^* g \)
g-summable ultrafilters

Definition.

An ultrafilter \( \mathcal{U} \in \omega^* \) is called \( g \)-summable ultrafilter if for every one-to-one \( f : \omega \to \mathbb{N} \) there exists \( U \in \mathcal{U} \) such that \( f[U] \in \mathcal{I}_g \).
**Definition.**

An ultrafilter \( \mathcal{U} \in \omega^* \) is called \( g \)-summable ultrafilter if for every one-to-one \( f : \omega \to \mathbb{N} \) there exists \( U \in \mathcal{U} \) such that \( f[U] \in \mathcal{I}_g \).

Rapid ultrafilter is \( g \)-summable for every \( \mathcal{I}_g \).
**g-summable ultrafilters**

**Definition.**

An ultrafilter \( \mathcal{U} \in \omega^* \) is called \( g \)-summable ultrafilter if for every one-to-one \( f : \omega \rightarrow \mathbb{N} \) there exists \( U \in \mathcal{U} \) such that \( f[U] \in \mathcal{I}_g \).

Rapid ultrafilter is \( g \)-summable for every \( \mathcal{I}_g \).

**Theorem (J.B.)**

There exists \( \mathcal{U} \in \omega^* \) such that \( \mathcal{U} \) is an \( \frac{1}{n} \)-summable ultrafilter.
**Theorem 3.**

If Conjecture 2. holds then $g$-summable ultrafilters exist in ZFC for every $k \in \mathbb{N}$ and $g \in \ell^k$. 
Theorem 3.
If Conjecture 2. holds then
\( g \)-summable ultrafilters exist in ZFC for every \( k \in \mathbb{N} \) and \( g \in \ell^k \).

Open questions:
Do \( \frac{1}{\ln n} \)-summable ultrafilters exist in ZFC?
Theorem 3.
If Conjecture 2. holds then $g$-summable ultrafilters exist in ZFC for every $k \in \mathbb{N}$ and $g \in \ell^k$.

Open questions:
Do $\frac{1}{\ln n}$-summable ultrafilters exist in ZFC?
Do weak $\mathcal{I}_g$-ultrafilters exist in ZFC? (open also for $g(n) = \frac{1}{n}$)
References


J. Flašková, Rapid ultrafilters and summable ideals, *arXiv:1208.3690*