Nowhere dense sets corresponding to summable ideals

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Čech-Stone compactification of $\omega$

Definition.
Čech-Stone compactification of $\omega$ is a compact topological space $\beta\omega$ such that:

- $\omega$ is a dense subspace of $\beta\omega$
- every (continuous) function $f : \omega \to [0, 1]$ can be extended to a continuous function $\beta f : \beta\omega \to [0, 1]$. 
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- every (continuous) function $f : \omega \to [0, 1]$ can be extended to a continuous function $\beta f : \beta\omega \to [0, 1]$

$\omega^* = \beta\omega \setminus \omega$ is called remainder of $\beta\omega$
Ultrafilters on $\omega$

Definition.
A family $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is called a filter on $\omega$ if:

- $\mathcal{F} \neq \emptyset$ and $\emptyset \notin \mathcal{F}$
- if $F_1, F_2 \in \mathcal{F}$ then $F_1 \cap F_2 \in \mathcal{F}$
- if $F \in \mathcal{F}$ and $F \subseteq G \subseteq X$ then $G \in \mathcal{F}$. 
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- if $F \in \mathcal{F}$ and $F \subseteq G \subseteq X$ then $G \in \mathcal{F}$.

If moreover $\mathcal{F}$ satisfies
- for every $M \subseteq \omega$ either $M \in \mathcal{F}$ or $\omega \setminus M \in \mathcal{F}$
then $\mathcal{F}$ is called an ultrafilter.
Topology on $\omega^*$

Points in $\beta\omega$ may be identified with ultrafilters on $\omega$:
- points in $\omega \leftrightarrow$ fixed ultrafilters
- points in $\omega^* \leftrightarrow$ free ultrafilters
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Topology on $\omega^*$ is generated by clopen sets:
$$A^* = \{U : A \in U\}, \quad A \in [\omega]^\omega$$
Topology on $\omega^*$

Points in $\beta\omega$ may be identified with ultrafilters on $\omega$:
- points in $\omega$ ↔ fixed ultrafilters
- points in $\omega^*$ ↔ free ultrafilters

Topology on $\omega^*$ is generated by clopen sets:

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Some more facts about $\omega^*$:
- zero-dimensional
- cardinality $2^c$
- dense-in-itself
Ideals on $\omega$

Definition.
An ideal on $\omega$ is a family $\mathcal{I} \subseteq \mathcal{P}(\omega)$ such that:

- $\mathcal{I} \neq \mathcal{P}(\omega)$ and $\emptyset \in \mathcal{I}$
- if $A_1, A_2 \in \mathcal{I}$ then $A_1 \cup A_2 \in \mathcal{I}$
- if $A \in \mathcal{I}$ and $B \subseteq A \subseteq \omega$ then $B \in \mathcal{I}$. 
Ideals on \( \omega \)

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An ideal on \( \omega \) is a family \( \mathcal{I} \subseteq \mathcal{P}(\omega) \) such that:

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- if \( A \in \mathcal{I} \) and \( B \subseteq A \subseteq \omega \) then \( B \in \mathcal{I} \).

**Examples:**

- \( \text{Fin} = \) finite subsets of \( \omega \)
- \( \mathcal{I}_{1/n} = \{ A \subseteq \mathbb{N} : \sum_{a \in A} \frac{1}{a} < \infty \} \)
- \( \mathcal{Z}_0 = \{ A \subseteq \mathbb{N} : \limsup_{n \to \infty} \frac{|A \cap n|}{n} = 0 \} \)
Summable ideals

Definition.

Given a function \( g : \omega \to [0, \infty) \) such that
\[
\sum_{n \in \omega} g(n) = \infty
\]
then the family
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\mathcal{I}_g = \{ A \subseteq \omega : \sum_{a \in A} g(a) < +\infty \}
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is a proper ideal which we call summable ideal
determined by function \( g \).
Summable ideals

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We will consider only tall summable ideals.
Ideals and open sets

Definition.
An ideal $\mathcal{I}$ on $\omega$ is tall (dense) if for every infinite set $A \subseteq \omega$ there exists $B \in [A]^\omega$ such that $B \in \mathcal{I}$. 
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A summable ideal is tall if and only if $\lim_{n \to \infty} g(n) = 0$. 
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A summable ideal is tall if and only if $\lim_{n \to \infty} g(n) = 0$.

To every ideal $\mathcal{I}$ on $\omega$ assign an open set

$$\sigma(\mathcal{I}) = \bigcup \{ A^* : A \in \mathcal{I} \}$$

and a closed set

$$\delta(\mathcal{I}) = \omega^* \setminus \sigma(\mathcal{I})$$
Ideals and (nowhere) dense sets

\[ \sigma(I) = \{ U \in \omega^* : U \cap I \neq \emptyset \} \]
\[ \delta(I) = \{ U \in \omega^* : U \cap I = \emptyset \} = \{ U \in \omega^* : I^* \subseteq U \} \]
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Lemma.

For an ideal \( I \) on \( \omega \) the following are equivalent:

- \( I \) is tall
- \( \sigma(I) \) is dense in \( \omega^* \)
- \( \delta(I) \) is nowhere dense in \( \omega^* \)
Ideals and (nowhere) dense sets

$$\sigma(I) = \{U \in \omega^* : U \cap I \neq \emptyset\}$$
$$\delta(I) = \{U \in \omega^* : U \cap I = \emptyset\} = \{U \in \omega^* : I^* \subseteq U\}$$

Lemma.

For an ideal $I$ on $\omega$ the following are equivalent:

- $I$ is tall
- $\sigma(I)$ is dense in $\omega^*$
- $\delta(I)$ is nowhere dense in $\omega^*$

If $I \subseteq J$ then $\sigma(I) \subseteq \sigma(J)$ and $\delta(I) \supseteq \delta(J)$. 
The problem

Problem S.4. (van Douwen [1978])
Is it true in ZFC that $\bigcup_{\pi \in S_\omega} \beta_\pi[A] \neq \omega^*$ whenever $A \subseteq \omega^*$ is nowhere dense? What if $A = \delta(\mathcal{Z}_0)$ or $A = \delta(\mathcal{I}_{1/n})$?
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Problem 235. (Hart, van Mill [1990])
For what nowhere dense sets $A \subseteq \omega^*$ do we have $\bigcup_{\pi \in S_\omega} \beta\pi[A] \neq \omega^*$?
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Theorem (Gryzlov)
For $A = \delta(\mathcal{Z}_0)$ we have $\bigcup_{\pi \in S_\omega} \beta_\pi [A] \neq \omega^*$. 
Results

Theorem 1.
For $A = \delta(I_{1/n})$ we have $\bigcup_{\pi \in S_{\omega}} \beta\pi[A] \neq \omega^*$. 
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For $A = \delta(I_{1/n})$ we have $\bigcup_{\pi \in S_\omega} \beta \pi[A] \neq \omega^*$. 

If $U \in \omega^* \setminus \bigcup_{\pi \in S_\omega} \beta \pi[A]$ then for all $\pi \in S_\omega$ there exists set $U \in \mathcal{U}$ with $\pi[U] \in I_{1/n}$. 
Results

Theorem 1.
For $A = \delta(I_{1/n})$ we have $\bigcup_{\pi \in S_\omega} \beta \pi [A] \neq \omega^*.$

If $U \in \omega^* \setminus \bigcup_{\pi \in S_\omega} \beta \pi [A]$ then for all $\pi \in S_\omega$ there exists set $U \in \mathcal{U}$ with $\pi[U] \in I_{1/n}.$

Definition A.
An ultrafilter $\mathcal{U}$ on $\omega$ is called a summable ultrafilter if for every one-to-one function $f : \omega \to \mathbb{N}$ there exists $U \in \mathcal{U}$ such that $f[U] \in I_{1/n}.$
Results

Theorem 1.
For $A = \delta(\mathcal{I}_{1/n})$ we have $\bigcup_{\pi \in S_\omega} \beta\pi[A] \neq \omega^*$. 

If $U \in \omega^* \setminus \bigcup_{\pi \in S_\omega} \beta\pi[A]$ then for all $\pi \in S_\omega$ there exists set $U \in \mathcal{U}$ with $\pi[U] \in \mathcal{I}_{1/n}$.

Definition A.
An ultrafilter $\mathcal{U}$ on $\omega$ is called a **summable ultrafilter** if for every one-to-one function $f : \omega \to \mathbb{N}$ there exists $U \in \mathcal{U}$ such that $f[U] \in \mathcal{I}_{1/n}$.

Theorem 1*.
Summable ultrafilters exist in ZFC.
Results

Definition B.
An ultrafilter $\mathcal{U}$ on $\omega$ is called a $g$-summable ultrafilter if for every one-to-one function $f : \omega \to \mathbb{N}$ there exists $U \in \mathcal{U}$ such that $f[U] \in I_g$. 
Results

Definition B.
An ultrafilter $\mathcal{U}$ on $\omega$ is called a $g$-summable ultrafilter if for every one-to-one function $f : \omega \to \mathbb{N}$ there exists $U \in \mathcal{U}$ such that $f[U] \in \mathcal{I}_g$.

Corollary 2.
If $g : \omega \to [0, \infty)$ satisfies $\frac{1}{n} \gg g(n)$ then $g$-summable ultrafilters exist in ZFC.
Results

Definition B.
An ultrafilter $\mathcal{U}$ on $\omega$ is called a $g$-summable ultrafilter if for every one-to-one function $f : \omega \to \mathbb{N}$ there exists $U \in \mathcal{U}$ such that $f[U] \in \mathcal{I}_g$.

Corollary 2.
If $g : \omega \to [0, \infty)$ satisfies $\frac{1}{n} \gg g(n)$ then $g$-summable ultrafilters exist in ZFC.

Theorem 3.
If $g(n) = \frac{\ln p \cdot n}{n}$, $p \in \omega$, then $g$-summable ultrafilters exist in ZFC (i.e. $\delta(\mathcal{I}_g)$ solves Problem 235).
Theorem 4.

\((\text{MA}_{\text{ctble}})\) If \(I_g\) is a summable ideal and \(A = \delta(I_g)\) then \(\bigcup_{\pi \in S_\omega} \beta\pi[A] \neq \omega^*\).
More results and questions

Theorem 4.

(\text{MA}_\text{ctble}) \text{ If } I_g \text{ is a summable ideal and } A = \delta(I_g) \text{ then } \bigcup_{\pi \in S_\omega} \beta\pi[A] \neq \omega^*.

Question.

Do $g$-summable ultrafilters exist in ZFC for every tall summable ideal $I_g$? What if $g(n) = \frac{1}{\sqrt{n}}$ or $\frac{1}{\ln n}$?
More results and questions

Theorem 4.

(MA_{ctble}) If $I_g$ is a summable ideal and $A = \delta(I_g)$ then \( \bigcup_{\pi \in S_\omega} \beta_\pi[A] \neq \omega^* \).

Question.

Do $g$-summable ultrafilters exist in ZFC for every tall summable ideal $I_g$? What if $g(n) = \frac{1}{\sqrt{n}}$ or $\frac{1}{\ln n}$?

Question.

Is there a $g$-summable ultrafilter which is not an $h$-summable ultrafilter if $I_g$ and $I_h$ are two incomparable summable ultrafilters?
Construction

Theorem 1*.
Summable ultrafilters exist in ZFC.
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Definition.
A family $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is called

- a $k$-linked family if $F_1 \cap \ldots \cap F_k$ is infinite whenever $F_i \in \mathcal{F}$, $i \leq k$.

- a centered system if $\mathcal{F}$ is $k$-linked for every $k$ i.e., if any finite subfamily of $\mathcal{F}$ has an infinite intersection.
Construction

We say that $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is a **summable family** if for every one-to-one function $f : \omega \to \mathbb{N}$ there is $A \in \mathcal{F}$ such that $f[A] \in \mathcal{I}_{1/n}$.
Construction

We say that $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is a **summable family** if for every one-to-one function $f : \omega \to \mathbb{N}$ there is $A \in \mathcal{F}$ such that $f[A] \in \mathcal{I}_{1/n}$.

**Proposition 5.**

For every $k \in \mathbb{N}$ there exists a summable $k$-linked family $\mathcal{F}_k \subseteq \mathcal{P}(\omega)$. 
Construction

Lemma 6.
If $F_k \subseteq P(\omega)$ is a $k$-linked family then
$\mathcal{F} = \{ F \subseteq \omega : (\forall k)(\exists U^k \in F_k) U^k \subseteq^* F \}$
is a centered system.
Construction

Lemma 6.
If $\mathcal{F}_k \subseteq \mathcal{P}(\omega)$ is a $k$-linked family then
\[ \mathcal{F} = \{ F \subseteq \omega : (\forall k)(\exists U^k \in \mathcal{F}_k) U^k \subseteq^* F \} \]
is a centered system.

If every $\mathcal{F}_k$ is summable then $\mathcal{F}$ is summable.
Construction

Lemma 6.
If $\mathcal{F}_k \subseteq \mathcal{P}(\omega)$ is a $k$-linked family then
\[ \mathcal{F} = \{ F \subseteq \omega : (\forall k)(\exists U^k \in \mathcal{F}_k) U^k \subseteq^* F \} \]
is a centered system.

If every $\mathcal{F}_k$ is summable then $\mathcal{F}$ is summable.

More generally, if $\mathcal{I}$ is a $P$-ideal and for every one-to-one function $f \in \omega \mathbb{N}$ and for every $k \in \mathbb{N}$ there exists $U^k \in \mathcal{F}_k$ such that $f[U^k] \in \mathcal{I}$ then there exists $U \in \mathcal{F}$ such that $f[U] \in \mathcal{I}$. 
References


