Van der Waerden spaces and their relatives

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For a given maximal almost disjoint (MAD) family $\mathcal{A}$ of infinite subsets of $\mathbb{N}$ we define the space $\Psi(\mathcal{A})$ as follows:

- The underlying set is $\mathbb{N} \cup \{p_A : A \in \mathcal{A}\}$.
- Every point in $\mathbb{N}$ is isolated.
- Every point $p_A$ has neighborhood base of all sets $\{p_A\} \cup A \setminus K$ where $K$ is a finite subset of $A$. 

$\Psi$-spaces
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Note: $\Psi(\mathcal{A})$ is regular, first countable and separable.
Sequentially compact spaces

All topological spaces are Hausdorff.

Definition.
A topological space $X$ is called **sequentially compact** if for every sequence $\langle x_n \rangle_{n \in \omega}$ in $X$ there exists a converging subsequence $\langle x_{n_k} \rangle_{k \in \omega}$.
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Is it possible to choose the subsequence in such a way that the set of indices is "large"?
AP-sets and van der Waerden spaces

A ⊆ \mathbb{N} is an AP-set if A contains arithmetic progressions of arbitrary length.

- (van der Waerden theorem)
  Sets that are not AP-sets form an ideal
- van der Waerden ideal is an $F_\sigma$-ideal
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- van der Waerden ideal is an $F_\sigma$-ideal

Definition A. (Kojman)

A topological space $X$ is called van der Waerden if for every sequence $\langle x_n \rangle_{n \in \omega}$ in $X$ there exists a converging subsequence $\langle x_{n_k} \rangle_{k \in \omega}$ so that $\{n_k : k \in \omega\}$ is an AP-set.
Sequentially compact $\neq$ van der Waerden

Every van der Waerden space is sequentially compact.
Sequently compact $\neq$ van der Waerden

Every van der Waerden space is sequentially compact.

Theorem (Kojman)
There exists a compact, sequentially compact, separable space which is first-countable at all points but one, which is not van der Waerden.
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Every van der Waerden space is sequentially compact.

**Theorem (Kojman)**

There exists a compact, sequentially compact, separable space which is first-countable at all points but one, which is not van der Waerden.

**Proof.** Consider the one-point compactification of $\Psi(\mathcal{A})$ for a suitable MAD family $\mathcal{A}$. 
Van der Waerden spaces – a sufficient condition

Theorem (Kojman)
If a Hausdorff space $X$ satisfies the following condition

$(\ast)$ The closure of every countable set in $X$ is compact and first-countable.

Then $X$ is van der Waerden.
Van der Waerden spaces – a sufficient condition

Theorem (Kojman)
If a Hausdorff space $X$ satisfies the following condition

(*) The closure of every countable set in $X$ is compact and first-countable.

Then $X$ is van der Waerden.

For example, compact metric spaces or every successor ordinal with the order topology satisfy (*).
$\mathcal{I}_{1/n}$-spaces

$\mathcal{I}_{1/n} = \{ A \subseteq \mathbb{N} : \sum_{a \in A} \frac{1}{a} < \infty \}$

The summable ideal $\mathcal{I}_{1/n}$ is an $F_\sigma$-ideal and $P$-ideal.
\( \mathcal{I}_{1/n} \)-spaces

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\mathcal{I}_{1/n} = \{ A \subseteq \mathbb{N} : \sum_{a \in A} \frac{1}{a} < \infty \}
\]

The summable ideal \( \mathcal{I}_{1/n} \) is an \( F_\sigma \)-ideal and \( P \)-ideal.

**Definition B.**

A topological space \( X \) is called \( \mathcal{I}_{1/n} \)-space if for every sequence \( \langle x_n \rangle_{n \in \omega} \) in \( X \) there exists a converging subsequence \( \langle x_{n_k} \rangle_{k \in \omega} \) so that \( \{ n_k : k \in \omega \} \) does not belong to \( \mathcal{I}_{1/n} \).
Sequentially compact $\neq \mathcal{I}_{1/n}$-space

Theorem 1.
There exists a compact, sequentially compact, separable space which is first-countable at all points but one, which is not an $\mathcal{I}$-space.
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There exists a compact, sequentially compact, separable space which is first-countable at all points but one, which is not an $I$-space.

Proof. Consider the one-point compactification of $\Psi(\mathcal{A})$ for a suitable MAD family $\mathcal{A}$. 

Sequentially compact $\neq \mathcal{I}_{1/n}$-space
I-spaces – a sufficient condition

Theorem 2.
If a Hausdorff space $X$ satisfies the following condition

(*) The closure of every countable set in $X$ is compact and first-countable.

Then $X$ is a $I_{1/n}$-space.
\( I_{1/n} \)-spaces – a sufficient condition

Theorem 2.

If a Hausdorff space \( X \) satisfies the following condition

\[ (*) \text{ The closure of every countable set in } X \text{ is compact and first-countable.} \]

Then \( X \) is a \( I_{1/n} \)-space.

Theorems 1. and 2. remain true if the summable ideal is replaced by an arbitrary tall \( F_\sigma \)-ideal on \( \omega \) which contains all finite sets.
\[ \mathcal{I}_{1/n} \text{ vs van der Waerden spaces} \]

Erdős-Turán Conjecture.
Every set \( A \notin \mathcal{I}_{1/n} \) is an AP-set.

If Erdős-Turán Conjecture is true then every \( \mathcal{I}_{1/n} \)-space is van der Waerden.
$I_{1/n}$ vs van der Waerden spaces

Erdős-Turán Conjecture.
Every set $A \notin I_{1/n}$ is an AP-set.

If Erdős-Turán Conjecture is true then every $I_{1/n}$-space is van der Waerden.

Theorem 3.
$(MA_{\sigma-cent.})$ There exists a van der Waerden space which is not an $I_{1/n}$-space.
\( \mathcal{I}_{1/n} \) vs van der Waerden spaces

Erdős-Turán Conjecture.
Every set \( A \not\in \mathcal{I}_{1/n} \) is an AP-set.

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Theorem 3.

\((\text{MA}_{\sigma-\text{cent.}})\) There exists a van der Waerden space which is not an \( \mathcal{I}_{1/n} \)-space.

Theorem 3. is true for an arbitrary \( F_\sigma \ P \)-ideal on \( \omega \).
Outline of the proof

Lemma
Assume $A \subseteq \mathbb{N}$ is an AP-set and $f : \mathbb{N} \to \mathbb{N}$.
There is an AP-set $C \subseteq A$ such that

1. either $f$ is constant on $C$
2. or $f$ is finite-to-one on $C$ and $f[C] \in \mathcal{I}_{1/n}$. 
Outline of the proof

Lemma

Assume $A \subseteq \mathbb{N}$ is an AP-set and $f : \mathbb{N} \rightarrow \mathbb{N}$. There is an AP-set $C \subseteq A$ such that

1. either $f$ is constant on $C$
2. or $f$ is finite-to-one on $C$ and $f[C] \in \mathcal{I}_{1/n}$.

Proposition

$(\text{MA}_{\sigma-\text{cent.}})$ There exists a MAD family $\mathcal{A} \subseteq \mathcal{I}_{1/n}$ so that for every AP-set $B \subseteq \mathbb{N}$ and every finite-to-one function $f : B \rightarrow \mathbb{N}$ there exists an AP-set $C \subseteq B$ and $A \in \mathcal{A}$ so that $f[C] \subseteq A$. 
Strongly van der Waerden spaces

Definition C. (Kojman)
A topological space $X$ is called strongly van der Waerden if for every AP-set $A \subseteq \mathbb{N}$ and every sequence $\langle x_n \rangle_{n \in A}$ in $X$ there exists a converging subsequence $\langle x_n \rangle_{n \in B}$ where $B \subseteq A$ is an AP-set.

Proposition (Kojman)
A topological space $X$ is van der Waerden if and only if it is strongly van der Waerden.
Product of van der Waerden spaces

Proposition (Kojman)
The product of two van der Waerden spaces is van der Waerden.
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Any finite or countable product of van der Waerden spaces is again van der Waerden.
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Question: What is the minimal number of van der Waerden spaces such that their product is not van der Waerden?
Product of van der Waerden spaces

Proposition (Kojman)
The product of two van der Waerden spaces is van der Waerden.

Any finite or countable product of van der Waerden spaces is again van der Waerden.

Question: What is the minimal number of van der Waerden spaces such that their product is not van der Waerden?
The upper bound is certainly less or equal to $h$. 
Strongly $\mathcal{I}_{1/n}$-spaces

**Definition D.**
A topological space $X$ is called strongly $\mathcal{I}_{1/n}$-space if for every $\mathcal{I}_{1/n}$-positive set $A \subseteq \mathbb{N}$ and every sequence $\langle x_n \rangle_{n \in A}$ in $X$ there exists a converging subsequence $\langle x_n \rangle_{n \in B}$ where $B \subseteq A$ is does not belong to $\mathcal{I}_{1/n}$. 
Strongly $\mathcal{I}_{1/n}$-spaces

Definition D.
A topological space $X$ is called strongly $\mathcal{I}_{1/n}$-space if for every $\mathcal{I}_{1/n}$-positive set $A \subseteq \mathbb{N}$ and every sequence $\langle x_n \rangle_{n \in A}$ in $X$ there exists a converging subsequence $\langle x_n \rangle_{n \in B}$ where $B \subseteq A$ is does not belong to $\mathcal{I}_{1/n}$.

Question: Is every $\mathcal{I}_{1/n}$-space a strongly $\mathcal{I}_{1/n}$-space?
Some questions

Question: Is the product of two $\mathcal{I}_{1/n}$-spaces an $\mathcal{I}_{1/n}$-space?

If $\mathcal{I}_{1/n}$-spaces and strongly $\mathcal{I}_{1/n}$-spaces coincide then the answer is obviously positive.
Some questions

Question: Is the product of two $\mathcal{I}_{1/n}$-spaces an $\mathcal{I}_{1/n}$-space?

If $\mathcal{I}_{1/n}$-spaces and strongly $\mathcal{I}_{1/n}$-spaces coincide then the answer is obviously positive.

Question: What is the minimal number of $\mathcal{I}_{1/n}$-spaces such that their product is not $\mathcal{I}_{1/n}$-space?