On Constrained Open Mapping Theorems

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Theorem

Let \((X, \| \cdot \|_X), (Y, \| \cdot \|_Y)\) be Banach spaces, let \(T : X \rightrightarrows Y\), and let \((x_0, y_0) \in \text{gph } T\). Suppose that \(\text{dom } T\) is bounded, that \(\text{gph } T\) is locally star-shaped at \((x_0, y_0)\) and that \(\text{rge } T\) is locally conic at \(y_0\). Then \(T\) is relatively open (on its domain) at \((x_0, y_0)\) with a linear rate.

There are \(\varepsilon_0 > 0\) and \(c > 0\) such that
\[
T(B_X(x_0, \varepsilon)) \supset \text{rge } T \cap B_Y(y_0, c\varepsilon)
\]
whenever \(\varepsilon \in (0, \varepsilon_0]\).
Theorem

Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be Banach spaces, let $T : X \ni Y$, and let $(x_0, y_0) \in \text{gph } T$. Suppose that $\text{dom } T$ is bounded, that $\text{gph } T$ is locally star-shaped at $(x_0, y_0)$ and that $\text{rge } T$ is locally conic at $y_0$. Then $T$ is relatively open (on its domain) at $(x_0, y_0)$ with a linear rate.

There is $\lambda_0 > 0$ such that for each $(x, y) \in \text{gph } T$ and each $\lambda \in [0, \lambda_0]$ one has

$$(1 - \lambda)(x_0, y_0) + \lambda(x, y) \in \text{gph } T.$$
Theorem

Let \((X, \|\cdot\|_X), (Y, \|\cdot\|_Y)\) be Banach spaces, let \(T : X \rightrightarrows Y\), and let \((x_0, y_0) \in \text{gph } T\). Suppose that \(\text{dom } T\) is bounded, that \(\text{gph } T\) is locally star-shaped at \((x_0, y_0)\) and that \(\text{rge } T\) is locally conic at \(y_0\). Then \(T\) is relatively open (on its domain) at \((x_0, y_0)\) with a linear rate.

There is a neighborhood \(W\) of \(y_0\) in \(Y\) along with a shifted cone \(C\) with vertex \(y_0\) such that

\[
\text{rge } T \cap W = C \cap W.
\]
Nice statement which is FALSE in general:

Let $(X, \| \cdot \|_X), (Y, \| \cdot \|_Y)$ be Banach spaces, let $T : X \rightrightarrows Y$, and let $(x_0, y_0) \in \text{gph } T$. Suppose that $\text{dom } T$ is bounded, that $\text{gph } T$ is locally star-shaped at $(x_0, y_0)$ and that $\text{rge } T$ is locally conic at $y_0$. Then $T$ is relatively open (on its domain) at $(x_0, y_0)$ with a linear rate.

$$K := \{(x, y, z) \in \mathbb{R}^3 : (x - z)^2 + y^2 \leq z^2, z \geq 0\}, \quad T := P|_K,$$

$x_0 := (0, 0, 0)$, and $y_0 := (0, 0)$. 

\[\begin{array}{c}
\text{K} \\
\text{P} \\
\text{X} \\
\text{Y} \\
\text{Z}
\end{array}\]
Nice statement which is **FALSE** in general:

Let \((X, \|\cdot\|_X), (Y, \|\cdot\|_Y)\) be Banach spaces, let \(T : X \rightrightarrows Y\), and let \((x_0, y_0) \in \text{gph} \ T\). Suppose that \(\text{dom} \ T\) is bounded, that \(\text{gph} \ T\) is locally star-shaped at \((x_0, y_0)\) and that \(\text{rge} \ T\) is locally conic at \(y_0\). Then \(T\) is relatively open (on its domain) at \((x_0, y_0)\) with a linear rate.

\[T(x) := \{x, 1\} \text{ if } x \in [0, 1] \text{ and } T(x) := \emptyset \text{ otherwise}, \quad x_0 := 0, \text{ and } y_0 := 1.\]
An analogue of Robinson-Ursescu theorem
Lyusternik-Graves theorem with convex constraint
A particular case - Strict differentiability

Nice statement which is **FALSE** in general:

Let \((X, \| \cdot \|_x), (Y, \| \cdot \|_y)\) be Banach spaces, let \(T : X \rightrightarrows Y\), and let \((x_0, y_0) \in \text{gph} \ T\). Suppose that \(\text{dom } T\) is bounded and that \(\text{gph } T\) is locally star-shaped at \((x_0, y_0)\). Then \(T\) is relatively open (on its domain) at \((x_0, y_0)\) with a linear rate.

\[
T(r) := \{(x, y) \in \mathbb{R}^2 : (x - r)^2 + y^2 \leq r^2\} \quad \text{if} \quad r \in [0, 1] \quad \text{and} \quad T(r) := \emptyset \quad \text{otherwise.}
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\[ x_0 = (0,0) \]
Theorem

Let \((X, \|\cdot\|_X), (Y, \|\cdot\|_Y)\) be Banach spaces, \(K \subset X\) and \(D \subset Y\) be closed, bounded and convex sets containing \(x_0 \in X\) and \(0 \in Y\), respectively. Suppose that \(F : K \to Y\) is continuous and \(T : K - x_0 \rightrightarrows Y\) is a "good approximation of \(F\) at \(x_0\) on \(K\) with respect to \(D\)". Then there are \(c > 0\) and \(\varepsilon_1 > 0\) such that

\[
F(K \cap B_X(x_0, \varepsilon)) + R^{-1}\varepsilon D \supset B_Y(y_0, c\varepsilon) \quad \text{whenever} \quad \varepsilon \in (0, \varepsilon_1).
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Let \((X, \|\cdot\|_X), (Y, \|\cdot\|_Y)\) be Banach spaces, \(K \subset X\) and \(D \subset Y\) be closed, bounded and convex sets containing \(x_0 \in X\) and \(0 \in Y\), respectively. Suppose that \(F : K \to Y\) is continuous and \(T : K - x_0 \rightrightarrows Y\) is a "good approximation of \(F\) at \(x_0\) on \(K\) with respect to \(D\)". Then there are \(c > 0\) and \(\varepsilon_1 > 0\) such that

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\]
To be a good approximation at the reference point means:

- The graph of $T$ is star-shaped at $(0,0)$;
- $\beta : (0, +\infty) \rightarrow [0, +\infty]$ is a non-decreasing function such that $\forall \varepsilon > 0 \ \forall k_1, k_2 \in K \cap B_x(x_0, \varepsilon) : k_1 - k_2 \in K - x_0$ and $\forall z \in T(k_1 - k_2)$ we have
  \[ \|F(k_1) - F(k_2) - z\|_Y \leq \beta(\varepsilon)\|k_1 - k_2\|_X; \]
- There is $r > 0$ such that $T(K - x_0) + D \supset B_Y(0, r)$;
- There is $R > 0$ such that $K \subset B_x(x_0, R)$;
- There is $\varepsilon_0 > 0$ such that $\rho := r^{-1}R\beta(\varepsilon_0) < 1$. 
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\[ c := (1 - \varrho)rR^{-1} \] and $\varepsilon_1 := \min\{\varepsilon_0, R\}$. 

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$c := (1 - \varrho)rR^{-1}$ and $\varepsilon_1 := \min \{\varepsilon_0, R\}$
Suppose that $\alpha > \beta > 0$ and put

$$F(x) = \begin{cases} 
\alpha x & \text{for } x \geq 0; \\
\beta x & \text{for } x < 0.
\end{cases}$$
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$$F(x) = \begin{cases} 
\alpha x & \text{for } x \geq 0; \\
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\end{cases}$$
Corollary

Let \((X, \|\cdot\|_X), (Y, \|\cdot\|_Y)\) be Banach spaces, let \(K \subset X\) be closed convex with \(x_0 \in K\), let \(F : K \to Y\) be continuous. Suppose that \(T : (K - x_0) \to Y\) is positively homogenous such that for each \(\delta > 0\) there is \(\alpha > 0\) such that

\[
\|F(k_1) - F(k_2) - T(k_1 - k_2)\|_Y \leq \delta \|k_1 - k_2\|_X
\]

whenever \(k_1, k_2 \in K \cap B_X(x_0, \alpha)\) with \(k_1 - k_2 \in K - x_0\).

If \(\overline{T(K - x_0)} \supset B_Y(0, r)\) for some \(r > 0\) then there are \(c > 0\) and \(\varepsilon_1 > 0\) such that

\[
F(K \cap B_X(x_0, \varepsilon)) \supset B_Y(F(x_0), c\varepsilon) \quad \text{whenever} \quad \varepsilon \in (0, \varepsilon_1].
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Theorem

Let \((X, \|\cdot\|_X), (Y, \|\cdot\|_Y)\) be Banach spaces, \(K \subset X\) and \(D \subset Y\) be closed, bounded and convex sets containing \(x_0 \in X\) and \(0 \in Y\), respectively. Suppose that \(F : K \to Y\) is continuous and \(T : X \rightrightarrows Y\) is a "good approximation of \(F\) around \(x_0\) on \(K\) with respect to \(D\)". Then

\[(A1)\] For each \(\overline{c} \in (0, (1 - \varrho)r/R)\) there is a neighborhood \(U_1\) of \(x_0\) in \(K\) and \(\varepsilon_1 > 0\) such that for each \(x \in U_1\) and each \(\varepsilon \in (0, \varepsilon_1]\) one has

\[F(K \cap B_X(x, \varepsilon)) + R^{-1}\varepsilon D \supset B_Y(F(x), \overline{c}\varepsilon);\]
Theorem

Let \((X, \| \cdot \|_X), (Y, \| \cdot \|_Y)\) be Banach spaces, \(K \subset X\) and \(D \subset Y\) be closed, bounded and convex sets containing \(x_0 \in X\) and \(0 \in Y\), respectively. Suppose that \(F : K \to Y\) is continuous and \(T : X \rightrightarrows Y\) is a "good approximation of \(F\) around \(x_0\) on \(K\) with respect to \(D\)." Then

\[(A2)\] There are neighborhoods \(U_2\) of \(x_0\) in \(K\) and \(V_2\) of \(F(x_0)\) in \(Y\), \(c_2 > 0\), and a continuous \(G : U_2 \times V_2 \to K\) such that for each \((x, y) \in U_2 \times V_2\) it holds

\[y \in F(G(x, y)) + c_2\|y - F(x)\|_Y D\quad \text{and}\]

\[G(x, y) \in x + c_2\|y - F(x)\|_Y (K - x).\]
To be a good approximation around the reference point means

- The graph of $T$ is closed and convex and there are $L > 0$ and a neighborhood $W$ of $x_0$ in the affine hull of $K$ with $T(K - v) \cap B_Y(0, L) \neq \emptyset$ for all $w \in W$.

- $\beta : (0, +\infty) \to [0, +\infty]$ is a non-decreasing function such that $\forall \varepsilon > 0 \ \forall k_1, k_2 \in K \cap B_X(x_0, \varepsilon)$ and $\forall z \in T(k_1 - k_2)$ we have $\|F(k_1) - F(k_2) - z\|_Y \leq \beta(\varepsilon)\|k_1 - k_2\|_X$;

- There is $r > 0$ such that $T(K - x_0) + D \supset B_Y(0, r)$;

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- $\beta : (0, +\infty) \rightarrow [0, +\infty]$ is a non-decreasing function such that $\forall \varepsilon > 0 \ \forall k_1, k_2 \in K \cap B_x(x_0, \varepsilon)$ and $\forall z \in T(k_1 - k_2)$ we have $\|F(k_1) - F(k_2) - z\|_Y \leq \beta(\varepsilon)\|k_1 - k_2\|_X$;
- There is $r > 0$ such that $T(K - x_0) + D \supset B_Y(0, r)$;
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- $\beta : (0, +\infty) \to [0, +\infty]$ is a non-decreasing function such that $\forall \varepsilon > 0 \ \forall k_1, k_2 \in K \cap B_X(x_0, \varepsilon)$ and $\forall z \in T(k_1 - k_2)$ we have $\|F(k_1) - F(k_2) - z\|_Y \leq \beta(\varepsilon)\|k_1 - k_2\|_X$.

- There is $r > 0$ such that $T(K - x_0) + D \supset B_Y(0, r)$.

- There is $R > 0$ such that $K \subset B_X(x_0, R)$.

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On Constrained Open Mapping Theorems
Theorem

Let \((X, \| \cdot \|_X), (Y, \| \cdot \|_Y)\) be Banach spaces, let \(R > 0\), and let \(K \subset B_X(x_0, R)\) be closed convex set with \(x_0 \in K\). Let \(F : K \to Y\) be continuous such that there is a compact convex \(T \subset \mathcal{L}(X, Y)\) along with \(\delta > 0\) such that for each distinct \(k_1, k_2 \in K\) one has

\[
\|F(k_1) - F(k_2) - T(k_1 - k_2)\|_Y < \delta \|k_1 - k_2\|_X \quad \text{for some } T \in T,
\]

and that there is \(r > \delta R\) such that \(T(K - x_0) \supset B_Y(0, r)\) for each \(T \in T\). Then for each \(c \in (0, r/R - \delta)\) there is a neighborhood \(U\) of \(x_0\) in \(K\) such that

\[
F(K \cap B_X(x, \varepsilon)) \supset B_Y(F(x), c \varepsilon) \quad \text{whenever } x \in U, \varepsilon \in (0, R].
\]
**Theorem**

Let \((X, \| \cdot \|_X), (Y, \| \cdot \|_Y)\) be Banach spaces, let \(R > 0\), and let \(K \subset B_X(x_0, R)\) be closed convex set with \(x_0 \in K\). Let \(F : K \to Y\) be continuous such that there is a compact convex \(T \subset \mathcal{L}(X, Y)\) along with \(\delta > 0\) such that for each distinct \(k_1, k_2 \in K\) one has

\[
\| F(k_1) - F(k_2) - T(k_1 - k_2) \|_Y < \delta \| k_1 - k_2 \|_X
\]

for some \(T \in T\), and that there is \(r > \delta R\) such that \(T(K - x_0) \supset B_Y(0, r)\) for each \(T \in T\). Then for each \(c \in (0, r/R - \delta)\) there is a neighborhood \(U\) of \(x_0\) in \(K\) such that

\[
F(K \cap B_X(x, \epsilon)) \supset B_Y(F(x), c \epsilon)
\]

whenever \(x \in U, \epsilon \in (0, R]\).
Corollary

Let \((X, \| \cdot \|_X), (Y, \| \cdot \|_Y)\) be Banach spaces, let \(K\) be a closed convex subset of \(X\) which contains \(x_0 \in X\), and let \(F : K \to Y\) be a continuous mapping. Suppose that \(F\) possesses the strict pre-derivative at \(x_0\) relative to \(K\) generated by a subset \(T_0\) of \(L(X, Y)\) such that \(\chi(T_0) < \sigma(T_0, K, x_0)\). Then \(F\) is open around \(x_0\) with a linear rate.
Corollary

Let \((X, \| \cdot \|_X), (Y, \| \cdot \|_Y)\) be Banach spaces, let \(K\) be a closed convex subset of \(X\) which contains \(x_0 \in X\), and let \(F : K \rightarrow Y\) be a continuous mapping. Suppose that \(F\) possesses the strict pre-derivative at \(x_0\) relative to \(K\) generated by a subset \(T_0\) of \(\mathcal{L}(X, Y)\) such that \(\chi(T_0) < \sigma(T_0, K, x_0)\). Then \(F\) is open around \(x_0\) with a linear rate.

There is \(A : X \Rightarrow Y\) such that for each \(c > 0\) there exists \(\Delta > 0\) such that for each \(x_1, x_2 \in K \cap B_X(x_0, \Delta)\) one has

\[
F(x_1) \in F(x_2) + A(x_1 - x_2) + c\|x_1 - x_2\|_X B_Y(0, 1),
\]

with \(A(x) := \{ T(x) : T \in T_0 \}, \ x \in X\).
Corollary

Let \((X, \| \cdot \|_X), (Y, \| \cdot \|_Y)\) be Banach spaces, let \(K\) be a closed convex subset of \(X\) which contains \(x_0 \in X\), and let \(F : K \to Y\) be a continuous mapping. Suppose that \(F\) possesses the strict pre-derivative at \(x_0\) relative to \(K\) generated by a subset \(T_0\) of \(L(X, Y)\) such that \(\chi(T_0) < \sigma(T_0, K, x_0)\). Then \(F\) is open around \(x_0\) with a linear rate.

\[
\chi(T_0) := \inf \left\{ r > 0 : T_0 \subset \bigcup \left\{ B_{L(X, Y)}(L, r) : L \in \mathcal{F} \right\}, \mathcal{F} \subset T_0 \text{ finite} \right\}.
\]
Corollary

Let \((X, \| \cdot \|_X), (Y, \| \cdot \|_Y)\) be Banach spaces, let \(K\) be a closed convex subset of \(X\) which contains \(x_0 \in X\), and let \(F : K \to Y\) be a continuous mapping. Suppose that \(F\) possesses the strict pre-derivative at \(x_0\) relative to \(K\) generated by a subset \(T_0\) of \(\mathcal{L}(X, Y)\) such that \(\chi(T_0) < \sigma(T_0, K, x_0)\). Then \(F\) is open around \(x_0\) with a linear rate.

\[
\sigma(T_0, K, x_0) = \inf \{ \sigma(T, K, x_0) : T \in T_0 \}, \text{ where}
\]

\[
\sigma(T, K, x_0) := \sup \{ r > 0 : B_Y(T(x_0), r) \subset T(K \cap B_X(x_0, 1)) \}.
\]
An analogue of Robinson-Ursescu theorem
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A particular case - Strict differentiability
Theorem

Suppose that \((X, \|\cdot\|_X), (Y, \|\cdot\|_Y)\) are Banach spaces, \(K \subset X\) is closed convex with \(x_0 \in K\), and \(F : K \to Y\) has a strict derivative \(T \in L(X, Y)\) at \(x_0\) with respect to \(K\), \(D \subset Y\) is closed convex with \(0 \in D\) and \(T(x_0) \in \text{core}[T(K) + D]\).

(S1) There is a neighborhood \(U_1\) of \(x_0\) in \(K\), constants \(c_2 > 0\) and \(\varepsilon_1 > 0\) such that for each \(x \in U_1\) and each \(\varepsilon \in (0, \varepsilon_1]\) we have

\[
F(K \cap B_X(x, \varepsilon)) + \varepsilon D \supset B_Y(F(x), c_2 \varepsilon);
\]

\[
\forall \delta > 0 \exists \alpha > 0 \forall x_1, x_2 \in K \cap B_X(x_0, \alpha) : \|F(x_1) - F(x_2) - T(x_1 - x_2)\|_Y \leq \delta \|x_1 - x_2\|_X.
\]
Theorem

Suppose that $(X, \| \cdot \|_X), (Y, \| \cdot \|_Y)$ are Banach spaces, $K \subset X$ is closed convex with $x_0 \in K$, and $F : K \rightarrow Y$ has a strict derivative $T \in L(X, Y)$ at $x_0$ with respect to $K$, $D \subset Y$ is closed convex with $0 \in D$ and $T(x_0) \in \text{core}[T(K) + D]$.

(S1) There is a neighborhood $U_1$ of $x_0$ in $K$, constants $c_2 > 0$ and $\varepsilon_1 > 0$ such that for each $x \in U_1$ and each $\varepsilon \in (0, \varepsilon_1]$ we have

$$F(K \cap B_X(x, \varepsilon)) + \varepsilon D \supset B_Y(F(x), c_2 \varepsilon);$$
Theorem

Suppose that \((X, \|\cdot\|_X), (Y, \|\cdot\|_Y)\) are Banach spaces, \(K \subset X\) is closed convex with \(x_0 \in K\), and \(F : K \rightarrow Y\) has a strict derivative \(T \in L(X, Y)\) at \(x_0\) with respect to \(K\), \(D \subset Y\) is closed convex with \(0 \in D\) and \(T(x_0) \in \text{core}[T(K) + D]\).

(S2) There are neighborhoods \(U_2\) of \(x_0\) in \(K\) and \(V_2\) of \(F(x_0)\) in \(Y\), \(c_3 > 0\), and a continuous mapping \(G : U_2 \times V_2 \rightarrow K\) such that for each \(x \in U_2\) and each \(y \in V_2\) one has

\[
y \in F(G(x, y)) + c_3\|y - F(x)\|_Y D \quad \text{and} \quad G(x, y) \in x + c_3\|y - F(x)\|_Y (K - x);
\]
Suppose that \((X, \|\cdot\|_X), (Y, \|\cdot\|_Y)\) are Banach spaces, \(K \subset X\) is closed convex with \(x_0 \in K\), and \(F : K \to Y\) has a strict derivative \(T \in L(X, Y)\) at \(x_0\) with respect to \(K\), \(D \subset Y\) is closed convex with \(0 \in D\) and \(T(x_0) \in \text{core}[T(K) + D]\).

(S3) If \(T\) is injective, then there are neighborhoods \(U_3\) of \(x_0\) in \(K\) and \(V_3\) of \(F(x_0)\) in \(Y\) such that the equation \(F(x) = y\) has exactly one solution \(x \in U_3\) whenever \(y \in V_3\);
Theorem

Suppose that \((X, \|\cdot\|_X), (Y, \|\cdot\|_Y)\) are Banach spaces, \(K \subset X\) is closed convex with \(x_0 \in K\), and \(F : K \to Y\) has a strict derivative \(T \in L(X, Y)\) at \(x_0\) with respect to \(K\), \(D \subset Y\) is closed convex with \(0 \in D\) and \(T(x_0) \in \text{core}[T(K) + D]\).

(S4) If there is \(u \in K\) different from \(x_0\) such that \(T(u) = T(x_0)\), then for each neighborhood \(U_4\) of \(x_0\) in \(K\) there is a neighborhood \(V_4\) of \(F(x_0)\) in \(Y\) such that for each \(y \in V_4\) there is a continuous non-constant mapping \(g_y : [0, 1] \to U_4\) such that

\[F(g_y([0, 1])) = \{y\};\]
Theorem

Suppose that \((X, \| \cdot \|_X), (Y, \| \cdot \|_Y)\) are Banach spaces, \(K \subset X\) is closed convex with \(x_0 \in K\), and \(F : K \to Y\) has a strict derivative \(T \in L(X, Y)\) at \(x_0\) with respect to \(K\). \(D \subset Y\) is closed convex with \(0 \in D\) and \(T(x_0) \in \operatorname{core}[T(K) + D]\).

(S5) The following conditions are equivalent:

(i) \(F\) is open on \(K\) around \(x_0\) with a linear rate;

(ii) \(F\) is open on \(K\) at \(x_0\) with a linear rate;

(iii) \(F\) is almost open on \(K\) at \(x_0\) with a linear rate;

(iv) \(T(x_0)\) is a core point of \(T(K)\).
Theorem

Suppose that \((X, \| \cdot \|_X), (Y, \| \cdot \|_Y)\) are Banach spaces, \(K \subset X\) is closed convex with \(x_0 \in K\), and \(F : K \to Y\) has a strict derivative \(T \in L(X, Y)\) at \(x_0\) with respect to \(K\), \(D \subset Y\) is closed convex with \(0 \in D\) and \(T(x_0) \in \text{core}[T(K) + D]\).

(S6) If \(K\) is a neighborhood of \(x_0\) (i.e., \(T(X) = Y\)), then the following assertions are equivalent:

(i) There is a continuous linear selection \(S\) for \(T^{-1}\);

(ii) \(T^{-1}(0)\) is complemented in \(X\);

(iii) There is a selection \(G\) for \(F^{-1}\) such that \(G(F(x_0)) = x_0\). Moreover, \(G\) is continuous on a neighborhood of \(F(x_0)\) and Fréchet differentiable at \(F(x_0)\).
An analogue of Robinson-Ursescu theorem
Lyusternik-Graves theorem with convex constraint
A particular case - Strict differentiability

References


