THIN ULTRAFILTERS

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Abstract. Two ideals on $\omega$ are presented which determine the same class of $\mathcal{I}$-ultrafilters. It is proved that the existence of these ultrafilters is independent of ZFC and the relation between this class and other well-known classes of ultrafilters is shown.

Let $\mathcal{I}$ be a family of subsets of a set $X$ such that $\mathcal{I}$ contains all singletons and is closed under subsets. Given an ultrafilter $\mathcal{U}$ on $\omega$, we say that $\mathcal{U}$ is an $\mathcal{I}$-ultrafilter if for any mapping $F : \omega \to X$ there is $A \in \mathcal{U}$ such that $F(A) \in \mathcal{I}$.

Some concrete examples of $\mathcal{I}$-ultrafilters are nowhere dense ultrafilters, measure zero ultrafilters or countably closed ultrafilters defined by taking $X = 2^\omega$ and $\mathcal{I}$ to be the nowhere dense sets, the sets with closure of measure zero, or the sets with countable closure respectively. The class of $\alpha$-ultrafilters was defined for an indecomposable countable ordinal $\alpha$ by taking $X = \omega_1$ and $\mathcal{I}$ to be the subsets of $\omega_1$ with order type less than $\alpha$.

Many interesting results concerning these ultrafilters were obtained by Baumgartner [1], Brendle [4], Barney [2]. It was proved by Shelah [7] that consistently there are no nowhere dense ultrafilters, consequently all mentioned ultrafilters (except the $\alpha$-ultrafilters for which the question is still open) may not exist.

In this paper, we focus on free ultrafilters on $\omega$ defined by taking $X = \omega$ and $\mathcal{I}$ to be two different collections of subsets of natural numbers.
I. Basic facts and definitions.

It was noticed in [1] that for a given family \( \mathcal{I} \) the \( \mathcal{I} \)-ultrafilters are closed downward under the Rudin-Keisler ordering \( \leq_{RK} \) (recall that \( \mathcal{U} \leq_{RK} \mathcal{V} \) if there is a function \( f : \omega \rightarrow \omega \) whose Stone extension \( \beta f : \beta \omega \rightarrow \beta \omega \) maps \( \mathcal{V} \) on \( \mathcal{U} \), see [5]).

Replacing \( f \mid U \in \mathcal{I} \) with \( f \mid U \in \langle \mathcal{I} \rangle \) in the definition of \( \mathcal{I} \)-ultrafilter, where \( \langle \mathcal{I} \rangle \) is the ideal generated by \( \mathcal{I} \), gives the same concept (see [2]), i.e. \( \mathcal{I} \)-ultrafilters and \( \langle \mathcal{I} \rangle \)-ultrafilters coincide.

Obviously, if \( U \) is an \( \mathcal{I} \)-ultrafilter then \( U \cap \mathcal{I} \neq \emptyset \) (the converse is not true) and if \( \mathcal{I} \subseteq \mathcal{J} \) then every \( \mathcal{I} \)-ultrafilter is a \( \mathcal{J} \)-ultrafilter.

Lemma. If \( \mathcal{C} \) is a class of ultrafilters closed downward under \( \leq_{RK} \) and \( \mathcal{I} \) an ideal on \( \omega \) then the following are equivalent:

(i) There exists \( \mathcal{U} \in \mathcal{C} \) which is not an \( \mathcal{I} \)-ultrafilter
(ii) There exists \( \mathcal{V} \in \mathcal{C} \) which extends \( \mathcal{I}^* \), the dual filter of \( \mathcal{I} \)

Proof. Every ultrafilter extending \( \mathcal{I}^* \) is not an \( \mathcal{I} \)-ultrafilter, so (ii) implies (i) trivially. To prove (i) implies (ii) assume that \( \mathcal{U} \in \mathcal{C} \) is not an \( \mathcal{I} \)-ultrafilter. Hence there is a function \( f \in \omega^\omega \) such that \( (\forall A \in \mathcal{I}) f^{-1}[A] \notin \mathcal{U} \). Let \( \mathcal{V} = \{ V \subseteq \omega : f^{-1}[V] \in \mathcal{U} \} \). Obviously \( \mathcal{V} \) extends \( \mathcal{I}^* \) and \( \mathcal{V} \leq_{RK} \mathcal{U} \). Since \( \mathcal{C} \) is closed downward under \( \leq_{RK} \) and \( \mathcal{U} \in \mathcal{C} \) we get \( \mathcal{V} \in \mathcal{C} \). \( \Box \)

A set \( A \subseteq \omega \) with enumeration \( A = \{ a_n : n \in \omega \} \) is called almost thin if \( \limsup_n \frac{a_n}{a_{n+1}} < 1 \) and thin if \( \lim_n \frac{a_n}{a_{n+1}} = 0 \) (see [3]).

We will denote the ideal generated by finite and thin sets by \( \mathcal{A} \). The corresponding \( \mathcal{I} \)-ultrafilters will be called thin ultrafilters and almost thin ultrafilters respectively. We prove in Section I. that in fact these two classes of ultrafilters coincide, although the ideals \( \mathcal{I} \) and \( \mathcal{A} \) differ as the set \( \{ 2^n : n \in \omega \} \in \mathcal{A} \setminus \mathcal{I} \).

We show in Section II. the relation between thin ultrafilters and selective ultrafilters and the relation between thin ultrafilters and \( Q \)-points in Section III. Let us recall the definitions:

A free ultrafilter \( \mathcal{U} \) is called a selective ultrafilter if for all partitions of \( \omega \), \( \{ R_i : i \in \omega \} \), either for some \( i, R_i \in \mathcal{U} \), or \( \exists U \in \mathcal{U} \) such that \( (\forall i \in \omega) \| U \cap R_i \| \leq 1 \).
A free ultrafilter $\mathcal{U}$ is called a $Q$-point if for all partitions of $\omega$ consisting of finite sets, $\{Q_i : i \in \omega\}$, $\exists U \in \mathcal{U}$ such that $(\forall i \in \omega)$ $|U \cap Q_i| \leq 1$.

II. Thin ultrafilters and almost thin ultrafilters.

The existence of thin ultrafilters is independent of ZFC. It is easy to construct a thin ultrafilter if we assume the Continuum Hypothesis (in Section II. we prove that a strictly weaker assumption that selective ultrafilters exist is sufficient) and we prove in Section III. that every thin ultrafilter is a $Q$-point whose existence is not provable in ZFC (see [6]).

**Proposition 1.** (CH) There is a thin ultrafilter.

**Proof.** Enumerate $\omega = \{f_\alpha : \alpha < \omega_1\}$. By transfinite induction on $\alpha < \omega_1$ we construct countable filter bases $\mathcal{F}_\alpha$ satisfying

(i) $\mathcal{F}_0$ is the Fréchet filter

(ii) $\mathcal{F}_\alpha \subseteq \mathcal{F}_\beta$ whenever $\alpha \leq \beta$

(iii) $\mathcal{F}_\gamma = \bigcup_{\alpha < \gamma} \mathcal{F}_\alpha$ for $\gamma$ limit

(iv) $(\forall \alpha) (\exists F \in \mathcal{F}_{\alpha + 1}) f_\alpha[F] \in \mathcal{T}$

Suppose we have constructed $\mathcal{F}_\alpha$. If there exists a set $F \in \mathcal{F}_\alpha$ such that $f_\alpha[F] \in \mathcal{T}$ then put $\mathcal{F}_{\alpha + 1} = \mathcal{F}_\alpha$. If $f_\alpha[F] \notin \mathcal{T}$ (in particular, $f_\alpha[F]$ is infinite) for every $F \in \mathcal{F}_\alpha$ then enumerate $\mathcal{F}_\alpha = \{F_n : n \in \omega\}$ and construct by induction a set $U = \{u_n : n \in \omega\}$ which we extend the filter base by:

Choose arbitrary $u_0 \in F_0$ such that $f_\alpha(u_0) > 0$ (such an element exists since $f_\alpha[F_0]$ is infinite). If $u_0, u_1, \ldots, u_{k-1}$ are already known we can choose $u_k \in \bigcap_{i \leq k} F_i$ so that $f_\alpha(u_k) > k \cdot f_\alpha(u_{k-1})$.

It is obvious that $U \subseteq^* F_n$ for all $n \in \omega$. We can check immediately that $f_\alpha[U]$ is thin:

$$\lim_{n \to \infty} \frac{f_\alpha(u_n)}{f_\alpha(u_{n+1})} \leq \lim_{n \to \infty} \frac{f_\alpha(u_n)}{(n + 1) \cdot f_\alpha(u_n)} = \lim_{n \to \infty} \frac{1}{n + 1} = 0$$

To complete the induction step let $\mathcal{F}_{\alpha + 1}$ be the countable filter base generated by $\mathcal{F}_\alpha$ and the set $U$.

It is clear that any ultrafilter extending $\bigcup_{\alpha < \omega_1} \mathcal{F}_\alpha$ is thin. $\square$
Since $\mathcal{H} \subset \mathcal{A}$ every thin ultrafilter has to be almost thin. The following proposition states that the converse also holds true and the classes of almost thin and thin ultrafilters coincide.

**Proposition 2.** Every almost thin ultrafilter is a thin ultrafilter.

**Proof.** Because of the lemma from Section I. it suffices to prove that every almost thin ultrafilter contains a thin set. To that end assume that $\mathcal{U}$ is an almost thin ultrafilter and $U_0 \in \mathcal{U}$ is an almost thin set which is not thin with enumeration $U_0 = \{ u_n : n \in \omega \}$. Denote $\limsup_n \frac{u_n}{u_{n+1}} = q_0 < 1$. We may assume that the set of even numbers belongs to $\mathcal{U}$ (otherwise the role of even and odd numbers interchange).

Define $g : \omega \to \omega$ so that $g(u_n) = 2n$, $g[\omega \setminus U_0] = \{ 2n+1 : n \in \omega \}$. Since $\mathcal{U}$ is an almost thin ultrafilter there exists $U_1 \in \mathcal{U}$ such that $g[U_1]$ is almost thin. Let $U = U_0 \cap U_1 = \{ u_{n_k} : k \in \omega \}$. Almost thin sets are closed under subsets, therefore $g[U] = \{ g(u_{n_k}) : k \in \omega \} \subseteq g[U_1]$ is almost thin and $1 > \limsup_k \frac{g(u_{n_k})}{g(u_{n_k+1})} = \limsup_k \frac{2n_k}{2n_k+1}$.

We know that there is $n_0$ such that $(\forall n \geq n_0) \frac{u_n}{u_{n+1}} \leq \frac{q_0+1}{2}$ and that there is $k_0$ such that $(\forall k \geq k_0) n_k \geq n_0$. Hence for $k \geq k_0$ we have

$$\frac{u_{n_k}}{u_{n_k+1}} = \frac{u_{n_k}}{u_{n_k+1}} \cdot \frac{u_{n_k+1}}{u_{n_k+2}} \cdot \cdots \frac{u_{n_k+1-1}}{u_{n_k+2}} \leq \left( \frac{q_0+1}{2} \right)^{n_{k+1}-n_k}$$

It follows from $\limsup_k \frac{n_k}{n_{k+1}} < 1$ that $\lim_k (n_{k+1} - n_k) = +\infty$. Hence

$$\lim_{k \to \infty} \frac{u_{n_k}}{u_{n_{k+1}}} \leq \lim_{k \to \infty} \left( \frac{q_0+1}{2} \right)^{n_{k+1}-n_k} = 0$$

and the set $U \in \mathcal{U}$ is thin. □

**III. Thin ultrafilters and selective ultrafilters.**
**Proposition 3.** Every selective ultrafilter is thin.

**Proof.** Selective ultrafilters are minimal points in Rudin-Keisler ordering (hence downward closed under $\leq_{RK}$) and we may apply the lemma from Section I. It suffices to prove that there is no selective ultrafilter extending the dual filter of $\mathcal{T}$, i.e. that every selective ultrafilter contains a thin set.

**Claim:** Every selective ultrafilter contains a thin set.

Assume $U$ is a selective ultrafilter and consider the partition of $\omega$, $\{R_n : n \in \omega\}$, where $R_0 = \{0\}$ and $R_n = [n!, (n + 1)!)$ for $n > 0$. Since $U$ is selective there exists $U_0 \in U$ such that $|U_0 \cap R_n| \leq 1$ for every $n \in \omega$. Since $U$ is an ultrafilter either $A_0 = \bigcup\{R_n : n \text{ is even}\}$ or $A_1 = \bigcup\{R_n : n \text{ is odd}\}$ belongs to $U$. Without loss of generality, assume $A_0 \in U$. Enumerate $U = U_0 \cap A_0 \in \mathcal{U}$ as $\{u_k : k \in \omega\}$.

If $u_k \in [(2m_k)!, (2m_k + 1)!)$ then $u_{k+1} \geq (2m_k + 2)!$ and we have $\frac{u_k}{u_{k+1}} \leq \frac{(2m_k+1)!}{(2m_k+2)!} = \frac{1}{2m_k+2} \leq \frac{1}{2k+2}$. Hence $U$ is thin. $\square$

**Proposition 4.** (CH) Not every thin ultrafilter is selective.

**Proof.** Fix a partition $\{R_n : n \in \omega\}$ of $\omega$ into infinite sets and enumerate $\omega = \{f_\alpha : \alpha < \omega_1\}$. By transfinite induction on $\alpha < \omega_1$ we will construct countable filter bases $\mathcal{F}_\alpha$ satisfying

1. $\mathcal{F}_0$ is the countable filter base generated by Fréchet filter and $\{\omega \setminus R_n : n \in \omega\}$
2. $\mathcal{F}_\alpha \subseteq \mathcal{F}_\beta$ whenever $\alpha \leq \beta$
3. $\mathcal{F}_\gamma = \bigcup_{\alpha < \gamma} \mathcal{F}_\alpha$ for $\gamma$ limit
4. $(\forall \alpha) \ (\forall F \in \mathcal{F}_\alpha) \ \{n : |F \cap R_n| = \omega\}$ is infinite
5. $(\forall \alpha) \ (\exists F \in \mathcal{F}_{\alpha+1}) \ f_\alpha[F] \notin \mathcal{I}$

Suppose we know already $\mathcal{F}_\alpha$. If there is a set $F \in \mathcal{F}_\alpha$ such that $f_\alpha[F] \in \mathcal{I}$ then put $\mathcal{F}_{\alpha+1} = \mathcal{F}_\alpha$. If $(\forall F \in \mathcal{F}_\alpha) \ f_\alpha[F] \notin \mathcal{I}$ then one of the following cases occurs.

**Case A.** $(\forall F \in \mathcal{F}_\alpha) \ \{n : |f_\alpha[F \cap R_n]| = \omega\}$ is infinite

Fix an enumeration $\{F_m : m \in \omega\}$ of $\mathcal{F}_\alpha$. According to the assumption the set $I_m = \{n : |f_\alpha[\bigcap_{i \leq m} F_i \cap R_n]| = \omega\}$ is infinite for all $m \in \omega$. Let us enumerate the set $\{(m, n) : n \in I_m, m \in \omega\}$ as $\{p_k : k \in \omega\}$ so that each ordered pair $(m, n)$ occurs infinitely many times.
times. By induction we will construct a set \( U = \{ u_k : k \in \omega \} \) which we may add to \( \mathcal{F}_\alpha \).

Consider \( p_0 = \langle m, n \rangle \). Since \( |f_\alpha[\bigcap_{i \leq m} F_i \cap R_n]| = \omega \) we can choose \( u_0 \in \bigcap_{i \leq m} F_i \cap R_n \) such that \( f_\alpha(u_0) > 0 \). Suppose we know \( u_0, \ldots, u_{k-1} \) such that \( u_{i+1} > u_i, u_i \in \bigcap_{j \leq m} F_j \cap R_n \) where \( p_i = \langle m, n \rangle \) and \( f_\alpha(u_{i+1}) > (i + 1) \cdot f_\alpha(u_i) \) for \( i < k - 1 \). Consider \( p_k = \langle m, n \rangle \). Since \( \bigcap_{i \leq m} F_i \cap R_n \) and its image under \( f_\alpha \) is infinite we may choose \( u_k \in \bigcap_{i \leq m} F_i \cap R_n \) so that \( u_k > u_{k-1} \) and \( f_\alpha(u_k) > k \cdot f_\alpha(u_{k-1}) \).

It remains to verify that \( f_\alpha[U] \) is thin and \( \mathcal{F}_\alpha \cup \{ U \} \) generates a filter base satisfying (iv).

- \( f_\alpha[U] \) is thin:

\[
\lim_{k \to \infty} \frac{f_\alpha(u_k)}{f_\alpha(u_{k+1})} \leq \lim_{k \to \infty} \frac{f_\alpha(u_k)}{(k + 1) \cdot f_\alpha(u_k)} = \lim_{k \to \infty} \frac{1}{k + 1} = 0
\]

- \( (\forall F \in \mathcal{F}_\alpha) \{ n : |U \cap F \cap R_n| = \omega \} \) is infinite:

For every \( F \in \mathcal{F}_\alpha \) there is \( m_F \in \omega \) such that \( U \cap F \cap R_n \supseteq U \cap \bigcap_{i \leq m_F} F_i \cap R_n \) which is infinite whenever \( n \in I_{m_F} \) since \( U \cap \bigcap_{i \leq m_F} F_i \cap R_n \supseteq \{ u_k : p_k = \langle m_F, n \rangle \} \).

To complete the induction step let \( \mathcal{F}_{\alpha+1} \) be the countable filter base generated by \( \mathcal{F}_\alpha \) and \( U \).

Case B. \( (\exists F_0 \in \mathcal{F}_\alpha) \{ n : |f_\alpha[F_0 \cap R_n]| = \omega \} \) is finite

Enumerate \( \mathcal{F}_\alpha \setminus \{ F_0 \} = \{ F_m : m > 0 \} \). Since \( F_0 \) satisfies (iv) there is \( n_0 \in \omega \) such that \( |f_\alpha[F_0 \cap R_{n_0}]| < \omega \) and \( |F_0 \cap R_{n_0}| = \omega \). It follows that \( \exists z_0 \in \omega \) such that \( f_\alpha^{-1}\{\{z_0\}\} \cap F_0 \cap R_{n_0} \) is infinite. The set \( \bigcap_{i \leq m} F_i \) satisfies (iv) for any \( m \in \omega \) and for all but finitely many \( n \) the set \( f_\alpha[\bigcap_{i \leq m} F_i \cap R_n] \) is finite. So we may choose \( n_m > n_{m-1} \) such that \( f_\alpha[\bigcap_{i \leq m} F_i \cap R_{n_m}] \) is finite and \( \bigcap_{i \leq m} F_i \cap R_{n_m} \) infinite.

We find \( z_m \) such that \( f_\alpha^{-1}\{\{z_m\}\} \cap \bigcap_{i \leq m} F_i \cap R_{n_m} \) is infinite.

Consider the sequence \( \langle z_m : m \in \omega \rangle \). We can find a subsequence \( \langle z_{m_j} : j \in \omega \rangle \) which is either constant or satisfies \( z_{m_{j+1}} > (j + 1) \cdot z_{m_j} \) for every \( j \in \omega \). Set \( U = \bigcup_{j \in \omega} f_\alpha^{-1}\{\{z_{m_j}\}\} \). It is obvious that \( f_\alpha[U] \in \mathcal{F} \) and \( (\forall F \in \mathcal{F}_\alpha) \{ n : |U \cap F \cap R_n| = \omega \} \) is infinite.
To complete the induction step let $\mathcal{F}_{\alpha+1}$ be the countable filter base generated by $\mathcal{F}_\alpha$ and $U$.

Filter base $\mathcal{F} = \bigcup_{\alpha<\omega} \mathcal{F}_\alpha$ satisfies (iv) and every ultrafilter which extends $\mathcal{F}$ is thin because of condition (v).

Claim: Every filter satisfying (iv) may be extended to an ultrafilter satisfying (iv).
Whenever $\mathcal{F}$ is a filter satisfying (iv) and $A \subseteq \omega$ then either for every $F \in \mathcal{F}$ exist infinitely many $n \in \omega$ such that $|A \cap F \cap R_n| = \omega$, so the filter generated by $\mathcal{F}$ and $A$ satisfies (iv) or there is $F_0 \in \mathcal{F}$ such that for all but finitely many $n \in \omega$ we have $|A \cap F_0 \cap R_n| < \omega$. Then since for every $F \in \mathcal{F}$ exist infinitely many $n \in \omega$ for which $|F \cap F_0 \cap R_n| = \omega$ the filter generated by $\mathcal{F}$ and $\omega \setminus A$ satisfies (iv). Hence for every subset of $\omega$ we may extend $\mathcal{F}$ either by the set itself or its complement. Consequently, $\mathcal{F}$ may be extended to an ultrafilter satisfying (iv).

IV. Thin ultrafilters and $Q$-points.

Proposition 5. Every thin ultrafilter is a $Q$-point.

Proof. Let $\mathcal{U}$ be a thin ultrafilter and $\mathcal{Q} = \{Q_n : n \in \omega\}$ a partition of $\omega$ into finite sets. Enumerate $Q_n = \{q_n^i : i = 0, \ldots, k_n\}$ (where $k_n = |Q_n| - 1$).

Define a one-to-one function $f : \omega \rightarrow \omega$ in the following way:

$f(q_0^0) = 0$, $f(q_n^{n+1}) = (n + 2) \cdot \max\{f(q_k^n), k_{n+1}\}$ for $n \in \omega$ and $f(q_n^i) = f(q_n^0) + i$ for $i \leq k_n, n \in \omega$.

Notice that $f|_{Q_n}$ is strictly increasing for every $n$.

Since $\mathcal{U}$ is a thin ultrafilter there exists $U_0 \in \mathcal{U}$ such that $f[U_0] = \{v_m : m \in \omega\}$ is a thin set. Hence there is $m_0 \in \omega$ such that $\frac{v_m}{v_{m+1}} < \frac{1}{2}$ for every $m \geq m_0$.

Since $f$ is one-to-one and $\mathcal{Q}$ a partition of $\omega$ we find $K \subseteq \omega$ of size at most $m_0$ such that $\{f^{-1}(v_i) : i < m_0\} \subseteq \bigcup_{n \in K} Q_n$. The latter set is finite. Therefore $U = U_0 \setminus \bigcup_{n \in K} Q_n \in \mathcal{U}$.

Claim: $(\forall n \in \omega) |U \cap Q_n| \leq 1$

The intersection is clearly empty for $n \in K$. Assume for the contrary that for some $n \notin K$ there are two distinct elements $u_1, u_2 \in U \cap Q_n$,
\[ u_1 < u_2. \] Then \( f(u_1) = v_m \) for some \( m \geq m_0 \) and \( f(u_2) = v_n \) for some \( n \geq m + 1 \). We get \( \frac{v_m}{v_{m+1}} \geq \frac{f(u_1)}{f(u_2)} \geq \frac{f(v_n)}{f(q_n)} = \frac{f(q_n)^{1+M}}{(n+1)M+M} = \frac{n+1}{n+2} \) where \( M = \max\{f(q_{k_n-1}), k_n\} \). But \( \frac{n+1}{n+2} \geq \frac{1}{2} \), a contradiction. \( \square \)

Note that while proving that every selective ultrafilter contains a thin set we actually proved that every \( Q \)-point contains a thin set as the considered partition consists of finite sets. However, \( Q \)-points need not be thin ultrafilters.

**Proposition 6.** (CH) *Not every \( Q \)-point is a thin ultrafilter.*

**Proof.** Fix \( \{R_n : n \in \omega\} \) a partition of \( \omega \) into infinite sets. Let \( \{\mathcal{Q}_\alpha : \alpha < \omega_1\} \) be the list of all partitions of \( \omega \) into finite sets. By transfinite induction on \( \alpha < \omega_1 \) we will construct countable filter bases \( \mathcal{F}_\alpha \) so that the following are satisfied:

(i) \( \mathcal{F}_0 \) is the Fréchet filter

(ii) \( \mathcal{F}_\alpha \subseteq \mathcal{F}_\beta \) whenever \( \alpha < \beta \)

(iii) \( \mathcal{F}_\gamma = \bigcup_{\alpha < \gamma} \mathcal{F}_\alpha \) for \( \gamma \) limit

(iv) \( (\forall \alpha) \ (\forall F \in \mathcal{F}_\alpha) \ (\forall n) \ |F \cap R_n| = \omega \)

(v) \( (\forall \alpha) \ (\exists F \in \mathcal{F}_{\alpha+1}) \ (\forall Q \in \mathcal{Q}_\alpha) \ |F \cap Q| \leq 1 \)

Suppose we already know \( \mathcal{F}_\alpha \). If there is a set \( F \in \mathcal{F}_\alpha \) such that \( |F \cap Q| \leq 1 \) for each \( Q \in \mathcal{Q}_\alpha \) then let \( \mathcal{F}_{\alpha+1} = \mathcal{F}_\alpha \). If \( (\forall F \in \mathcal{F}_\alpha) \ (\exists Q \in \mathcal{Q}_\alpha) \ |F \cap Q| > 1 \), we construct by induction a set \( U \) compatible with \( \mathcal{F}_\alpha \) such that \( |U \cap Q| \leq 1 \) for each \( Q \in \mathcal{Q}_\alpha \).

Enumerate \( \mathcal{F}_\alpha = \{F_k : k \in \omega\} \), \( \mathcal{Q}_\alpha = \{Q_n : n \in \omega\} \) and list \( \{R_n : n \in \omega\} = \{M_k : k \in \omega\} \) so that each \( R_n \) is listed infinitely often. To start the construction of \( U \) choose \( u_0 \in M_0 \) arbitrarily. Since \( \bigcup \mathcal{Q}_\alpha = \omega \) there exists \( n_0 \in \omega \) such that \( u_0 \in Q_{n_0} \).

Suppose we know already \( u_0, \ldots, u_{k-1} \) and \( n_0, \ldots, n_{k-1} \) such that \( u_i \in Q_{n_i} \) for \( i < k \). Since \( \bigcup_{i<k} Q_{n_i} \) is finite and \( \bigcap_{i<k} F_i \cap M_k \) is infinite according to (iv) we may choose \( u_k \in (\bigcap_{i<k} F_i \cap M_k) \setminus \bigcup_{i<k} Q_{n_i} \). Let \( Q_{n_k} \) be the unique element of \( \mathcal{Q}_\alpha \) which contains \( u_k \).

Finally, let \( U = \{u_k : k \in \omega\} \). For every \( n \in \omega \) we have either \( U \cap Q_n = \{u_k\} \) (if \( n = n_k \)) or \( U \cap Q_n = \emptyset \). It remains to check that \( U \) is compatible with \( \mathcal{F}_\alpha \) and satisfies (iv). However, for every
\( F \in \mathcal{F}_\alpha \) and for every \( R_n \) there is \( n_F \in \omega \) such that \( U \cap F \cap R_n \supseteq \bigcap_{i<n_F} F_i \cap R_n \supseteq \{u_k : k \geq n_F, M_k = R_n\} \). The latter set is infinite. Hence the countable filter base generated by \( \mathcal{F}_\alpha \) and \( U \) satisfies (ii), (iv), (v) as required and it may be taken as \( \mathcal{F}_{\alpha+1} \).

Because of condition (v) every ultrafilter which extends the filter base \( \mathcal{F} = \bigcup_{\alpha<\omega_1} \mathcal{F}_\alpha \) is a \( Q \)-point. Let \( \mathcal{G} = \{\bigcup_{n \in M} R_n : \omega \setminus M \in \mathcal{T}\} \). Condition (iv) in induction assumption guarantees that \( \mathcal{F} \cup \mathcal{G} \) generates a free filter on \( \omega \). Every ultrafilter \( \mathcal{U} \) which extends \( \mathcal{F} \cup \mathcal{G} \) is a \( Q \)-point but not a thin ultrafilter because for every \( U \in \mathcal{U} \) \( f[U] \notin \mathcal{F} \) where \( f : \omega \to \omega \) is defined by \( f | R_n = n \). \( \square \)

REFERENCES