



# On sums and products of certain $\mathcal{I}$ -ultrafilters

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# $\mathcal{I}$ -ultrafilters

## Definition A. (Baumgartner)

Let  $\mathcal{I}$  be a family of subsets of a set  $X$  such that  $\mathcal{I}$  contains all singletons and is closed under subsets. An ultrafilter  $\mathcal{U}$  on  $\omega$  is called an  $\mathcal{I}$ -ultrafilter if for every  $F : \omega \rightarrow X$  there exists  $A \in \mathcal{U}$  such that  $F[A] \in \mathcal{I}$ .

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- if  $\mathcal{I} \subseteq \mathcal{J}$  then every  $\mathcal{I}$ -ultrafilter is a  $\mathcal{J}$ -ultrafilter

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- if  $\mathcal{I} \subseteq \mathcal{J}$  then every  $\mathcal{I}$ -ultrafilter is a  $\mathcal{J}$ -ultrafilter
- $\mathcal{I}$ -ultrafilters and  $\langle \mathcal{I} \rangle$ -ultrafilters coincide

where  $\langle \mathcal{I} \rangle$  is the ideal generated by  $\mathcal{I}$

# $\mathcal{I}$ -ultrafilters for $X = 2^\omega$

family  $\mathcal{I}$

converging sequences  
and finite sets

discrete sets

scattered sets

$\{A : \mu(\bar{A}) = 0\}$

nowhere dense sets

corresponding  $\mathcal{I}$ -ultrafilters

$P$ -points

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**Theorem (Shelah)**

There may be no nowhere dense ultrafilters.

# $\mathcal{I}$ -ultrafilters for $X = \omega$

## Theorem 1.

If  $\mathcal{I}$  is a maximal ideal and  $\chi(\mathcal{I}) = \mathfrak{c}$  then  $\mathcal{I}$ -ultrafilters exist in ZFC.

# $\mathcal{I}$ -ultrafilters for $X = \omega$

## Theorem 1.

If  $\mathcal{I}$  is a maximal ideal and  $\chi(\mathcal{I}) = \mathfrak{c}$  then  $\mathcal{I}$ -ultrafilters exist in ZFC.

$\mathcal{I} \subseteq \mathcal{P}(\omega)$  is a **dense ideal** if for every  $A \in [\omega]^\omega$  there exists an infinite set  $B \subseteq A$  such that  $B \in \mathcal{I}$ .

## Proposition 2.

If  $\mathcal{I}$  is not a dense ideal then there are no  $\mathcal{I}$ -ultrafilters.

# $\mathcal{I}$ -ultrafilters for $X = \omega$

Theorem 3.

( $\mathfrak{p} = \mathfrak{c}$ ) If  $\mathcal{I}$  is a dense ideal then  $\mathcal{I}$ -ultrafilters exist.

# $\mathcal{I}$ -ultrafilters for $X = \omega$

## Theorem 3.

( $\mathfrak{p} = \mathfrak{c}$ ) If  $\mathcal{I}$  is a dense ideal then  $\mathcal{I}$ -ultrafilters exist.

## Theorem 4.

If  $\mathcal{I}$  is a (dense)  $F_\sigma$ -ideal or analytic  $P$ -ideal then every selective ultrafilter is an  $\mathcal{I}$ -ultrafilter.

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If  $\mathcal{I}$  is a (dense)  $F_\sigma$ -ideal or analytic  $P$ -ideal then every selective ultrafilter is an  $\mathcal{I}$ -ultrafilter.

## Theorem 5.

Let  $\mathcal{I}$  be a  $P$ -ideal. If there is an  $\mathcal{I}$ -ultrafilter then there is an  $\mathcal{I}$ -ultrafilter that is not a  $P$ -point.

# Small subsets of $\omega$ (or $\mathbb{N}$ )

Let  $A$  be a subset of  $\omega$  with an increasing enumeration  $A = \{a_n : n \in \omega\}$ . We say that  $A$  is

**thin** if  $\lim_{n \rightarrow \infty} \frac{a_n}{a_{n+1}} = 0$

**almost thin** if  $\lim_{n \rightarrow \infty} \frac{a_n}{a_{n+1}} < 1$

**(SC)-set** if  $\lim_{n \rightarrow \infty} a_{n+1} - a_n = \infty$

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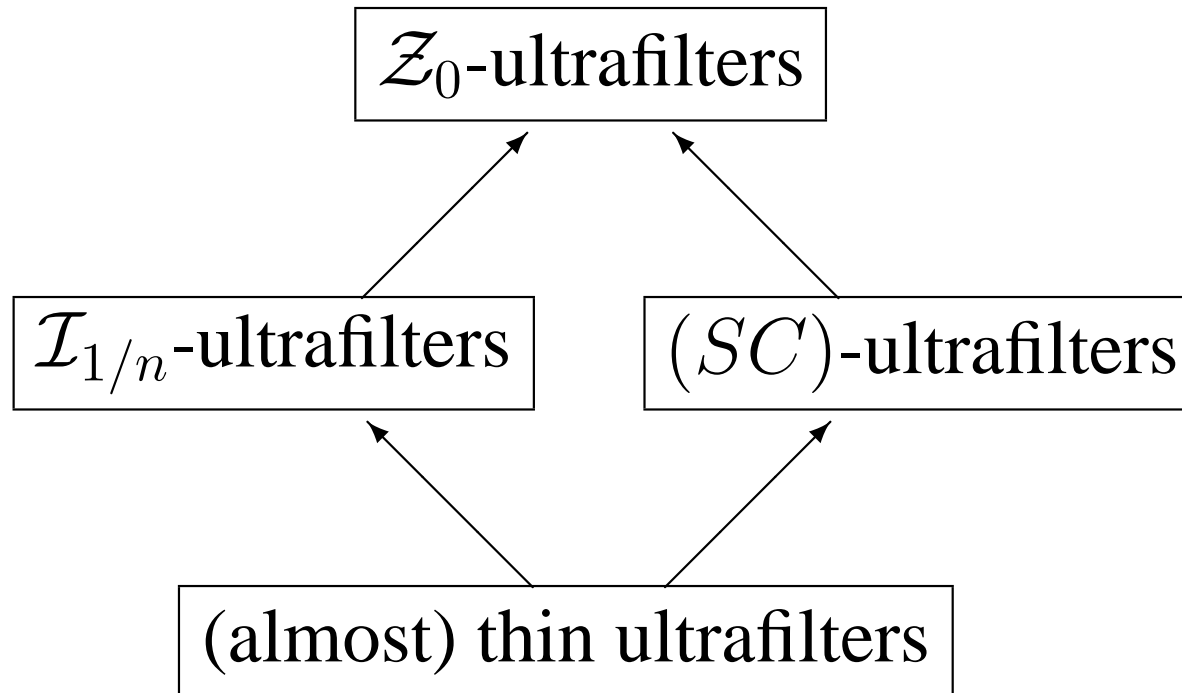
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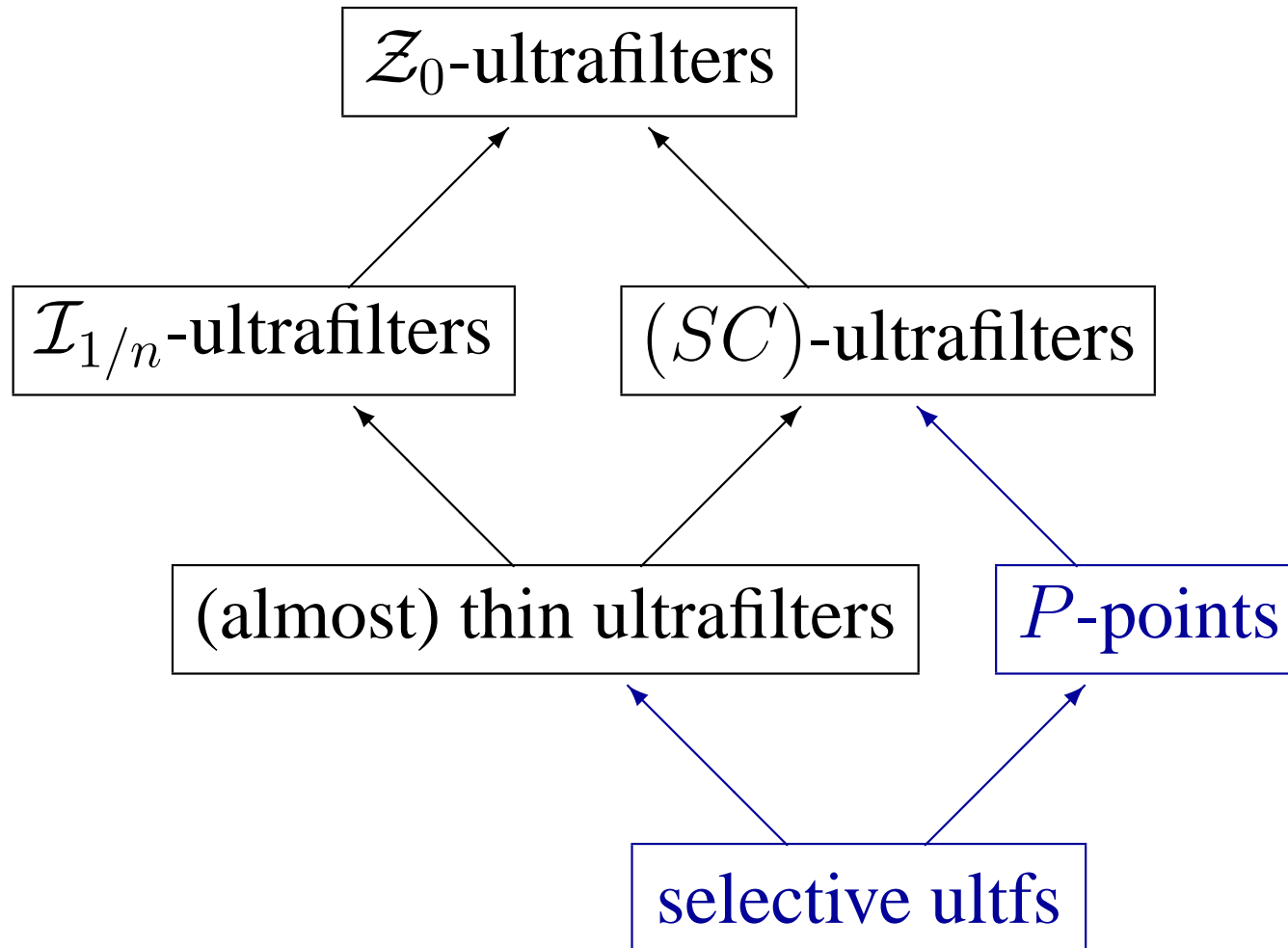
$\mathcal{I}_{1/n} = \{A \subseteq \mathbb{N} : \sum_{a \in A} \frac{1}{a} < \infty\}$

$\mathcal{Z}_0 = \{A \subseteq \mathbb{N} : \limsup_{n \rightarrow \infty} \frac{|A \cap n|}{n} = 0\}$

# Connections



# Connections



assuming  $\text{MA}_{\text{ctble}}$   
no arrow can be added

# Ultrafilter sums and products

## Definition B.

Let  $\mathcal{U}$  and  $\mathcal{V}_n$ ,  $n \in \omega$ , be ultrafilters on  $\omega$ .

$\mathcal{U}$ -sum of ultrafilters  $\mathcal{V}_n$ ,  $\sum_{\mathcal{U}} \langle \mathcal{V}_n : n \in \omega \rangle$ ,  
is an ultrafilter on  $\omega \times \omega$  defined by

$M \in \sum_{\mathcal{U}} \langle \mathcal{V}_n : n \in \omega \rangle$  if and only if  
 $\{n : \{m : \langle n, m \rangle \in A\} \in \mathcal{V}_n\} \in \mathcal{U}$ .

If  $\mathcal{V}_n = \mathcal{V}$  for every  $n \in \omega$  then we write

$\sum_{\mathcal{U}} \langle \mathcal{V}_n : n \in \omega \rangle = \mathcal{U} \cdot \mathcal{V}$  and the ultrafilter  $\mathcal{U} \cdot \mathcal{V}$   
is called the **product of ultrafilters  $\mathcal{U}$  and  $\mathcal{V}$** .

# Ultrafilter sums and products

## Definition C. (Baumgartner)

Let  $\mathcal{C}$  and  $\mathcal{D}$  be two classes of ultrafilters.

We say that  $\mathcal{C}$  is closed under  $\mathcal{D}$ -sums provided that whenever  $\{\mathcal{V}_n : n \in \omega\} \subseteq \mathcal{C}$  and  $\mathcal{U} \in \mathcal{D}$  then  $\sum_{\mathcal{U}} \langle \mathcal{V}_n : n \in \omega \rangle \in \mathcal{C}$ .

- If  $\mathcal{D}$  is a class of  $\mathcal{I}$ -ultrafilters then we say that  $\mathcal{C}$  is closed under  $\mathcal{I}$ -sums.

# Some results

## Theorem 6.

Let  $\mathcal{I}$  be a  $P$ -ideal on  $\omega$  (or  $\mathbb{N}$ ).

If  $\mathcal{U}$  is an  $\mathcal{I}$ -ultrafilter and  $\{n : \mathcal{V}_n \text{ is } \mathcal{I}\text{-ultrafilter}\} \in \mathcal{U}$   
then  $\sum_{\mathcal{U}} \langle \mathcal{V}_n : n \in \omega \rangle$  is an  $\mathcal{I}$ -ultrafilter.

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## Corollary 7.

$\mathcal{I}$ -ultrafilters are closed under  $\mathcal{I}$ -sums if  $\mathcal{I}$  is a  $P$ -ideal.

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$\mathcal{I}$ -ultrafilters are closed under  $\mathcal{I}$ -sums if  $\mathcal{I}$  is a  $P$ -ideal.

## Proposition 8.

For arbitrary  $\mathcal{U} \in \omega^*$  the ultrafilter  $\mathcal{U} \cdot \mathcal{U}$  is not an  $(SC)$ -ultrafilter.

# Weak $\mathcal{I}$ -ultrafilters

## Definition A. (Baumgartner)

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# Weak $\mathcal{I}$ -ultrafilters

## Definition D.

Let  $\mathcal{I}$  be a family of subsets of a set  $X$  such that  $\mathcal{I}$  contains all singletons and is closed under subsets. An ultrafilter  $\mathcal{U}$  on  $\omega$  is called an **weak  $\mathcal{I}$ -ultrafilter** if for every **finite-to-one**  $F : \omega \rightarrow X$  there exists  $A \in \mathcal{U}$  such that  $F[A] \in \mathcal{I}$ .

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An ultrafilter  $\mathcal{U}$  on  $\omega$  is called an **weak  $\mathcal{I}$ -ultrafilter** if for every **finite-to-one**  $F : \omega \rightarrow X$  there exists  $A \in \mathcal{U}$  such that  $F[A] \in \mathcal{I}$ .

An ultrafilter  $\mathcal{U}$  on  $\omega$  is called an  **$\mathcal{I}$ -close ultrafilter** if for every **one-to-one**  $F : \omega \rightarrow X$  there exists  $A \in \mathcal{U}$  such that  $F[A] \in \mathcal{I}$ .

# More results

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For arbitrary  $\mathcal{U} \in \omega^*$  the ultrafilter  $\mathcal{U} \cdot \mathcal{U}$  is not an  $(SC)$ -close ultrafilter.

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Let  $\mathcal{I}$  be a  $P$ -ideal on  $\omega$  (or  $\mathbb{N}$ ).

- If  $\{n : \mathcal{V}_n \text{ is } \mathcal{I}\text{-ultrafilter}\} \in \mathcal{U}$  and  $\mathcal{U}$  is an  $\mathcal{I}$ -ultrafilter then  $\sum_{\mathcal{U}} \langle \mathcal{V}_n : n \in \omega \rangle$  is an  $\mathcal{I}$ -ultrafilter.

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- If  $\{n : \mathcal{V}_n \text{ is weak } \mathcal{I}\text{-ultrafilter}\} \in \mathcal{U}$  then  $\sum_{\mathcal{U}} \langle \mathcal{V}_n : n \in \omega \rangle$  is a **weak**  $\mathcal{I}$ -ultrafilter.
- If  $\{n : \mathcal{V}_n \text{ is } \mathcal{I}\text{-close ultrafilter}\} \in \mathcal{U}$  then  $\sum_{\mathcal{U}} \langle \mathcal{V}_n : n \in \omega \rangle$  is an  **$\mathcal{I}$ -close** ultrafilter.

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## Theorem 10.

Assume  $\mathcal{I}$  is an ideal on  $\mathbb{N}$ .

If  $\sum_{\mathcal{U}} \langle \mathcal{V}_n : n \in \omega \rangle$  is an  $\mathcal{I}$ -ultrafilter  
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and  $\mathcal{U}$  is an  $\mathcal{I}$ -ultrafilter.

# More results

## Theorem 11.

Assume  $\mathcal{I}$  is an ideal on  $\mathbb{N}$  and there exists  $g : \mathbb{N} \rightarrow \mathbb{N}$  with  $g(n) > n$  for every  $n \in \mathbb{N}$  and  $A \notin \mathcal{I}$  implies  $g[A] \notin \mathcal{I}$  for every  $A \subseteq \mathbb{N}$ .

If  $\sum_{\mathcal{U}} \langle \mathcal{V}_n : n \in \omega \rangle$  is a **weak**  $\mathcal{I}$ -ultrafilter then  $\{n : \mathcal{V}_n \text{ is a **weak** } \mathcal{I}\text{-ultrafilter}\} \in \mathcal{U}$ .

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## Theorem 12.

Assume  $\mathcal{I}$  is an ideal on  $\mathbb{N}$  and  $A \notin \mathcal{I}$  implies  $A + 1 \notin \mathcal{I}$  and  $2A \notin \mathcal{I}$  for each  $A \subseteq \mathbb{N}$ .

If  $\sum_{\mathcal{U}} \langle \mathcal{V}_n : n \in \omega \rangle$  is an  **$\mathcal{I}$ -close** ultrafilter then  $\{n : \mathcal{V}_n \text{ is an **\mathcal{I}**$ -close ultrafilter}\} \in \mathcal{U}.

# A "problematic" ideal

Van der Waerden ideal is the family

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- If  $\sum_{\mathcal{U}} \langle \mathcal{V}_n : n \in \omega \rangle$  is an  $\mathcal{W}$ -close ultrafilter then  $\{n : \mathcal{V}_n \text{ is an } \mathcal{W}\text{-close ultrafilter}\} \in \mathcal{U}.$

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- If  $\sum_{\mathcal{U}} \langle \mathcal{V}_n : n \in \omega \rangle$  is an  $\mathcal{W}$ -close ultrafilter then  $\{n : \mathcal{V}_n \text{ is an } \mathcal{W}\text{-close ultrafilter}\} \in \mathcal{U}.$

Question:

Are  $\mathcal{W}$ -ultrafilters closed under  $\mathcal{W}$ -sums or products?

# References

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Flašková, J., Ultrafilters and small sets, Ph.D. Thesis, Charles University, Prague, 2006.