

Uniformity of the van der Waerden ideal

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AP-sets and van der Waerden theorem

Definition.

A set $A \subseteq \mathbb{N}$ is called an **AP-set** if it contains arbitrary long arithmetic progressions.

Van der Waerden Theorem.

If an AP-set is partitioned into finitely many pieces then at least one of them is again an AP-set.

Sets which are not AP-sets form a proper ideal on \mathbb{N}
— van der Waerden ideal denoted by \mathcal{W}

Van der Waerden ideal \mathcal{W}

The van der Waerden ideal \mathcal{W} is

- **a tall ideal** — because every infinite $A \subseteq \mathbb{N}$ contains an infinite subset with no arithmetic progressions of length 3
- **F_σ -ideal** — because $\mathcal{W} = \bigcup_{n \in \mathbb{N}} \mathcal{W}_n$ where

$$\mathcal{W}_n = \{A \subseteq \mathbb{N} : A \text{ contains no a. p. of length } n\}$$
- **not a P -ideal** — consider for example the sets

$$A_k = \{2^n + k : n \in \omega\} \text{ for } k \in \omega$$

Van der Waerden ideal \mathcal{W}

Szemerédi Theorem.

$$\mathcal{W} \subseteq \mathcal{Z} \quad \text{where } \mathcal{Z} = \{A \subseteq \mathbb{N} : \limsup_{n \rightarrow \infty} \frac{|A \cap n|}{n} = 0\}$$

Erdős Conjecture.

$$\mathcal{W} \subseteq \mathcal{I}_{1/n} \quad \text{where } \mathcal{I}_{1/n} = \{A \subseteq \mathbb{N} : \sum_{a \in A} \frac{1}{a} < \infty\}$$

Cardinal invariants of ideals on ω

Definition (Hernández-Hernández, Hrušák).

Let \mathcal{I} be a tall ideal on ω containing the ideal of finite sets. Define the following cardinals associated with \mathcal{I} :

$$\text{add}^*(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge (\forall I \in \mathcal{I})(\exists \mathbf{A} \in \mathcal{A})(\mathbf{A} \not\subseteq^* I)\}$$

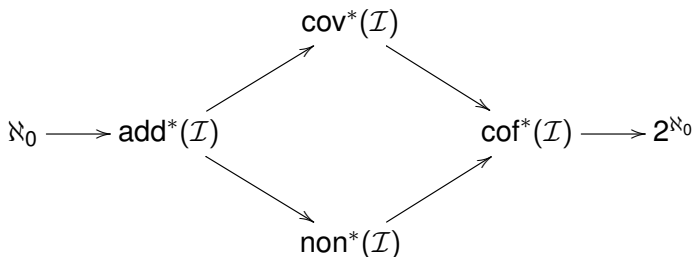
$$\text{cov}^*(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge (\forall X \in [\omega]^{\aleph_0})(\exists \mathbf{A} \in \mathcal{A})(|\mathbf{A} \cap X| = \aleph_0)\}$$

$$\text{cof}^*(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge (\forall I \in \mathcal{I})(\exists \mathbf{A} \in \mathcal{A})(I \subseteq^* \mathbf{A})\}$$

$$\text{non}^*(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq [\omega]^{\aleph_0} \wedge (\forall I \in \mathcal{I})(\exists \mathbf{A} \in \mathcal{A})(|\mathbf{A} \cap I| < \aleph_0)\}$$

Cardinal invariants of ideals on ω

The inequalities holding among these cardinals are summarized in the following diagram:



Lower bound for $\text{non}^*(\mathcal{W})$

Theorem 1. $\text{non}^*(\mathcal{W}) \geq \text{cov}(\mathcal{M})$

Sketch of the proof:

1. $\text{cov}(\mathcal{M}) = \min\{|\mathcal{F}| : \mathcal{F} \text{ s.t. } (\forall g \in \omega^\omega)(\exists f \in \mathcal{F})(\forall^\infty n)f(n) \neq g(n)\}$
2. $\omega = \bigcup_{n \in \omega} I_n$ where $I_n = [2^n; 2^{n+1})$
3. For every $A \in \mathcal{A} \subseteq [\omega]^{\aleph_0}$ define $f_A : \omega \rightarrow \omega$

$$f_A = \begin{cases} \min(I_n \cap A) & \text{if } I_n \cap A \neq \emptyset \\ \text{undefined} & \text{otherwise} \end{cases}$$

4. If $|\mathcal{A}| < \text{cov}(\mathcal{M})$ then $(\exists g \in \omega^\omega)(\forall A \in \mathcal{A})(\exists^\infty n)f_A(n) = g(n)$
5. $I = \{g(n) : n \in \omega\} \in \mathcal{W}$ and $|I \cap A| = \aleph_0$ for every $A \in \mathcal{A}$

Upper bound for $\text{non}^*(\mathcal{W})$

Theorem 2. $\text{non}^*(\mathcal{W}) \leq \mathfrak{r}$

Sketch of the proof:

1. Identify \mathbb{N} with $\Delta = \{\langle m, n \rangle \in \omega \times \omega : n \leq m\}$
2. Let \mathcal{R} be a hereditarily reaping family of size \mathfrak{r}
3. For $R \in \mathcal{R}$ and $n \in \omega$ put

$$A_{R,n} = \{\langle m, n \rangle \in \Delta : m \in R, m \geq n\}$$

4. Show that for every $I \in \mathcal{I}$ there exists $R \in \mathcal{R}$, $k \in \mathbb{N}$ with

$$|A_{R,k} \cap I| < \aleph_0$$

More bounds for $\text{non}^*(\mathcal{W})$

If $\mathcal{I} \subseteq \mathcal{J}$ are two tall ideals on \mathbb{N} then $\text{non}^*(\mathcal{I}) \leq \text{non}^*(\mathcal{J})$.

Theorem (Hernández-Hernández, Hrušák).

$$\text{non}^*(\mathcal{Z}) \leq \max\{\mathfrak{d}, \text{non}(\mathcal{N})\}$$

Corollary 3. $\text{non}^*(\mathcal{W}) \leq \max\{\mathfrak{d}, \text{non}(\mathcal{N})\}$

Questions about upper bounds for $\text{non}^*(\mathcal{W})$

Question A. Does $\text{non}^*(\mathcal{W}) \leq \mathfrak{d}$ hold in ZFC?

NO. In the dual Hechler model $\mathfrak{d} = \aleph_1$ and $\text{non}^*(\mathcal{W}) = \aleph_2$.

Question B. Does $\text{non}^*(\mathcal{W}) \leq \text{non}(\mathcal{N})$ hold in ZFC?

VERY LIKELY YES. Because $\text{non}^*(\mathcal{I}_{1/n}) \leq \text{non}(\mathcal{N})$ (H.-H., Hr.) and $\text{non}^*(\mathcal{W}) \leq \text{non}^*(\mathcal{I}_{1/n})$ if Erdős Conjecture is true.

Questions about lower bounds for $\text{non}^*(\mathcal{W})$

Theorem (Hernández-Hernández, Hrušák).

$$\text{non}^*(\mathcal{Z}) \geq \min\{\mathfrak{d}, \text{cov}(\mathcal{N})\}$$

Question C. Does $\text{non}^*(\mathcal{W}) \geq \min\{\mathfrak{d}, \text{cov}(\mathcal{N})\}$ hold in ZFC?

What about other small cardinals — \mathfrak{b} , \mathfrak{h} , \mathfrak{p} etc.?

Additivity number of \mathcal{W}

The additivity number of an ideal \mathcal{I} is uncountable

if and only if

the ideal \mathcal{I} is a P -ideal.

Observation 4.

$$\text{add}^*(\mathcal{W}) = \aleph_0$$

Cofinality number of \mathcal{W}

Proposition 5. $\text{cof}^*(\mathcal{W}) = 2^{\aleph_0}$

Sketch of the proof:

1. Show that there exists a perfect set $P \subseteq {}^\omega\omega$ such that every $f \in P$ satisfies $f(n+1) > 2f(n)$ for every $n \in \omega$ and whenever $f_0, f_1, \dots, f_k \in P$ are distinct, there exist infinitely many $n \in \omega$ such that $\{f_0(n), f_1(n), \dots, f_k(n)\}$ is a set of $k+1$ successive integers.
2. $A_f = \{f(n) : n \in \omega\} \in \mathcal{W}$ for every $f \in P$
3. $\{f \in P : A_f \subseteq^* B\}$ is finite for every $B \in \mathcal{W}$

Covering number of \mathcal{W}

Theorem (Hernández-Hernández, Hrušák).

$$\text{cov}^*(\mathcal{Z}) \geq \min\{\mathfrak{b}, \text{cov}(\mathcal{N})\}$$

Corollary 6. $\text{cov}^*(\mathcal{W}) \geq \min\{\mathfrak{b}, \text{cov}(\mathcal{N})\}$

Conjectures.

1. $\text{cov}^*(\mathcal{W}) \leq \text{non}(\mathcal{M})$
2. $\text{cov}^*(\mathcal{W}) \geq \mathfrak{s}$
3. $\text{cov}^*(\mathcal{W}) \leq \max\{\mathfrak{b}, \text{non}(\mathcal{N})\}$

and many more...

References

F. Hernández-Hernández, M. Hrušák, Cardinal invariants of analytic P -ideals, *Canad. J. Math.* **59**(3), 575 – 595, 2007.

D. Meza Alcántara, Ideals and filters on countable sets, *Ph.D. thesis*, 2009.