Characterizing forbidden disjoint pairs for 2-factor of 2-connected graphs

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Joint work with Přemysl Holub, Zdeněk Ryjáček, Petr Vrána and Liming Xiong

Terminology

All graphs are finite, simple and undirected.
$\alpha(G)$: the independence number of G,
$\kappa(G)$: the connectivity of G,
$G[X]$: the subgraph of G induced by X,
$N_G(x)$: the neighborhood of x in G,
$N_C(S)$: the neighborhood of S in the subgraph C of G.

Terminology

- A graph G is hamiltonian: G has a spanning cycle.
- A 2-factor of a graph is a spanning subgraph whose components are cycles.
- The Ramsey number $r(k, l)$ is defined as the smallest integer such that every graph on $r(k, l)$ vertices contains either a clique of k vertices or an independent set of l vertices.

Terminology

- Let $\mathcal{H}$ be a set of graphs. A graph G is said to be $\mathcal{H}$-free if G does not contain F as an induced subgraph for all F in $\mathcal{H}$, and we call each graph F of $\mathcal{H}$ a forbidden subgraph.
- Let $\mathcal{H}$ be a set of two graphs. Then we call $\mathcal{H}$ a forbidden disjoint pair. Specially, if each of $\mathcal{H}$ is connected, then we call $\mathcal{H}$ a forbidden pair.

Terminology

- We use $H_1 \cup H_2$ to denote the disjoint union of two vertex-disjoint graphs H_1 and H_2.
- We use $H_1 \cup H_2$-free instead of $(H_1 \cup H_2)$-free.
- So, $H_1 \cup H_2$-free means we forb $H_1 \cup H_2$ as an induced subgraph; it does not mean forbidding H_1 and/or H_2.

Motivation

Theorem 1.1 (Bedrossian, 1991)

Let $\{R, S\}$ be a forbidden pair such that neither R nor S is an induced subgraph of P_3. Then every 2-connected graph $\mathcal{H}$-free graph G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and S is an induced subgraph of $P_6, N_{1,1,1}$, or $B_{1,2}$.
[a]

[a] Bedrossian, Forbidden subgraph and minimum degree conditions for Hamiltonicity, Ph.D. Thesis, Memphis State University, 1991

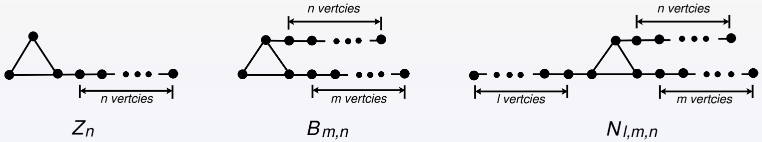

Figure: $Z_n, B_{m,n}$ and $N_{l,m,n}$

Motivation

Because some non-Hamiltonian 2-connected graph with small order, Faudree and Gould have characterized all forbidden pairs for hamiltonicity of 2-connected graphs with sufficiently large order.

Theorem 1.2 (Faudree and Gould, DM, 1997)

Let $\{R, S\}$ be a forbidden pair such that neither R nor S is an induced subgraph of P_3. Then every 2-connected graph $\mathcal{H}$-free graph with sufficiently large order is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and S is an induced subgraph of $P_6, N_{1,1,1}, B_{1,2}$ or Z_3. [a]

[a] R. J. Faudree and R. J. Gould, Characterizing forbidden pairs for Hamiltonian properties, Discrete Math. 173(1997)45-60

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Motivation

Faudree, Faudree and Ryjáček characterized all forbidden pairs for 2-factor of 2-connected graphs with sufficiently larger order.

Theorem 1.3 (Faudree, Faudree and Ryjáček, DM, 2008)

Let $\{R, S\}$ be a forbidden pair such that neither R nor S is an induced subgraph of P_3. Then every 2-connected graph $\{R, S\}$-free graph with sufficiently large order has a 2-factor (up to symmetry) $R = K_{1,3}$ and S is an induced subgraph of $P_7, B_{1,4}, N_{1,1,3}$, or $R = K_{1,4}$ and $S = P_4$. [a]

[a] J. R. Faudree, R. J. Faudree and Z. Ryjáček, Forbidden subgraphs that imply 2-factors, Discrete Math. 308(2008)1571-1582.

Motivation

Li and Vrána characterized all forbidden disjoint pairs for hamiltonicity of 2-connected graphs.

Theorem 1.4 (Li and Vrána, JGT, 2017)

Let $\{R, S\}$ be a forbidden disjoint pair such that neither R nor S is an induced subgraph of P_3 or $3K_1$. Then there is an integer n_0 such that every 2-connected $\{R, S\}$-free graph of order at least n_0 is hamiltonian, if and only if, one of the following is true (up to symmetry):

- $R = K_{1,3}$ and S is an induced subgraph of P_6, Z_3, $N_{0,1,2}$, $N_{1,1,1}$, $K_1 \cup Z_2$, $K_2 \cup Z_1$, or $K_3 \cup P_4$;
- $R = K_{1,k}$ with $k \geq 4$ and S is an induced subgraph of $2K_1 \cup K_2$;
- $R = kK_1$ with $k \geq 4$ and S is an induced subgraph of L_l with $l \geq 3$, or $2K_1 \cup K_{l'}$ with $l' \geq 2$.

[a]

Our results

We consider the following problem.

Problem: Determine the set H of forbidden disjoint pairs $\mathcal{H}$ such that every 2-connected $\mathcal{H}$-free graph G with sufficiently larger order has a 2-factor.

We first characterized forbidden subgraphs to guarantee a 2-connected graph with sufficiently large order has a 2-factor.

Theorem 2.1 (Holub, Ryjáček, Vrána, Wang and Xiong, 2017^+)

Let H be a graph. Then every 2-connected H-free graph G with order at least $r(4,28)$ has a 2-factor if and only if H is an induced subgraph of P_3 or $4K_1$.

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Let $\{R, S\}$ be a forbidden disjoint pair such that neither R nor S is an induced subgraph of P_3 or $4K_1$. Then there is an integer n_0 such that every 2-connected $\{R, S\}$-free graph of order at least n_0 has a 2-factor if and only if (up to symmetry):

- $R = K_{1,3}$ and S is an induced subgraph of P_7, $B_{1,4}$, $N_{1,1,3}$, $\triangle \cup Z_1$, $Z_1 \cup P_4$, or $Z_4 \cup K_1$;
- $R = K_{1,4}$ and S is an induced subgraph of $P_3 \cup 2K_1$;
- $R = K_{1,r}$ with $r \geq 5$ and S is an induced subgraph of $P_3 \cup K_1$ or $3K_1 \cup K_2$;
- $R = kK_1$ with $k \geq 5$ and S is an induced subgraph of L_l with $l \geq 3$, or $3K_1 \cup K_{l'}$ with $l' \geq 2$.

Sketch the proof of the pair $\{K_{1,4}, P_3 \cup 2K_1\}$

Next, we will sketch the proof of one pair $\{K_{1,4}, P_3 \cup 2K_1\}$. Before sketching the proof of the pair $\{K_{1,4}, P_3 \cup 2K_1\}$, we need the following theorems and lemma.

Theorem 2.3 (Chv$\acute{a}$tal and Erdös, 1972, DM)

Let G be a graph on at least three vertices with independent number α and connectivity κ, where $\alpha \leq \kappa$. Then G is hamiltonian.

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Theorem 2.4 (Aldred, Egawa, Fujisawa, Ota and Saito, 2011, JGT)

For $n \geq 4$, every $(n-1)$-connected $K_{1,n}$-free graph contains a 2-factor.

Lemma 2.5 (Holub, Ryjáček, Vrána, Wang and Xiong, 2017^+)

Let G be a 2-connected kK_1-free graph such that G can be partitioned into two sets X and Y satisfying the following:

(1) X contains a clique C such that each vertex of X has at least $k+6$ neighbors in C;

(2) $\alpha(G[Y]) \leq 2$.

Then G has a 2-factor.

Now, we will prove that every 2-connected $\{K_{1,4}, P_3 \cup 2K_1\}$-free graph with order at least $r(5, r(4, 28) + 4)$ has a 2-factor.

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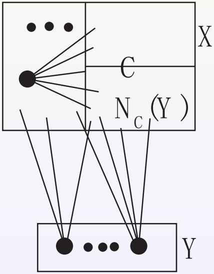
Sketch the proof of the pair $\{K_{1,4}, P_3 \cup 2K_1\}$

- We first claim that G is $5K_1$-free.
- Since G is $5K_1$-free and $n(G) \geq r(5, r(4,28)+4)$, G contains a clique C of size $r(4,28)+4$. Set $X = \{x \in V(G) : d_C(x) \geq 16\}$ and $Y = V(G) \backslash X$, clearly $C \subseteq X$. Then $\kappa(G[X]) \geq 16$ and $\alpha(G[X]) \leq \alpha(G) \leq 4$, $G[X]$ is hamiltonian by Chvátal-Erdös Theorem.
- We can prove that $\alpha(G[Y]) \leq 3$. If $\alpha(G[Y]) \leq 2$, then by Lemma 2.5, we are done. Hence we assume that $\alpha(G[Y]) = 3$.



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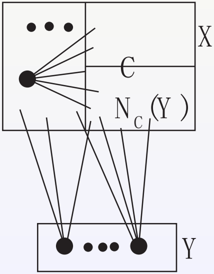

In the following, we need distinguish two cases.

Case 1. $G[Y]$ is disconnected.

By $\alpha(G[Y]) = 3$, $G[Y]$ has at most three components. We can prove that $G[Y]$ has exactly three components Y_1, Y_2, Y_3 and each $G[Y_i]$ is a clique.

For $i \in \{1, 2, 3\}$, let $P(z_1^i, z_2^i) = z_1^i P^i z_2^i$ denote a path of $G[X \cup Y_i]$ such that $V(P^i) = V(Y_i)$ and $z_1^i, z_2^i \in X$, and let $X_1 = V(P(z_1^1, z_2^1) \cap P(z_1^2, z_2^2) \cap P(z_1^3, z_2^3)$.

We need to consider the following six basic cases because any other cases can be changed to some basic case.

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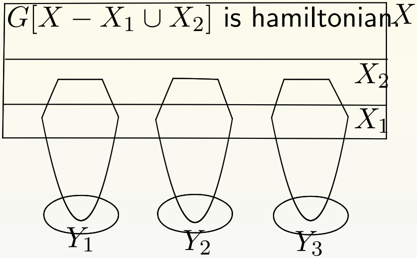
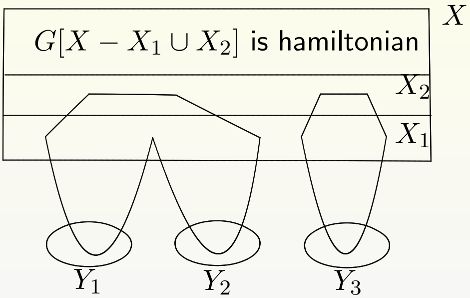
Subcase 1.1. $|X_1| = 6$.



Since $\kappa(G[X - X_1 \cup X_2]) \geq \kappa(G[X]) - |X_1 \cup X_2| \geq 16 - 12 = 4$ and $\alpha(G[X - X_1 \cup X_2]) \leq \alpha(G) \leq 4$, $G[X - X_1 \cup X_2]$ is hamiltonian by Chv$\acute{a}$tal-Erdös Theorem. Then G has a 2-factor with exactly four components.

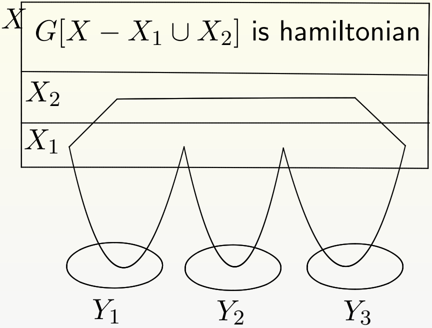
Sketch the proof of the pair $\{K_{1,4}, P_3 \cup 2K_1\}$

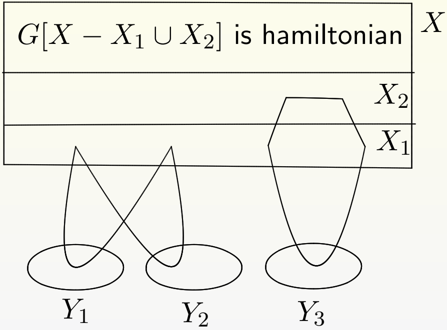
Subcase 1.2. $|X_1| = 5$.



Since $\kappa(G[X - X_1 \cup X_2]) \geq \kappa(G[X]) - |X_1 \cup X_2| \geq 16 - 9 = 7$ and $\alpha(G[X - X_1 \cup X_2]) \leq \alpha(G) \leq 4$, $G[X - X_1 \cup X_2]$ is hamiltonian by Chv$\acute{a}$tal-Erdös Theorem. Then G has a 2-factor with exactly three components

Subcase 1.3. $|X_1| = 4$.





Since $\kappa(G[X - X_1 \cup X_2]) \geq \kappa(G[X]) - |X_1 \cup X_2| \geq 16 - 9 = 7$ and $\alpha(G[X - X_1 \cup X_2]) \leq \alpha(G) \leq 4$, $G[X - X_1 \cup X_2]$ is hamiltonian by Chvátal-Erdös Theorem, and thus G has a 2-factor.

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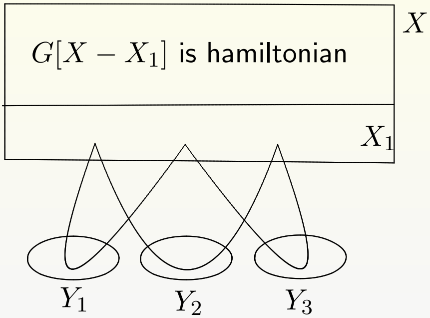
Subcase 1.4. $|X_1| = 3$.



Since $\kappa(G[X - X_1]) \geq \kappa(G[X]) - |X_1| \geq 16 - 3 = 13$ and $\alpha(G[X - X_1]) \leq \alpha(G) \leq 4$, $G[X - X_1]$ is hamiltonian by Chv$\acute{a}$tal-Erdös Theorem. Then G has a 2-factor with exactly two components.

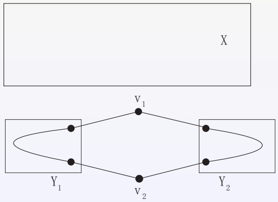
Sketch the proof of the pair $\{K_{1,4}, P_3 \cup 2K_1\}$

Case 2. $G[Y]$ is connected.

If $\kappa(G[Y]) \geq 3$, then $G[Y]$ is 3-connected $K_{1,4}$-free, $G[Y]$ has a 2-factor by Theorem 2.4, and thus G has a 2-factor.

Subcase 2.1. $\kappa(G[Y]) = 2$.

Let $\{v_1, v_2\}$ be a vertex-cut of $G[Y]$. By $\alpha(G[Y]) = 3$, $G[Y] - \{v_1, v_2\}$ has at most three components. If $G[Y] - \{v_1, v_2\}$ has exactly two components Y_1, Y_2, then we can prove that both $G[Y_1]$ and $G[Y_2]$ are cliques. Hence $G[Y]$ is hamiltonian, and thus G has a 2-factor with exactly two components.



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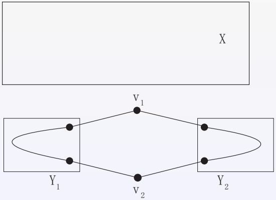
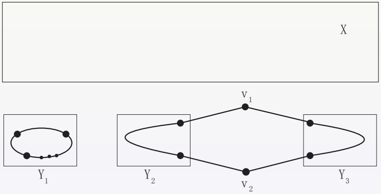


Sketch the proof of the pair $\{K_{1,4}, P_3 \cup 2K_1\}$

Hence we assume that $G[Y] - \{v_1, v_2\}$ has exactly three components Y_1, Y_2, Y_3. Then by $\alpha(G[Y]) = 3$, each $G[Y_i]$ is a clique.

- If some Y_i has at least three vertices, then $G[Y_i]$ has hamiltonian cycle. Thus we delete the set Y_i and the resulting graph of $G[Y] - Y_i$ has a hamiltonian cycle, and thus G has a 2-factor with exactly three components.



- Hence we assume that $|Y_i| \leq 2$ for $i \in \{1, 2, 3\}$.

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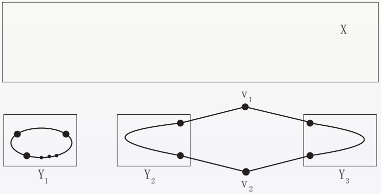

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Sketch the proof of the pair $\{K_{1,4}, P_3 \cup 2K_1\}$

- We claim that v_i is adjacent to every vertex in $Y_1 \cup Y_2 \cup Y_3$ for $i \in \{1, 2\}$. Since G is 2-connected, there exists two disjoint edges x_1y_2, x_2y_2 such that $x_1, x_2 \in X$ and $y_1, y_2 \in \{v_1, v_2\} \cup Y_1 \cup Y_2$.

We need to consider the following three basic cases because any other cases can be changed to some basic case.

Subcase 2.1.1. y_1 and y_2 are in the same Y_i.

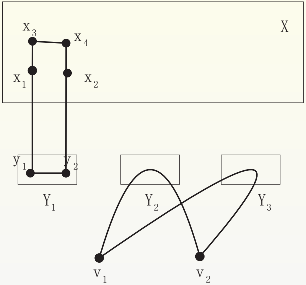

Figure: G has a 2-factor with exactly three components.

Since $\kappa(G[X - \{x_1, x_2, x_3, x_4\}]) \geq \kappa(G[X]) - |\{x_1, x_2, x_3, x_4\}| \geq 16 - 4 = 12$ and $\alpha(G[X - \{x_1, x_2, x_3, x_4\}]) \leq \alpha(G) \leq 4$, $G[X - \{x_1, x_2, x_3, x_4\}]$ is hamiltonian by Chvátal-Erdös Theorem. Then G has a 2-factor with exactly three components.

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Subcase 2.1.2. y_1 and y_2 are in different Y_i.

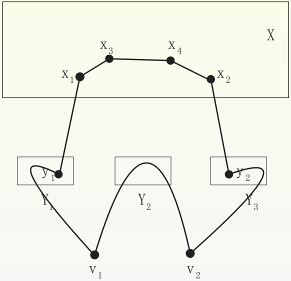

Figure: G has a 2-factor with exactly two components.

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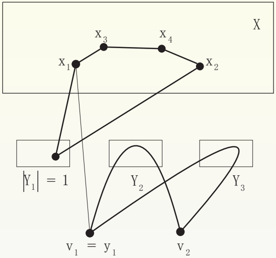
Subcase 2.1.3. $\{y_1, y_2\} \cap \{v_1, v_2\} \neq \emptyset$.



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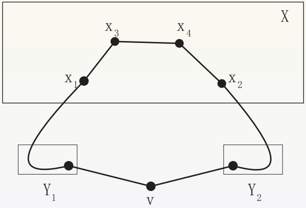
Sketch the proof of the pair $\{K_{1,4}, P_3 \cup 2K_1\}$

Subcase 2.2. $\kappa(G[Y]) = 1$.

Let v be a cut vertex of $G[Y]$. Then by $\alpha(G[Y]) = 3$, $G[Y] - v$ has at most three components.

If $G[Y] - v$ has exactly two components Y_1, Y_2, then we can prove that both $G[Y_1]$ and $G[Y_2]$ are cliques.



Since $\kappa(G[X - \{x_1, x_2, x_3, x_4\}]) \geq \kappa(G[X]) - |\{x_1, x_2, x_3, x_4\}| \geq 16 - 4 = 12$ and $\alpha(G[X - \{x_1, x_2, x_3, x_4\}]) \leq \alpha(G) \leq 4$, $G[X - \{x_1, x_2, x_3, x_4\}]$ is hamiltonian by Chvátal-Erdös Theorem. Then G has a 2-factor with exactly two components.

Hence we assume that $G[Y]-v$ has exactly three components Y_1, Y_2 and Y_3.

- We claim that each $G[Y_j]$ is a clique and v is adjacent to every vertex in $Y_1 \cup Y_2 \cup Y_3$.
- If $G[Y \cup N_C(Y)]$ is 2-connected, then there exists a subgraph C' of clique C such that $G[Y \cup N_C(Y) \cup C']$ is $4K_1$-free and $|G[Y \cup N_C(Y) \cup C']| \geq r(4, 28)$, and thus $G[Y \cup N_C(Y) \cup C']$ has a 2-factor by Theorem 2.1. Also we can prove that $G[X - N_C(Y) \cup C']$ is hamiltonian by Chv$\acute{a}$tal-Erdös Theorem. Then G has a 2-factor.

Hence we assume that $\kappa(G[Y \cup N_C(Y)]) = 1$. Then there exists a vertex $x \in X \backslash (N_C(Y))$ such that x is adjacent to some vertex in $Y_1 \cup Y_2 \cup Y_3$. Let $xy_1 \in E(G)$ with $y_1 \in Y_1$.

Sketch the proof of the pair $\{K_{1,4}, P_3 \cup 2K_1\}$

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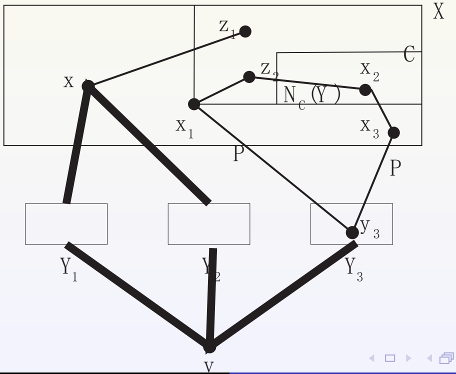
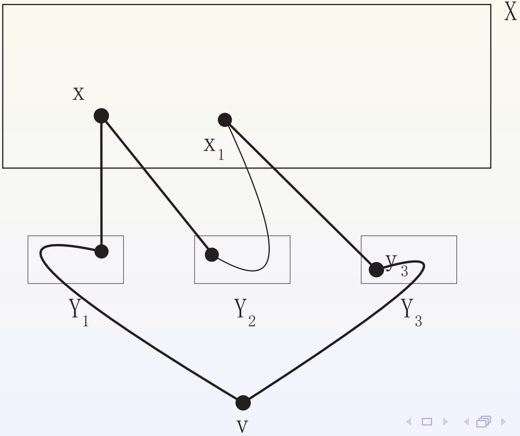
Hence we assume that $\kappa(G[Y \cup N_C(Y)]) = 1$. Then there exists a vertex $x \in X \backslash (N_C(Y))$ such that x is adjacent to some vertex in $Y_1 \cup Y_2 \cup Y_3$. Let $xy_1 \in E(G)$ with $y_1 \in Y_1$.

Subcase 2.2.1. x has a neighbor $z_1 \in C \backslash N_C(Y)$.

We can prove that x is adjacent to every vertex in $Y_1 \cup Y_2$. Let $z_2 \in V(C) \backslash (N_C(Y) \cup \{z_1\})$. Choose a shortest path P joining z_2 and some vertex $y_3 \in Y_3$ such that the internal vertices of P are in X. Clearly, the length of P is at most three.

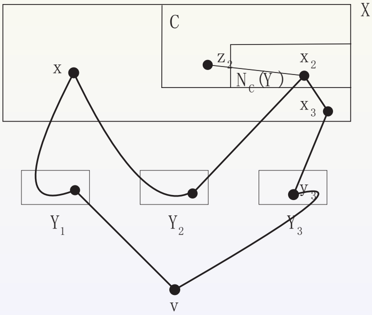


- If the length of P is two, denoted by $z_2x_1y_3$, then we can prove that x_1 is adjacent to every vertex in $Y_2 \cup Y_3$, and we can find a 2-factor of G with exactly two components.



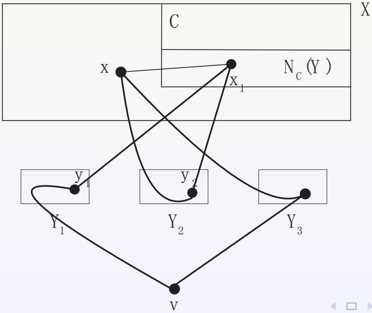
Sketch the proof of the pair $\{K_{1,4}, P_3 \cup 2K_1\}$

- If the length of P is three, denoted by $z_2x_2x_3y_3$, then we can prove that x_2 is adjacent to some vertex in $Y_1 \cup Y_2$, and thus we can find a 2-factor of G with exactly two components.



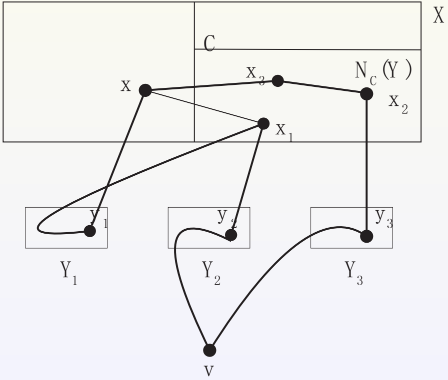
Subcase 2.2.2. $N_C(x) \subseteq N_C(Y)$.
Let x_1 be a neighbor of x in $N_C(Y)$.

- If x has a neighbor in $Y_2 \cup Y_3$, then we can prove that x is adjacent to every vertex in $Y_1 \cup Y_2 \cup Y_3$. Also we can prove that x_1 has two neighbors y_1 and y_2 in Y_1 and Y_2, respectively, then G has a 2-factor with exactly two components.



Sketch the proof of the pair $\{K_{1,4}, P_3 \cup 2K_1\}$

- If x has no neighbor in $Y_2 \cup Y_3$, then we can prove x_1 is adjacent to every vertex in $Y_1 \cup Y_2$. Also we can prove that there exists a vertex x_2 in $N_C(Y)$ has a neighbor in Y_3, then G has a 2-factor with exactly two components.



Thanks for your attention!