Approximation Methods for Optimal Active Fault Detection

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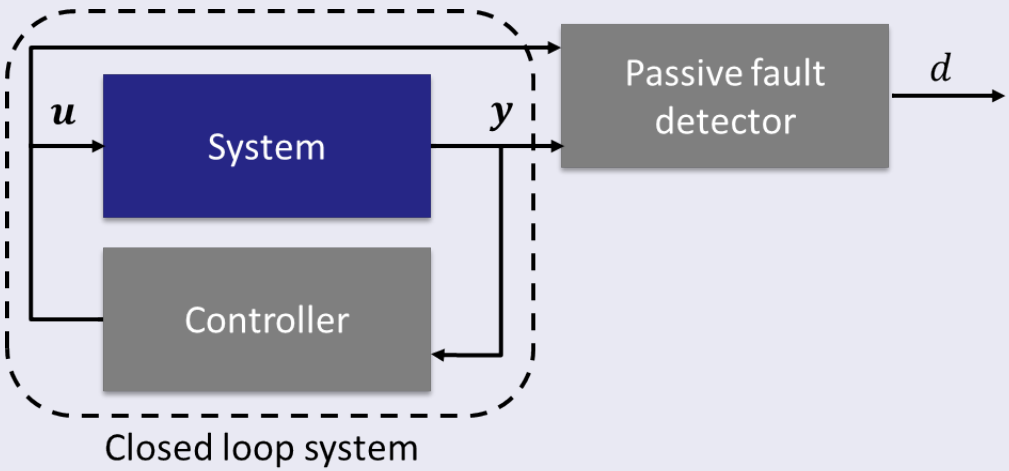
Outline

Introduction

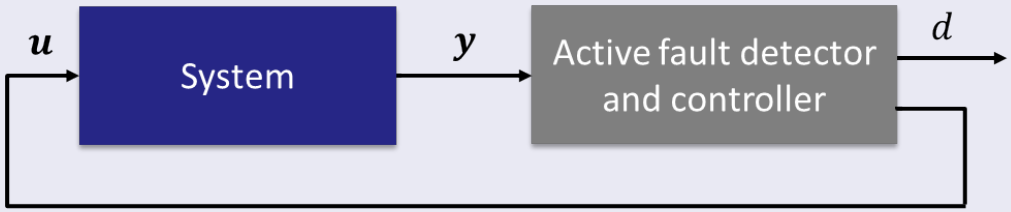
Passive and active fault detection



- **Passive detector** uses the input and output data [**u**, **y**] to generate a decision d about faults in the system, no input signal improving the quality of detection is generated.
- **Active detector** uses the output data **y** to generate a decision d and an input signal **u** that controls and probes the system.

Introduction

Passive and active fault detection



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Introduction

Goals of the paper

- To design active fault detector for non-linear systems over an infinite time horizon with a discounted criterion.
- To compare the value iteration algorithm and the policy iteration algorithm for designing approximate active fault detector.

Problem formulation

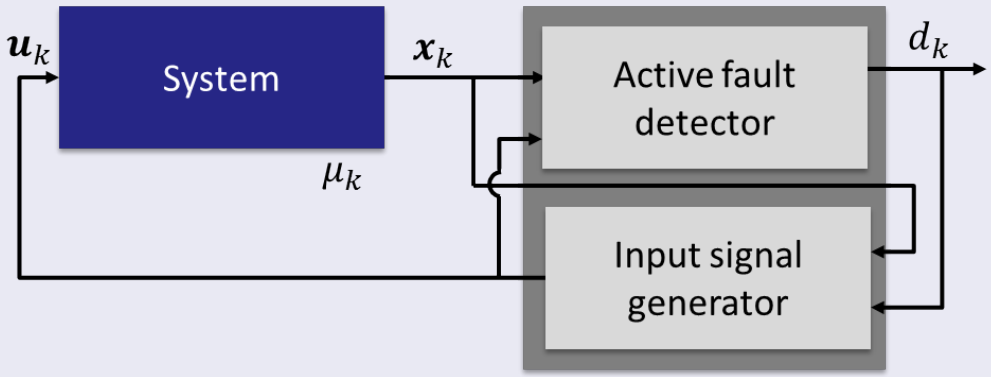
System description

$$\mathbf{x}_{k+1} = \mathbf{f}\left(\mathbf{x}_k, \mu_k, \mathbf{u}_k\right) + \mathbf{w}_k, \tag{1}$$

$\mathbf{x}_k^{\mathrm{a}} = [\mathbf{x}_k^{\mathsf{T}}, \mu_k]^{\mathsf{T}} \in \mathbb{R}^{n_x} \times \mathcal{M}$ - a hybrid state at time step k,
$\mathbf{x}_k \in \mathbb{R}^{n_x}$ - a known common part of state, $\mu_k \in \mathcal{M} = \{1, 2, \ldots, N\}$- an unknown index of the fault-free or faulty model,
$\mathbf{u}_k \in \mathcal{U} \subset \mathbb{R}^{n_u}$ - an input from the set $\mathcal{U} = \{\bar{\mathbf{u}}^1, \bar{\mathbf{u}}^2, \ldots, \bar{\mathbf{u}}^M\}$,
$\mathbf{w}_k \in \mathbb{R}^{n_x}$ - a state noise with the known pdf $p_{\mathbf{w}}(\mathbf{w}_k | \mathbf{x}_k^{\mathrm{a}})$, mean value $\mathbf{m}_{\mathbf{w}}(\mathbf{x}_k^{\mathrm{a}})$ and covariance matrix $\mathbf{P}_{\mathbf{w}}(\mathbf{x}_k^{\mathrm{a}})$,
$P_{i,j} = P(\mu_{k+1} = j | \mu_k = i)$ - known transition probabilities of switching between models,
$p(\mathbf{x}_0)$ - a known initial pdf of $\mathbf{x}_0$,
$P(\mu_0)$ - a known probability of μ_0.

Problem formulation

Active fault detector



Two actions: **decision** about faults and **input** to the system.

$$\begin{bmatrix} d_k \\ \mathbf{u}_k \end{bmatrix} = \boldsymbol{\rho}_k \left(\begin{bmatrix} \mathbf{x}_0^k \\ \mathbf{u}_0^{k-1} \end{bmatrix} \right), \tag{2}$$

$\boldsymbol{\rho}_k : \mathbb{R}^{n_x(k+1)} \times \mathcal{U}^k \mapsto \mathcal{M} \times \mathcal{U}$ - an unknown policy, $d_k \in \mathcal{M}$ - a decision (an estimate of μ_k), $\mathbf{x}_i^j = [\mathbf{x}_i^\mathsf{T}, \mathbf{x}_{i+1}^\mathsf{T}, \ldots, \mathbf{x}_j^\mathsf{T}]^\mathsf{T}$.

Problem formulation

Optimality criterion

- The main focus is on detection.
- A goal is to minimize the discounted criterion with a discount factor λ

$$J(\boldsymbol{\rho}) = \lim_{F\to\infty} \mathrm{E}\left\{\sum_{k=0}^{F} \lambda^k L^{\mathrm{d}}(\mu_k, d_k)\right\}, \ \lambda \in (0,1), \qquad (3)$$

the detection cost function $L^{\mathrm{d}}(\mu_k, d_k)$ penalizes wrong decisions.

Problem reformulation

Why problem reformulation?

- The original problem formulation belongs to a class of imperfect state information problems.
- Reformulation as a perfect state information problem to use standard approaches of dynamic programming.
- System, active fault detector and optimality criterion are reformulated using hyper-state $\mathbf{s}_k$.

Problem reformulation

System

Reformulation as a perfect state information problem

$$\mathbf{s}_{k+1} = \phi(\mathbf{s}_k, \mathbf{u}_k, \mathbf{x}_{k+1}), \tag{4}$$

$\mathbf{s}_k = [\mathbf{x}_k,\ \mathbf{b}_k]^T \in \mathcal{S}$ - a hyper-state (perfect state information),
$\mathbf{b}_k = [b_{k,1},\ \ldots,\ b_{k,i},\ \ldots,\ b_{k,N-1}]^T \in \mathcal{B}$ - a belief of models,
$b_{k,i} = P(\mu_k = i | \mathbf{x}_0^k, \mathbf{u}_0^{k-1})$,
ϕ - a non-linear vector function.

Problem reformulation

Active fault detector

Reformulated using the hyper-state $\mathbf{s}_k$ and stationary policies $\sigma : \mathcal{S} \mapsto \mathcal{M}$ and $\gamma : \mathcal{S} \mapsto \mathcal{U}$,

$$\begin{bmatrix} d_k \\ \mathbf{u}_k \end{bmatrix} = \begin{bmatrix} \sigma(\mathbf{s}_k) \\ \gamma(\mathbf{s}_k) \end{bmatrix} = \boldsymbol{\rho}(\mathbf{s}_k), \tag{5}$$

$\sigma : \mathcal{S} \mapsto \mathcal{M}$ detector, $\gamma : \mathcal{S} \mapsto \mathcal{U}$ input signal generator.

Problem reformulation

Optimality criterion

Reformulation using the hyper-state $\mathbf{s}_k$

$$\bar{J}(\boldsymbol{\rho}, \mathbf{s}_0) = \lim_{F \to \infty} \mathrm{E} \left\{ \sum_{k=0}^{F} \lambda^k \bar{L}^{\mathrm{d}}(\mathbf{s}_k, d_k) | \mathbf{s}_0 \right\}, \tag{6}$$

$\bar{L}^{\mathrm{d}} : \mathcal{S} \times \mathcal{M} \mapsto \mathbb{R}^+$ - a detection cost function that is equivalent to $L^{\mathrm{d}}(\mu_k, d_k)$ and can be expressed as

$$\bar{L}^{\mathrm{d}}(\mathbf{s}_k, d_k) = \mathrm{E}\{L^{\mathrm{d}}(\mu_k, d_k) | \mathbf{x}_0^k, \mathbf{u}_0^{k-1}, d_k\}. \tag{7}$$

Optimal active fault detector

Design

To find the Bellman function V^* that solves functional equation

$$V^*(\mathbf{s}_k) = \min_{d_k \in \mathcal{M}} \mathrm{E}\left\{ \bar{L}^{\mathrm{d}}(\mathbf{s}_k, d_k) | \mathbf{s}_k, d_k \right\} + \lambda \min_{\mathbf{u}_k \in \mathcal{U}} \mathrm{E}\left\{ V^*(\mathbf{s}_{k+1}) | \mathbf{s}_k, \mathbf{u}_k \right\}. \tag{8}$$

Optimal detector σ^* and optimal input signal generator γ^*

$$d_k^* = \sigma^*(\mathbf{s}_k) = \arg\min_{d_k \in \mathcal{M}} \bar{L}^{\mathrm{d}}(\mathbf{s}_k, d_k), \tag{9}$$

$$\mathbf{u}_k^* = \gamma^*(\mathbf{s}_k) = \arg\min_{\mathbf{u}_k \in \mathcal{U}} \mathrm{E}\left\{ V^*(\mathbf{s}_{k+1}) | \mathbf{s}_k, \mathbf{u}_k \right\}. \tag{10}$$

Analytical solution to the Bellman equation is almost impossible to find. Numerical methods are employed.

Approximate active fault detector

What to do to solve the Bellman equation

- The hyper-state space is quantized by a uniform grid with grid points $\bar{\mathbf{s}} \in \mathbb{R}^{n_s}$. Hyper-states projected to the grid using aggregation function.
- The Bellman function is approximated by a piecewise constant function $\bar{V}$ found by two numerical methods of dynamic programming: Value iteration and Policy iteration algorithms.
- Due to non-linearity of the system, the expectation in the Bellman equation is approximated by Unscented transform.

Approximate active fault detector

Grid design

Grid is defined by the Cartesian product
$\mathcal{S}^{\mathrm{g}} \triangleq \mathcal{S}_1^{\mathrm{g}} \times \mathcal{S}_2^{\mathrm{g}} \times \ldots \times \mathcal{S}_{n_s}^{\mathrm{g}} = \{\bar{\mathbf{s}}^r\}_{r=1}^{N_{\mathrm{g}}}$,
$\mathcal{S}_j^{\mathrm{g}}$ - a discrete set,
N_{g} - a number of grid points.

Aggregation design

Aggregation function $\mathbf{g} : \mathcal{S} \mapsto \mathcal{S}^{\mathrm{g}}$ project hyper-states to grid points

$$\bar{\mathbf{s}}_k = \mathbf{g}(\mathbf{s}_k) = \arg\min_{\boldsymbol{\xi} \in \mathcal{S}^{\mathrm{g}}} \|\mathbf{s}_k - \boldsymbol{\xi}\|_2. \tag{11}$$

Approximate active fault detector

Value iteration over grid points (VI)

Iterative computation of the recursive functional relation

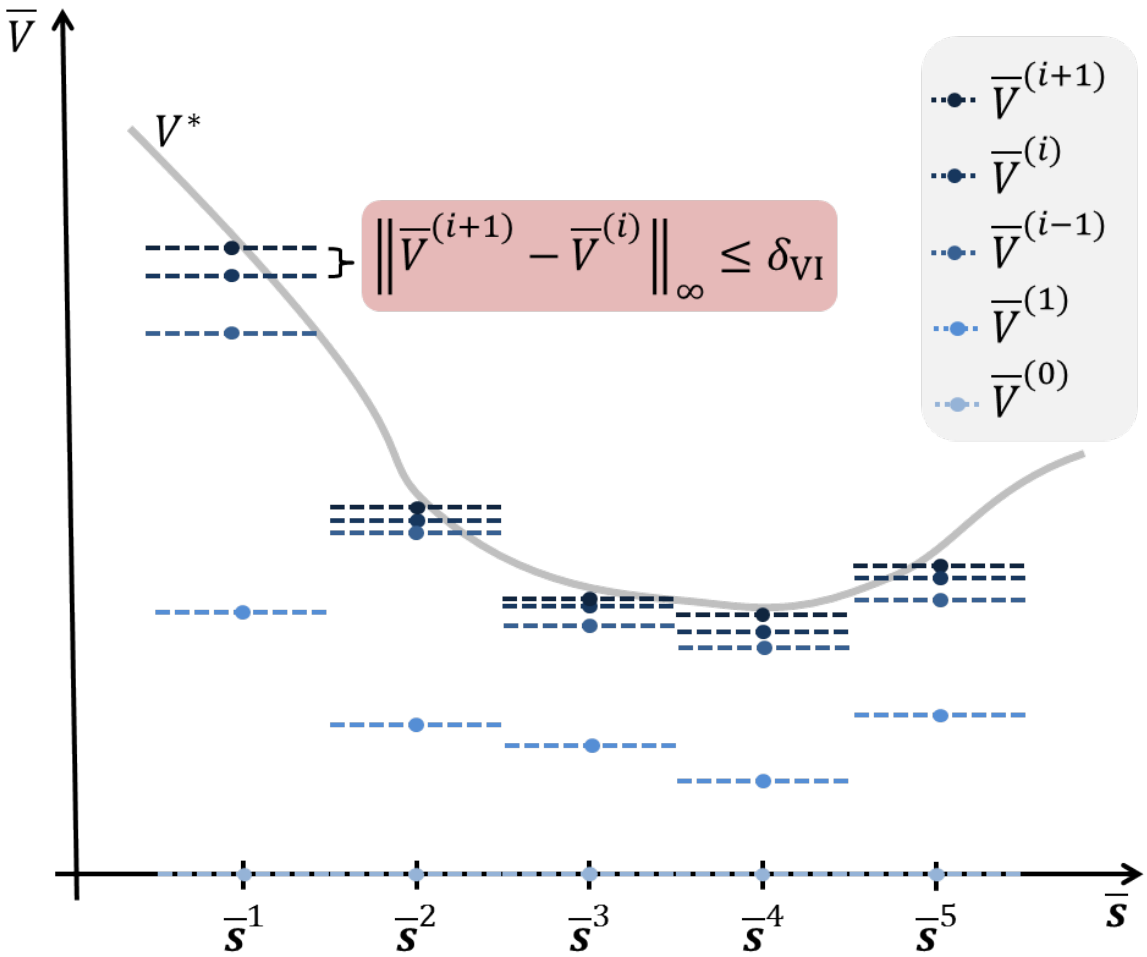
$$\bar{V}^{(i+1)}(\boldsymbol{\xi}) = \min_{d \in \mathcal{M}} \bar{L}^{\mathrm{d}}(\boldsymbol{\xi}, d) + \lambda \min_{\mathbf{u} \in \mathcal{U}} \mathsf{E}\left\{ \bar{V}^{(i)}(\boldsymbol{\xi}') | \boldsymbol{\xi}, \mathbf{u} \right\}, \tag{12}$$

$\bar{V}^{(i)} : \mathcal{S}^{\mathrm{g}} \mapsto \mathbb{R}$ - a value function approximation, $\bar{V}^{0} = \mathbf{0}$,
$\boldsymbol{\xi} \in \mathcal{S}^{\mathrm{g}}$ - a grid point,
$\boldsymbol{\xi}' = \mathbf{g}(\phi(\boldsymbol{\xi}, \mathbf{u}, \mathbf{x}')) \in \mathcal{S}^{\mathrm{g}}$ - the grid point resulting from using $\mathbf{u}$ at $\boldsymbol{\xi}$,
i - the iteration index.

- Theoretical computational complexity: $\mathcal{O}\left(MN_{\mathrm{g}}^{2}\right)$.

Approximate active fault detector

- Value iteration



Approximate active fault detector

Policy iteration over grid points (PI)

1. Evaluate the policy $\gamma^{(i)}$ to find the corresponding $\bar{V}_{\gamma^{(i)}}$,

$$\bar{V}_{\gamma^{(i)}}(\boldsymbol{\xi}) = \min_{d \in \mathcal{M}} \bar{L}^d(\boldsymbol{\xi}, d) + \lambda \, \mathrm{E}\left\{ \bar{V}_{\gamma^{(i)}}(\boldsymbol{\xi}') | \boldsymbol{\xi}, \gamma^{(i)}(\boldsymbol{\xi}) \right\}, \quad (13)$$

   solve the equation using GEM or iteratively.

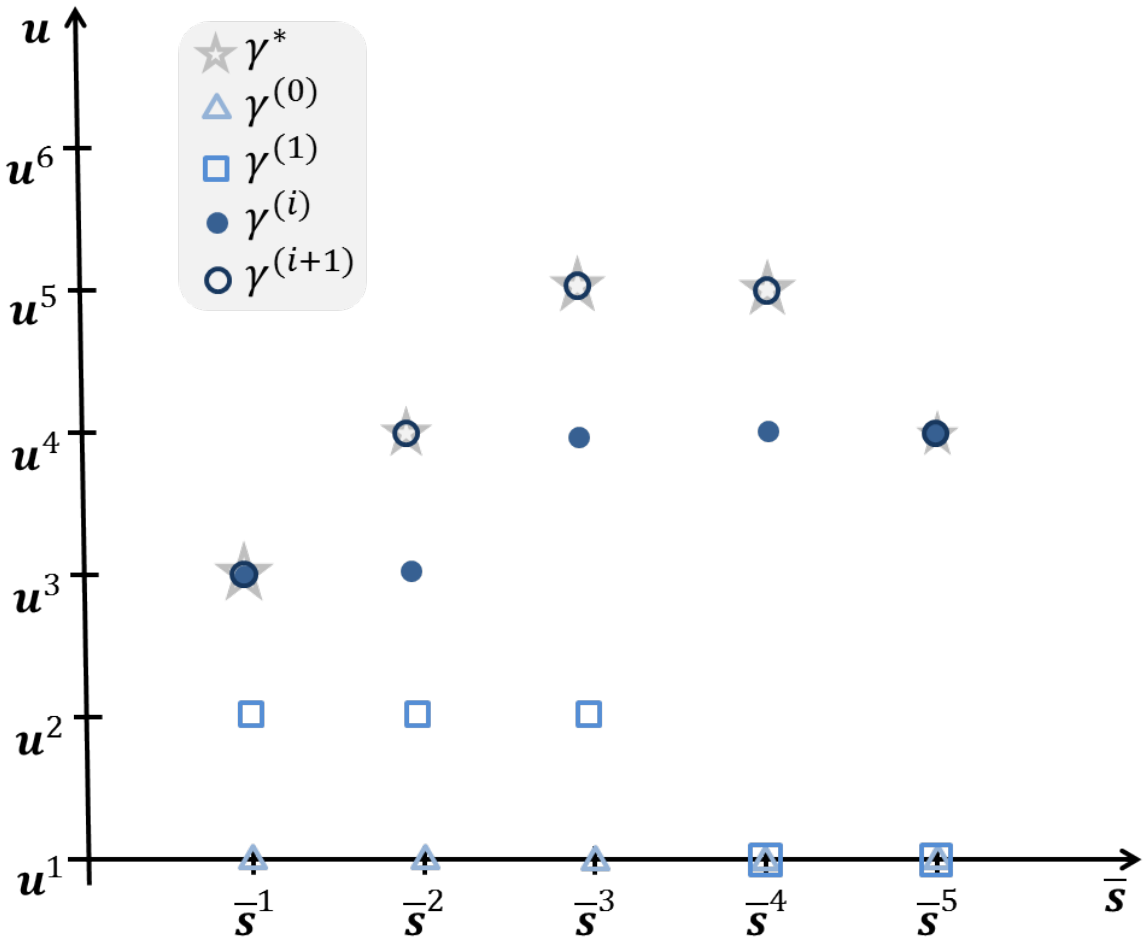
2. Improve the policy by solving optimization problem

$$\gamma^{(i+1)}(\boldsymbol{\xi}) = \arg\min_{\mathbf{u} \in \mathcal{U}} \lambda \, \mathrm{E}\left\{ \bar{V}_{\gamma^{(i)}}(\boldsymbol{\xi}') | \boldsymbol{\xi}, \mathbf{u} \right\}, \quad (14)$$

- Theoretical computational complexity: $\mathcal{O}\left(MN_{\mathrm{g}}^2\right)$ (policy improvement) + $\mathcal{O}\left(N_{\mathrm{g}}^3\right)$ (policy evaluation by GEM).

Approximate active fault detector

- Policy iteration (policy improvement)



Numerical example

Pendulum

- An intermittent fault in the rolling bearings resulting in a change of the viscous damping.
- Two non-linear discretized state-space models: fault-free model and faulty model with a different viscous damping coefficient.
- Task to compare the value iteration and policy iteration algorithms is terms of approximate fault detector and computational complexity.
- Two grids used - Coarse Grid (CG) with 3267 grid points, Fine Grid (FG) with 112413 grid points.

Numerical example

Criterion

Zero-one detection cost function

$$L^{\mathrm{d}}(\mu_k, d_k) = \begin{cases} 0 & \text{if } d_k = \mu_k, \\ 1 & \text{otherwise.} \end{cases}$$

Numerical example

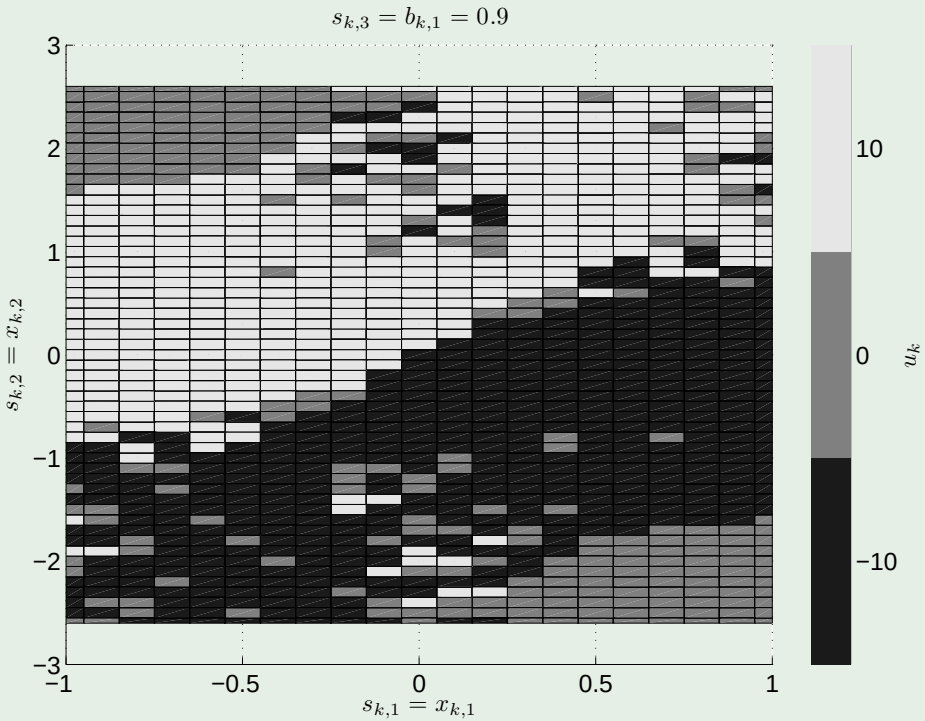
Computational complexity

The estimates of the criterion computed by 10000 MC simulations for a finite horizon of 500 steps and comparison with a simple random signal generator.

grid/alg	T_{iter} [s]	n_{iter}	T_{total} [s]	$\hat{J}^{\text{MC}}$	$3\sigma_{\hat{J}^{\text{MC}}}$
random	n/a	n/a	n/a	3.0536	0.0670
CG/VI	1.71	5	8.6	2.4759	0.0530
CG/PI	2.34	4	9.4	2.4760	0.0530
FG/VI	59.91	40	2396	1.2737	0.0290
FG/PI	79.16	11	871	1.2505	0.0285

Numerical example

Input signal generator



- Corresponding policies of VI and PI differ only in few grid points.

Conclusion

Conclusion

- Proposed a design of approximate active fault detector for non-linear stochastic systems over an infinite-time horizon.
- Two numerical methods of dynamic programming: value iteration and policy iteration algorithms were compared.
- The policy iteration has a higher computational requirements than the value iteration. However it is more time efficient due to faster convergence.
- The detection quality of both active fault detectors is comparable.
- Input signal generator of the approximate active fault detector provides better quality detection that a simple random input signal generator.