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Rozcvička

① Dokažte, že $1+2+\ldots+n = \binom{n+1}{2}$.

② Show that $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$ for all $n\in\mathbb{N}$.

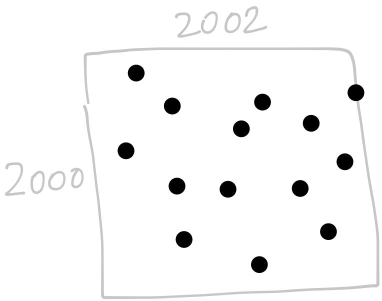
D2

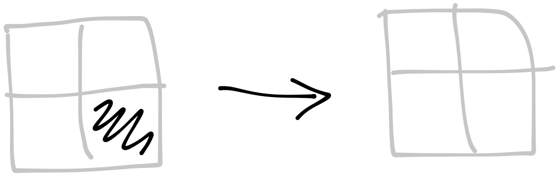
Dané přirozené číslo n má číslice, jejichž hodnoty se zleva doprava zvětšují. Ukažte, že ciferný součet čísla $9n$ je vždy roven devíti. Návod: $9n = 10n - n$.

D4

Lukáš a Marek, kteří se znají, se sešli na párty, na níž platilo: Mají-li někteří dva účastníci stejný počet známých, pak nemají žádného společného známého. Dokažte, že na párty je někdo, kdo tam má právě jednoho známého. Návod: Nejprve vyberte jednoho účastníka, který zná na párty největší počet osob.

2. Form a 2000×2002 screen with unit screens. Initially, there are more than 1999×2001 unit screens which are on. In any 2×2 screen, as soon as there are 3 unit screens which are off, the 4th screen turns off automatically. Prove that the whole screen can never be totally off.





D5

Ve společnosti lidí jsou některé dvojice spřátelené. Pro každé celé $k \geq 3$ řekneme, že společnost je k-dobrá, pokud lze každou k-tici lidí ze společnosti rozesadit kolem kruhového stolu tak, že se každí dva sousedé přátelí. Dokažte, že je-li společnost 6-dobrá, pak je i 7-dobrá. [A–67–III–1]

D3

Nalezněte největší možné přirozené číslo, jehož každá číslice (kromě obou krajních) je menší než aritmetický průměr číslic sousedních.

Fibonacciho čísla

The Fibonacci sequence

$$F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, \ldots$$

is defined as follows: $F_1 = F_2 = 1$ and $F_n = F_{n-2} + F_{n-1}$ for all $n \geq 3$. Prove that the following statements are true for all $n \in \mathbb{N}$.

(a) $F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$.

(b) $|F_{n+2}F_n - (F_{n+1})^2| = 1$.

(c) 5 divides n if and only if 5 divides F_n.

13. (IMO 2002) Let n be a positive integer. Each point (x, y) in the plane, where x and y are non-negative integers with $x + y < n$, is colored red or blue, subject to the following condition: if a point (x, y) is red, then so are all points (x', y') with $x' \leq x$ and $y' \leq y$. Let A be the number of ways to choose n blue points with distinct x-coordinates, and let B be the number of ways to choose n blue points with distinct y-coordinates. Prove that $A = B$.

[**Mathematical Induction**, *Slinko*] On each planet in a planetary system consisting of an odd number of planets there is a astronomer observing the nearest planet. The distances between each pair of planets are all different. Prove that at least one planet is not observed by an astronomer.

9. (IMO Shortlist 2002) Let n be a positive integer. A sequence of n positive integers (not necessarily distinct) is called *full* if it satisfied the following condition: for each positive integer $k \geq 2$, if the number k appears in the sequence then so does the number $k-1$, and moreover the first occurrence of $k-1$ comes before the last occurrence of k. For each n, show that there are $n!$ full sequences.

EXAMPLE 3.4. *[USA MATHEMATICAL OLYMPIAD, 1978] An integer n is called* good *if we can write*

$$a_1 + a_2 + \cdots + a_k = n,$$

where $a_1, a_2, \ldots, a_k$ are positive integers (not necessarily distinct) satisfying

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k} = 1.$$

Given the information that the integers 33 through to 73 are good, prove every integer greater than 32 is good.

D6

Ve skupině 90 dětí má každé aspoň 30 kamarádů (kamarádství je vzájemné). Dokažte, že je lze rozdělit do tří 30členných skupin tak, aby každé dítě mělo ve své skupině aspoň jednoho kamaráda. [A–61–III–5]

①

k rozmyšlení doma

2. (IMO 2015) We say that a finite set $\mathcal{S}$ of points in the plane is *balanced* if, for any two different points A and B in $\mathcal{S}$, there is a point C in S such that $AC = BC$. We say that $\mathcal{S}$ is *centre-free* if for any three different points A, B, and C in $\mathcal{S}$, there is no point P in $\mathcal{S}$ such that $PA = PB = PC$.

   (a) Show that for all integers $n \geq 3$, there exists a balanced set consisting of n points.

   (b) Determine all integers $n \geq 3$ for which there exists a balanced centre-free set consisting of n points.

②

3. (IMO 2013) Assume that k and n are two positive integers. Prove that there exist positive integers $m_1, \ldots, m_k$ such that

$$1 + \frac{2^k - 1}{n} = \left(1 + \frac{1}{m_1}\right) \ldots \left(1 + \frac{1}{m_k}\right)$$

③

3. (IMO 1977) In a finite sequence of real numbers the sum of any seven successive terms is negative and the sum of any eleven successive terms is positive. Determine the maximum number of terms in the sequence.

④

2. (IMO 1983) Is it possible to choose 1983 distinct positive integers, all less than or equal to 100,000, no three of which are consecutive terms of an arithmetic progression?