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6. 12. 2023

Geometrie

E1. *The midpoints P, Q, R, S of any quadrilateral in plane or space are vertices of a parallelogram.*

E2. *Reconstruct a pentagon from the midpoints* P, Q, R, S, T *of its sides.*

E3. *Let $ABCDEF$ be any hexagon, and let $A_1B_1C_1D_1E_1F_1$ be the hexagon of the centroids of the triangles ABC, BCD, CDE, DEF, EFA, FAB. Then the $A_1B_1C_1D_1E_1F_1$ has parallel and equal opposites sides.*

E4. *Let $ABCD$ be a quadrilateral, and let $A'B'C'D'$ be the quadrilateral of the centroids of BCD, CDA, DAB, ABC. Show that $ABCD$ can be transformed into $A'B'C'D'$ by a stretch from some point Z. Find Z and the stretch factor t.*

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E5. *Find the centroid S of n points $A_1, \ldots, A_n$ defined by*

$$\sum_{i=1}^{n} \overrightarrow{SA_i} = \overrightarrow{O}\,.$$

E6. *The diagonals of a quadrilateral are orthogonal if and only if the sums of the squares of opposite sides are equal.*

E7. *The diagonals of a quadrilateral are orthogonal iff its medians have equal length.*

E8. *Let A, B, C, D be four points in space. Then we always have*

$$|AB|^2 + |CD|^2 - |BC|^2 - |AD|^2 = 2\overrightarrow{AC} \cdot \overrightarrow{DB}.$$

E9. *Someone found in his attic an old description of a pirate, who died long ago. It read as follows: Go to the island X, start at the gallows, go to the elm tree, and count the steps. Then turn left by* 90°*, and go the same number of steps until point* g'*. Again, go from the gallows to the fig tree, and count the steps. Then turn right by* 90°*, and go the same number of steps to the point* g''*. A treasure is buried in the midpoint* t *of* $g'g''$*.*

A man went to the island and found the elm tree e and the fig tree f. But the gallows could not be traced. Find the treasure point t.

E10. Napoleonic Triangles. *If one erects regular triangles outwardly (inwardly) on the sides of a triangle, then their centers are vertices of a regular triangle (outer and inner Napoleonic triangles).*

E11. *Squares are erected outwardly on the sides of a quadrilateral. If the centers of the squares are* x, y, z, u, *then the segments* xz *and* yu *are perpendicular and of equal length.*

E12. *Squares* $cbqp$ *and* $acmn$ *are erected outwardly on the sides* bc *and* ac *of the triangle* abc*. Show that the midpoints* d, e *of these squares, the midpoint* g *of* ab*, and the midpoint* f *of* mp *are vertices of a square.*

E13. *Let* $a_1b_1c_1$ *and* $b_1b_2b_3$ *be two, positively oriented, regular triangles and let* c_i *be the midpoint of* a_ib_i*. Then* $c_1c_2c_3$ *is a regular triangle.*

E14. *Let $A,\ B,\ C,\ D$ be four points in a plane. Then*

$$|AB| \cdot |CD| + |BC| \cdot |AD| \geq |AC| \cdot |BD| \quad \textit{(Ptolemy's inequality).}$$

Problems

1. Show that
$$|AC|^2 + |BD|^2 = |AB|^2 + |BC|^2 + |CD|^2 + |DA|^2 \iff A + C = B + D.$$

2. Let $A,\ B,\ C,\ D$ be four space points. Prove the theorem: *If, for all points X in space, $|AX|^2 + |CX|^2 = |BX|^2 + |DX|^2$, then $ABCD$ is a rectangle.*

3. Rectangles $ABDE$, $BCFG$, $CAHI$ are erected outwardly on the sides of a triangle ABC. Show that the perpendicular bisectors of the segments HE, DG, FI are concurrent.

4. A regular n-gon $A_1 \cdots A_n$ is inscribed in a circle with center O and radius R. X is any point with $d = |OX|$. Then $\sum_{i=1}^{n} |A_i X|^2 = n(R^2 + d^2)$.

5. Let ABC be a regular triangle inscribed in a circle. Then $PA^n + PB^n + PC^n$ is independent of the choice of P on the circle for $n = 2,\ 4$.

6. For any point P of the circumcircle of the square $ABCD$, the sum $PA^n + PB^n + PC^n + PD^n$ is independent of the choice of P if $n = 2,\ 4,\ 6$.
7. Prove Euler's theorem: *In a quadrilateral $ABCD$ with medians MN and PQ,* $|AC|^2 + |BD|^2 = 2\left(|MN|^2 + |PQ|^2\right)$.
8. Find the locus of all points X, which satisfy $\overrightarrow{AX} \cdot \overrightarrow{CX} = \overrightarrow{CB} \cdot \overrightarrow{AX}$.
9. Three points $A,\ B,\ C$ are such that $|AC|^2 + |BC|^2 = |AB|^2/2$. What is the relative position of these points?
10. If M is a point and $ABCD$ a rectangle, then $\overrightarrow{MA} \cdot \overrightarrow{MC} = \overrightarrow{MB} \cdot \overrightarrow{MD}$.
11. The points E, F, G, H divide the sides of the quadrilateral $ABCD$ in the same ratios. Find the condition for $EFGH$ to be a parallelogram.
12. Let Q be an arbitrary point in the plane and M be the midpoint of AB. Then $|QA|^2 + |QB|^2 = 2|QM|^2 + |AB|^2/2$.
13. Let $A,\ B,\ C,\ D$ denote four points in space and AB the distance between A and B, and so on. Show that $AC^2 + BD^2 + AD^2 + BC^2 \geq AB^2 + CD^2$.