What's on the menu this week? – an appetizer

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Domažlice, March 2013

Contents

Why did you get invited for dinner?

- The overall motivation is to continue the workshops of 1995–2011 in Enschede, Nectiny, Hannover, Hajek and Domažlice in order to make **progress** on several intriguing conjectures.
- These highly interrelated conjectures involve line graphs, claw-free graphs, cubic graphs, snarks, and concepts like Hamilton cycles, Hamilton-connectedness, dominating closed trails (circuits), and dominating cycles.
- Perhaps there is a link to double cycle covers and nowhere zero flows, or other conjectures; there is a link to the P vs NP millennium problem.
- In order to introduce the workshop topics, I was asked to repeat some background, and present a **survey** on some of the conjectures and their relationships.
- More details can be found in the **survey paper** copied from Graphs and Combinatorics.

Starting from scratch: the first two conjectures

The following two conjectures were tossed in the eighties.

Matthews & Sumner, 1984:

Conjecture (MS-Conjecture)

Every 4-connected **claw-free graph** *is* **hamiltonian**.

Thomassen, 1986:

Conjecture (T-Conjecture)

Every 4-connected **line graph** *is* **hamiltonian**.

Let me start by explaining the **terminology** to understand the above statements and their relationship.

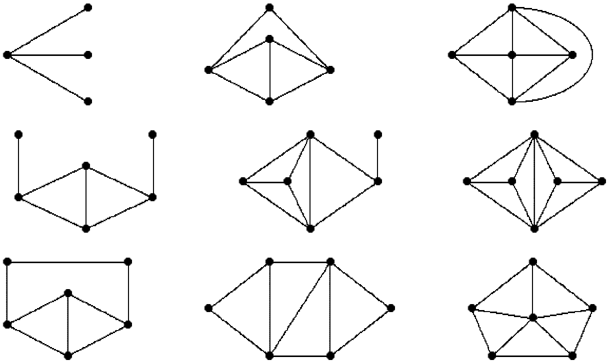
Basic graph terminology

- All graphs in this talk are finite, undirected, loopless and the majority is simple (in some results we allow multiple edges) .
- We denote a graph G as $G = (V, E)$, where $V = V(G)$ is the vertex set and $E = E(G)$ is the edge set.
- A graph is **hamiltonian** if it contains a cycle through all its vertices, i.e., a connected spanning 2-regular subgraph.
- If H is a graph, then the **line graph of** H, denoted by $L(H)$, is the graph on vertex set $E(H)$ in which two distinct vertices in $L(H)$ are adjacent if and only if their corresponding edges in H share an end vertex (with a straightforward extension in case of multiple edges).
- A graph is a **line graph** if it is isomorphic to $L(H)$ for some graph H.
- Which graphs are line graphs and which are not?

A forbidden subgraph characterization of line graphs

Theorem (Beineke, 1969)

A graph is a line graph if and only if *it does not contain a copy of any of the following nine graphs as an* **induced** *subgraph.*



Forbidden induced subgraphs – claw-free graphs

- Let G be a graph and let S be a nonempty subset of $V(G)$. Then the **subgraph of G induced by S**, denoted by $G[S]$, is the graph with vertex set S, and all edges of G with both end vertices in S.
- H is an **induced subgraph** of G if it is induced in G by some subset of $V(G)$.
- G is **H-free** if H is not an induced subgraph of G.
- In particular, a graph G is **claw-free** if G does not contain a copy of the **claw** $K_{1,3}$ as an induced subgraph.
- Direct inspection of Beineke's result shows:

  every line graph is claw-free

The two earlier conjectures revisited

Conjecture (MS)

Every 4-connected claw-free graph is hamiltonian.

Conjecture (T)

Every 4-connected line graph is hamiltonian.

- Since line graphs are claw-free the first conjecture is **stronger** than the second one.
- Or are they **equivalent**? (A question Herbert Fleischner posed during the EIDMA workshop on Hamiltonicity of 2-tough graphs, Hotel Hölterhof, Enschede, November 19-24, 1995.)

A useful tool: the closure

- To answer the question affirmatively, Zdeněk Ryjáček introduced a **closure concept** for claw-free graphs at the same workshop.
- It is based on adding edges to a graph G without destroying the (non)hamiltonicity (similar to the Bondy-Chvátal closure).
- The edges are added by looking at a vertex v and the subgraph of G induced by $N(v)$: the set of *neighbors* of v.
- If $G[N(v)]$ is **connected** and not a complete graph, all edges are added to turn $G[N(v)]$ into a **complete** graph.
- This procedure is repeated in the new graph, etc., until it is impossible to add any more edges.

The two conjectures are equivalent

Theorem (Ryjáček, 1997)

Let G be a **claw-free** *graph. Then*

- *the closure* $\mathrm{cl}(G)$ *is uniquely determined,*
- $\mathrm{cl}(G)$ *is hamiltonian if and only if G is hamiltonian,*
- $\mathrm{cl}(G)$ *is the* **line graph** *of a triangle-free graph.*

Corollary (using a result of Zhan, 1991)

Every 7-connected claw-free graph is hamiltonian.

The conjectures are false for 3-connected graphs. The best positive result to date is the following by Tomáš Kaiser and Petr Vrána (2012).

Theorem (Kaiser and Vrána, 2012)

Every **5-connected** *claw-free graph with* **minimum degree at least 6** *is hamiltonian.*

From line graphs to their root graphs

- Whenever we consider a **line graph** G, we can identify a graph H such that $G = L(H)$ (in polynomial time).
- If G is connected this H is unique, except for $G = K_3$: then H can be K_3 or $K_{1,3}$ (this is different for multigraphs).
- If we take $K_{1,3}$ in this exceptional case, we can talk of a unique H as the **root graph** of the connected line graph G isomorphic to $L(H)$.
- What is the counterpart in H of a **Hamilton cycle** in G?
- A *closed trail* (circuit) is a connected eulerian subgraph, i.e., a connected subgraph in which all degrees are even.
- A **dominating closed trail** (DCT or D-circuit) is a closed trail T such that every edge has at least one end vertex on T.

Hamilton cycles and dominating closed trails

There is an intimate relationship between DCTs in H and Hamilton cycles in $L(H)$.

Theorem (Harary and Nash-Williams, 1965)

Let H be a graph with at least three edges. Then $L(H)$ is hamiltonian if and only if H contains a DCT.

- What is the counterpart in H of 4-connectivity in $L(H)$? Note that 4-edge-connectivity is not the right answer!
- A graph H is *essentially 4-edge-connected* if it contains no edge-cut R such that $|R| < 4$ and at least two components of $H - R$ contain an edge.
- $L(H)$ is 4-connected if and only if H is essentially 4-edge-connected.

Another equivalent conjecture

The previous results and observations imply that the following conjecture is **equivalent** to the two we have seen before.

Conjecture (DCT-conjecture)

Every **essentially** *4-edge-connected graph has a DCT.*

- Note that 4-edge-connected graphs contain two edge-disjoint spanning trees.
- Hence 4-edge-connected graphs contain a **spanning** closed trail, in particular a DCT.
- So line graphs of 4-edge-connected graphs are hamiltonian (and *Hamilton-connected*).
- So the gap looks small ......

From root graphs to cubic graphs

- If H is *cubic*, i.e., 3-regular, then a DCT becomes a *dominating cycle* (abbreviated DC).
- A cubic graph is essentially 4-edge-connected if and only if it is cyclically 4-edge-connected.
- H is *cyclically 4-edge-connected* if H contains no edge-cut R such that $|R| < 4$ and at least two components of $H - R$ contain a cycle.

Conjecture (Ash & Jackson, 1984)

Every cyclically 4-edge-connected **cubic** *graph has a DC.*

Fleischner and Jackson (1989) proved that this conjecture is **equivalent** to the others.

Main ingredient: Let H be an essentially 4-edge-connected graph of minimum degree $\delta(G) \geq 3$ and let $v \in V(H)$ be of degree $d(v) \geq 4$. Then some *cubic inflation* of H at v is essentially 4-edge-connected.

Two weaker (?) conjectures from the same paper

Conjecture (Jaeger, $\leq$ 1989)

Every cyclically 4-edge-connected cubic graph G has a cycle C such that $G - V(C)$ is acyclic.

Conjecture (Bondy, $\leq$ 1989)

Every cyclically 4-edge-connected cubic graph G on n vertices has a cycle C of length at least $c \cdot n$, for some constant c with $0 < c < 1$.

It is obvious that the conjecture of Ash-Jackson implies the conjecture of Jaeger, and one can show that the conjecture of Jaeger implies the conjecture of Bondy.

Are they equivalent?

From cubic graphs to non-3-edge colorable cubic graphs

For *non-3-edge colorable* cubic graphs we have the following conjecture of Herbert Fleischner.

Conjecture (F-Conjecture)

Every cyclically 4-edge-connected cubic graph that is **not 3-edge-colorable** *has a DC.*

Kochol (2000) proved that it is **equivalent** to the others.

One direction is obvious. For the other direction, he was assuming a counterexample to the previous conjecture and used it as a black box building block. In combination with an auxiliary gadget that is almost cubic and not 3-edge-colorable he constructed a counterexample to the F-Conjecture.

From non-3-edge colorable cubic graphs to snarks

A *snark* is a cyclically 4-edge-connected cubic graph of **girth at least** 5 that is not 3-edge-colorable.

Conjecture (Snark-Conjecture)

Every snark has a DC.

The above conjecture is also **equivalent** to the others, as shown by B., Fijavž, Kaiser, Kužel, Ryjáček & Vrána (2008), using the constructive approach together with the concept of *contractible subgraphs*.

To date this is the **seemingly weakest** conjecture equivalent to the others.

Is there a link to the **Double Cycle Conjecture**?
Is there a link to **Nowhere Zero Flows**?

Let me turn to some **seemingly stronger** conjectures in the remainder of the talk.

Seemingly stronger versions for cubic graphs

Fouquet & Thuillier (1990) established a seemingly stronger but equivalent version of the Ash-Jackson-Conjecture.

It is stronger in the sense that they require a DC that contains two given disjoint edges, as follows.

Conjecture

In a cyclically 4-edge-connected cubic graph **any two disjoint edges** *are on a DC.*

The equivalence was extended by Fleischner & Kochol (2002) by requiring a DC through any two given edges.

Conjecture

In a cyclically 4-edge-connected cubic graph **any two edges** *are on a DC.*

Seemingly stronger versions for cubic graphs

There are several further equivalent versions involving **subgraphs** of cubic graphs.

Conjecture (Kužel, 2008)

Any subgraph H of an essentially 4-edge-connected cubic graph G with $\delta(H) = 2$ and $|V_2(H)| = 4$ is $V_2(H)$-dominated.

Conjecture (Kužel, Ryjáček & Vrána, 2012)

Any subgraph H of an essentially 4-edge-connected cubic graph G with $\delta(H) = 2$ and $|V_2(H)| = 4$ is strongly $V_2(H)$-dominated.

Here we assume that H has 4 vertices with degree 2, and we require that $G + \{e, f\}$ has a DCT containing any two new (parallel) matching edges e and f added between pairs of these 4 vertices, and for the second statement also that $G + e$ has a DCT containing any newly added edge e between a pair of these vertices.

Seemingly stronger versions for cubic graphs

Conjecture (B., Fijavž, Kaiser, Kužel, Ryjáček & Vrána, 2008)

Every cyclically 4-edge-connected cubic graph G contains a weakly $A_G(F)$-contractible subgraph F with $\delta(F) = 2$.

For a graph F and a set $A \subset V(F)$, F is called (weakly) A-contractible if for every (nonempty) even subset $X \subset A$ and for every partition P of X into 2-element subsets, the graph F^P has a DCT containing A and $E(P)$.

Here $E(P) = \{uv \mid u, v$ are in the same equivalence class of $P\}$ and F^P is the multigraph obtained from F by adding the edges of $E(P)$.

If a 2-connected cubic graph H has a weakly $A_H(F)$-contractible subgraph F, then H has a DC if and only if H/F has a DCT.

Here $A_H(F)$ are the vertices of F that have a neighbor in $V(H) \setminus V(F)$.

Back to line graphs – seemingly stronger versions

A graph is *Hamilton-connected* if there is a Hamilton path between any two vertices.

Kužel & Xiong (2004) established equivalence with the following conjecture.

Conjecture

Every 4-connected line graph of a multigraph is **Hamilton-connected***.*

Ryjáček & Vrána (2011) extended the equivalence to claw-free graphs.

Conjecture

Every 4-connected claw-free graph is Hamilton-connected.

Here the 2-closure played a crucial role: adding edges in the neighborhood if it induces a 2-connected graph. This does not always result in a line graph (of a multigraph): squares of cycles have to be treated differently.

A link to the P versus NP problem

At present the seemingly strongest version of the conjectures is by Kužel, Ryjáček & Vrána (2012).

A graph G is *1-Hamilton-connected* if for any vertex x of G there is a Hamilton path in $G - x$ between any two vertices. They also define a slightly stronger property called *2-edge-Hamilton-connectivity*.

Conjecture

Every 4-connected line graph of a multigraph is **1-Hamilton-connected** *(2-edge-Hamilton-connected).*

The first statement can be extended to claw-free graphs. This is based on a further extension of the multigraph closure technique: the line graph obtained is of a multigraph with at most two triangles or at most one double edge.

This suggests that Thomassen's Conjecture (and all equivalent versions) might fail.

How close are we to refuting the conjectures?

If the above conjecture is true, it implies that a line graph is 1-Hamilton-connected (2-edge-Hamilton-connected) **if and only if** it is 4-connected.

The connectivity of a (line) graph can be determined in **polynomial time**.

It is an **NP-complete** problem to decide whether a **line graph** is hamiltonian (Bertossi, 1981).

It is not difficult to show that deciding whether a given **graph** is 1-Hamilton-connected is also NP-complete.

It seems **likely** that deciding whether a given graph is 1-Hamilton-connected remains NP-complete when restricted to line graphs. If we can show this, it implies that

Thomassen's Conjecture cannot be true, unless P=NP.

How close are we to proving the conjectures?

The gap between the conjecture(s) and the positive results is narrowing.

If we drop the connectivity condition of the 2-regular spanning subgraph, we move from a Hamilton cycle to a *2-factor*. Enomoto, Jackson, Katerinis & Saito (1985) proved that every *2-tough* graph contains a 2-factor. This implies:

Theorem

Every 4-connected claw-free graph has a **2-factor***.*

It does not seem easy to use this as a starting point to show that there is a 2-factor with only one component, although there are some results that give upper bounds on the number of components. These results are beyond the scope of this talk.

Relaxing the 4-connectedness and adding something else

I do not want to discuss degree conditions in this talk, but here is a connectivity-only result.

If we add an 'essentially connectivity' condition there is this result due to Lai, Shao, Wu & Zhou (2006).

Theorem

Every 3-connected, **essentially 11-connected** *claw-free (line) graph is hamiltonian.*

Kaiser and Vrána claim that 11 can be replaced by 9, while 5 would be best possible (by the line graph of the Petersen graph in which the edges of a perfect matching are subdivided exactly once).

Question: how far can we decrease the 9 by raising the 3 to 4 in the theorem?

Restrictions on the root graph

Lai (1994) proved the following partial affirmative answer to Thomassen's Conjecture.

Theorem

Every 4-connected line graph of a planar graph is hamiltonian.

Kriesell (2001) proved a similar result on line graphs of claw-free (multi)graphs with the stronger conclusion of Hamilton-connectedness. In fact, he proved the following more general result.

Theorem

Let G be a graph such that $L(G)$ is 4-connected and every vertex of degree 3 in G is on an edge of multiplicity at least 2 or on a triangle of G. Then $L(G)$ is Hamilton-connected.

Restrictions on the root graph

For **quasi claw-free** graphs, i.e., in which all vertices u, v at distance 2 have a common neighbor w with $N(w) \subseteq N[u] \cup N[v]$, we have:

Theorem (Lai, Shao & Zhan, 2004)

Every 4-connected line graph of a **quasi claw-free** *graph is Hamilton-connected.*

There are many results along these lines.

In most proofs the root graphs are considered and the aim is to find a (closed or open) trail (internally) dominating all edges. A **common approach** is the following.

First the degree 1 vertices are deleted, then the degree 2 vertices are suppressed, and now one tries to show that the reduced graph has a suitable spanning (closed) trail. This is very similar to using two edge-disjoint spanning trees, or collapsibility, or advanced closure concepts.

Bon appétit, guten Appetit, dobrou chut', eet smakelijk!

My favourite desserts for the end of the week:

- we will have improved on the currently best result that 5-connected claw-free graphs with $\delta \geq 6$ are hamiltonian, or
- we will have established the complexity of 1-Hamilton-connectedness for line graphs, or
- we will have shown that all conjectures are equivalent to the famous Double Cycle Cover Conjecture, or
- . . . this one is up to you!

I hope you did not lose your appetite!