Supereularian graphs and hamiltonicity of line graphs

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Introduction

Backgrounds and notations

- Matthews and Sumner (1984) conjectured: Every 4-connected claw-free graph is hamiltonian.
- Thomassen (1986) conjectured: that every 4-connected line graph is hamiltonian.
- Zhan (1991): Every 7-connected line graph is hamiltonian connected.
- Ryjáček (1997): (i) The two conjectures are equivalent. (ii) Every 7-connected claw-free graph is hamiltonian.
- Kaiser and Vrana (2012): Every 5-connected line graph with minimum degree at least 6 is hamiltonian.
- Lai et al. (2006) considered whether the high essential connectivity of the 3-connected line graphs can guarantee the existence of the hamiltonian cycle in graphs and they showed that every 3-connected, essentially 11-connected line graph is hamiltonian.
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Background and notations

- **dominating trail (cycle):** A trail (cycle) is a *dominating trail (cycle)* if each edge of $G$ is incident with at least one vertex of the trail (cycle).

- **dominating trailable:** A graph is *dominating trailable* if for each pair of $x$ and $y$ of edges of $G$ there exists a dominating trail with end edges $x$ and $y$, respectively.

Spanning closed trail (or spanning eulerian subgraph), Supereulerian graphs, spanning trailable

Core of a graph: denoted by $G_0$, is obtained from $G$ by deleting all the vertices of degrees 1, and contracting exactly one of the two edges $xy$ or $yz$ for each path $xyz$ in $G$ with $d_G(y) = 2$.

If $G_0$ is spanning trailable (supereulerian), then $L(G)$ is hamiltonian connected (hamiltonian).

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Introduction

Catlin’s Reduction

Collapsible: Let $O(G)$ denote the set of odd degree vertices of $G$. A graph $G$ is collapsible if for any subset $R \subseteq V(G)$ with $|R| \equiv 0 \pmod{2}$, $G$ has a spanning connected subgraph $H_R$ such that $O(H_R) = R$. 

If $G$ is collapsible, then $G$ is supereulerian and then $L(G)$ is hamiltonian.

Let $G(e_1, e_2)$ be the graph obtained from $G$ by subdividing the edges $e_1$ and $e_2$. If $G(e_1, e_2)$ is collapsible for any two edges $e_1$ and $e_2$ of $G$, then $G$ is spanning trailable and then $L(G)$ is hamiltonian connected.

Reduction of $G$: The contraction $G/H$ is called the reduction of $G$ if $H$ is the maximal collapsible subgraph of $G$, i.e. there is no non-trivial collapsible subgraph in $G/H$. 

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Collapsible: Let $O(G)$ denote the set of odd degree vertices of $G$. A graph $G$ is collapsible if for any subset $R \subseteq V(G)$ with $|R| \equiv 0$ (mod 2), $G$ has a spanning connected subgraph $H_R$ such that $O(H_R) = R$.

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Catlin's Reduction

Catlin (1988) : If $H$ is a collapsible subgraph of $G$, then $G$ is collapsible if and only if $G/H$ is collapsible; $G$ is supereulerian if and only if $G/H$ is supereulerian.
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- If $G$ contains two edge disjoint trees, then $G$ is collapsible.
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- If $G$ contains two edge disjoint trees, then $G$ is collapsible.

- *Catlin*, (1988): If $G$ is reduced and if $|E(G)| \geq 3$, then $\delta(G) \leq 3$, and $2|V(G)| - |E(G)| \geq 4$. 

Introduction

**R-closure**

- Ryjáček (1997): R-closure, which turns the hamiltonicity of claw-free graphs to line graphs.
- Ryjáček and Vrána (2010): multigraphs closure, which turns the hamiltonian connectedness of claw-free graphs to line graphs.
Theorem

Every 3-connected essentially 11-connected line graph is hamiltonian connected.
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Used Claim in the proof
- Catlin’s Reduction mentioned above
- If the line graph $L(G)$ is 3-connected and essentially 11-connected, then the core of $G$ is such that $G_0(e_1, e_2)$ is collapsible.
Theorem: Edge-disjoint trees

If the line graph $L(G)$ is 3-connected and essentially 10-connected, then the core of $G$ contains two edge-disjoint spanning trees.
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If the line graph $L(G)$ is 3-connected and essentially 10-connected, then the core of $G$ contains two edge-disjoint spanning trees.

Used Theorem and a Claim

- Nash-Williams and Tutte: A finite graph $G$ can be decomposed into $k$ connected factors if and only if

$$|S| \geq k(\omega(G - S) - 1)$$

for each subset $S$ of $E(G)$.

- Let $G$ be a graph with minimum degree $\delta \geq 3$. If $d_G(e) \geq 7$ for every edge $e \in E(G[D_3(G) \cup T(G)])$, then $|E(G)| \geq 2|V(G)|$. ($D_3(G)$: the set of vertices of degree 3; $T(G)$ denotes $N(D_3(G))$)
Every 3-connected essentially 10-connected line graph is hamiltonian connected

**Used Theorem**

Catlin and Lai: Let $G$ be a graph and let $e_1, e_2 \in E(G)$. If $G$ has two edge-disjoint spanning trees, then exactly one of the following holds:

(a) $G$ has a spanning $(e_1, e_2)$-trail.
(b) $\{e_1, e_2\}$ is an edge-cut of $G$. 
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Theorem

Every 3-connected essentially 10-connected line graph is hamiltonian connected.
Let $L(G)$ be a 3-connected, essentially 4-connected line graph of a graph $G$. If $G$ has at most 9 (sharp) vertices of degree 3, then $L(G)$ is hamiltonian.
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If a $R$-closure of a claw-free graph $G$ contains at most 9 maximal 3-cliques, then $G$ is hamiltonian.
Problem 1

What is the minimum integer $k \geq 5$ such that every 3-connected, essentially $k$-connected line graph is hamiltonian (hamiltonian connected)?
Problems on 3-connected line graphs

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- Kužel and Xiong show that every 4-connected line is hamiltonian if and only if it is hamiltonian connected.
Problems on 3-connected line graphs

**Problem 1**
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- Kužel and Xiong show that every 4-connected line is hamiltonian if and only if it is hamiltonian connected.

**Problem 2**
What is the minimum integer $k \geq 5$ such that every 3-connected, essentially $k$-connected line graph graph is hamiltonian if and only if it is hamilton-connected?
Supereulerian and edge-degree condition

- Edge degree of \( e = uv \in E(G) \): \( d(e) = d(u) + d(v) - 2 \) (if \( e \) is a loop of \( G \), then \( u = v \))
- \( \xi(G) = \min\{d(e) : e \in E(G)\} \)

Problem 3

Is a 3-edge connected, essentially 4-edge connected graph with \( \xi \geq 7 \) is supereulerian?

Chen and shan: If Problem 3 is "YES", then every 4-connected claw-free graph with minimum degree 5 is hamiltonian.

If Problem 3 is "YES", then every 3-edge connected, essentially 5 edge-connected graph is supereulerian.
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- If Problem 3 is "YES", then every 3-edge connected, essentially 5 edge-connected graph is supereulerian.
Supereulerian and edge-degree condition

\( \lambda_3(G) \)

3-restricted edge connectivity \( \lambda_3(G) \): The minimum size of the edge cuts whose removal such that the resulting graph containing two components of order at least 3.

If a 3-edge connected graph \( G \) with \( \xi \geq 7 \) and \( \lambda_3 \geq 7 \), then \( G \) is collapsible.
Supereulerian and edge-degree condition

$\lambda_3(G)$

3-restricted edge connectivity $\lambda_3(G)$: The minimum size of the edge cuts whose removal such that the resulting graph containing two components of order at least 3.

If a 3-edge connected graph $G$ with $\xi \geq 7$ and $\lambda_3 \geq 7$, then $G$ is collapsible.

If a 3-edge connected simple graph $G$ with $\xi \geq 7$ and $\lambda_3 \geq 6$, then $G$ is collapsible.
Thanks for your attention!