Supereulerian graphs and hamiltonicity of line graphs

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Outlines

1. **Introduction**
   - Background and notations
   - Catlin's Reduction
   - Ryjáček's Closure

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   - 3-connected and essentially 11-connected line graphs
   - 3-connected and essentially 10-connected line graphs
   - 3-connected and essentially 4-connected line graphs
   - problems

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   - 3-connected and essentially 10-connected line graphs
   - 3-connected and essentially 4-connected line graphs
   - problems
3. **Supereulerian graphs and edge-degree condition**

Introduction

Backgrounds and notations

- Matthews and Sumner (1984) conjectured: Every 4-connected claw-free graph is hamiltonian.
- Thomassen (1986) conjectured: that every 4-connected line graph is hamiltonian.

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  (ii) Every 7-connected claw-free graph is hamiltonian.
- Kaiser and Vrána (2012): Every 5-connected line graph with minimum degree at least 6 is hamiltonian.
- Lai et al. (2006) considered whether the high essential connectivity of the 3-connected line graphs can guarantee the existence of the hamiltonian cycle in graphs and they showed that every 3-connected, essentially 11-connected line graph is hamiltonian.

Introduction

Background and notations

- dominating trail (cycle): A trail (cycle) is a *dominating trail* (*cycle*) if each edge of G is incident with at least one vertex of the trail (cycle).
- dominating trailable: A graph is *dominating trailable* if for each pair of x and y of edges of G there exists a dominating trail with end edges x and y, respectively.

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- If G_0 is spanning trailable (supereulerian), then $L(G)$ is hamiltonian connected (hamiltonian).

Introduction

Catlin's Reduction

- Collapsible: Let $O(G)$ denote the set of odd degree vertices of G. A graph G is *collapsible* if for any subset $R \subseteq V(G)$ with $|R| \equiv 0 (\mod 2)$, G has a spanning connected subgraph H_R such that $O(H_R) = R$.

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- If $G(e_1, e_2)$ is collapsible for any two edges e_1 and e_2 of G, then G is spanning trailable and then $L(G)$ is hamiltonian connected.

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- G is collapsible, then G is supereulerian and then $L(G)$ is hamiltonian.
- Let $G(e_1, e_2)$ be the graph obtained from G by subdividing the edges e_1 and e_2.
- If $G(e_1, e_2)$ is collapsible for any two edges e_1 and e_2 of G, then G is spanning trailable and then $L(G)$ is hamiltonian connected.
- Reduction of G: The contraction G/H is called the *reduction* of G if H is the maximal collapsible subgraph of G, i.e. there is no non-trivial collapsible subgraph in G/H.

Introduction

Catlin's Reduction

- *Catlin*(1988) : If H is a collapsible subgraph of G, then G is collapsible if and only if G/H is collapsible; G is supereulerian if and only if G/H is supereulerian.

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- If G contains two edge disjoint trees, then G is collapsible.

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- If G contains two edge disjoint trees, then G is collapsible.
- *Catlin*, (1988) :. If G is reduced and if $|E(G)| \geq 3$, then $\delta(G) \leq 3$, and $2|V(G)| - |E(G)| \geq 4$.

Introduction

R-closure

- Ryjáček (1997): R-closure, which turns the hamiltonicity of claw-free graphs to line graphs.
- Ryjáček and Vrána (2010): multigraphs closure, which turns the hamiltonian connectedness of claw-free graphs to line graphs.

3-connected essentially 11-connected line graph

Theorem

Every 3-connected essentially 11-connected line graph is hamiltonian connected.

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Every 3-connected essentially 11-connected line graph is hamiltonian connected.

Used Claim in the proof

- Catlin's Reduction mentioned above
- If the line graph $L(G)$ is 3-connected and essentially 11-connected, then the core of G is such that $G_0(e_1, e_2)$ is collapsible.

3-connected essentially 10-connected line graph

Theorem: Edge-disjoint trees

If the line graph $L(G)$ is 3-connected and essentially 10-connected, then the core of G contains two edge-disjoint spanning trees.

3-connected essentially 10-connected line graph

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If the line graph $L(G)$ is 3-connected and essentially 10-connected, then the core of G contains two edge-disjoint spanning trees.

Used Theorem and a Claim

- Nash-Williams and Tutte: A finite graph G can be decomposed into k connected factors if and only if

$$|S| \geq k(\omega(G - S) - 1)$$

  for each subset S of $E(G)$.
- Let G be a graph with minimum degree $\delta \geq 3$. If $d_G(e) \geq 7$ for every edge $e \in E(G[D_3(G) \cup T(G)])$, then $|E(G)| \geq 2|V(G)|$. ($D_3(G)$: the set of vertices of degree 3; $T(G)$ denotes $N(D_3(G))$)

Every 3-connected essentially 10-connected line graph is hamiltonian connected

Used Theorem

Catlin and Lai: Let G be a graph and let $e_1, e_2 \in E(G)$. If G has two edge-disjoint spanning trees, then exactly one of the following holds:
(a) G has a spanning (e_1, e_2)-trail.
(b) $\{e_1, e_2\}$ is an edge-cut of G.

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Theorem

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3-connected essentially 4-connected line graph

- Let $L(G)$ be a 3-connected, essentially 4-connected line graph of a graph G. If G has at most 9 (sharp) vertices of degree 3, then $L(G)$ is hamiltonian.

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- Let $L(G)$ be a 3-connected, essentially 4-connected line graph of a graph G. If G has at most 9 (sharp) vertices of degree 3, then $L(G)$ is hamiltonian.
- If a R-closure of a claw-free graph G contains at most 9 maximal 3-cliques, then G is hamiltonian.

Problems on 3-connected line graphs

Problem 1

What is the minimum integer $k \geq 5$ such that every 3-connected, essentially k-connected line graph graph is hamiltonian (hamiltonian connected)?

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Problem 2

What is the minimum integer $k \geq 5$ such that every 3-connected, essentially k-connected line graph graph is hamiltonian if and only if it is hamilton-connected?

Supereulerian and edge-degree condition

- Edge degree of $e = uv \in E(G)$: $d(e) = d(u) + d(v) - 2$ (if e is a loop of G, then $u = v$)
- $\xi(G) = \min\{d(e) : e \in E(G)\}$

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Is a 3-edge connected, essentially 4-edge connected graph with $\xi \geq 7$ is supereulerian?

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- If Problem 3 is "YES", then every 3-edge connected, essentially 5 edge-connected graph is supereuerlian.

Supereulerian and edge-degree condition

$\lambda_3(G)$

3-restricted edge connectivity $\lambda_3(G)$: The minimum size of the edge cuts whose removal such that the resulting graph containing two components of order at least 3.

If a 3-edge connected graph G with $\xi \geq 7$ and $\lambda_3 \geq 7$, then G is collapsible.

Supereulerian and edge-degree condition

$\lambda_3(G)$

3-restricted edge connectivity $\lambda_3(G)$: The minimum size of the edge cuts whose removal such that the resulting graph containing two components of order at least 3.

If a 3-edge connected graph G with $\xi \geq 7$ and $\lambda_3 \geq 7$, then G is collapsible.

If a 3-edge connected simple graph G with $\xi \geq 7$ and $\lambda_3 \geq 6$, then G is collapsible.

Thanks for your attention!